ON THE OSCILLATION OF SOLUTIONS OF A HIGHER ORDER NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH AN OSCILLATING COEFFICIENT

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ABSTRACT

In this paper, we are concerned with the oscillation of the solutions of a certain more general higher order nonlinear neutral type functional differential equation with an oscillating coefficient of the form

\[ y(t) + \sum_{i=1}^{m} P_i(t) y(\tau_i(t)) + \sum_{i=1}^{m} Q_i(t) f_i(y(\sigma_i(t))) = 0 \]

where \( n \geq 2 \); \( P_i(t), Q_i(t), \tau_i(t) \in C[t_0, +\infty) \) for \( i = 1, 2, \cdots, m \); \( P_i(t) \) is an oscillatory function for \( i = 1, 2, \cdots, m \); \( Q_i(t) \) is positive valued for \( i = 1, 2, \cdots, m \). \( \sigma_i(t) \in C'[t_0, +\infty), \sigma'_i(t) > 0, \]

\( \sigma_i(t) \leq t; \sigma_i(t) \rightarrow +\infty \) as \( t \rightarrow \infty \) for \( i = 1, 2, \cdots, m \); \( \tau_i(t) \rightarrow +\infty \)

as \( t \rightarrow \infty \) for \( i = 1, 2, \cdots, m \); \( f_i(u) \in C(R, R) \) is a nondecreasing function, \( uf_i(u) > 0 \) for \( u \neq 0 \) and \( i = 1, 2, \cdots, m \). We obtained two sufficient criteria for oscillatory behaviour of its solutions.

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ÖZET

Bu çalışmada; \( n \geq 2 \), \( i = 1, 2, \cdots, m \) için \( P_i(t), Q_i(t), \tau_i(t) \in C[t_0, +\infty) \); \( i = 1, 2, \cdots, m \) için \( P_i(t) \) ler sınırlı fonksiyonlar; \( i = 1, 2, \cdots, m \) için \( Q_i(t) \) ler pozitif değerli fonksiyonlar; \( \sigma_i(t) \in C'[t_0, +\infty) \), \( \sigma_i'(t) > 0 \), \( \sigma_i(t) \leq t \) ve \( t \to \infty \) iken \( i = 1, 2, \cdots, m \) için \( \tau_i(t) \to +\infty \); \( i = 1, 2, \cdots, m \) için \( f_i(u) \in C(R, R) \) ler azalmayan fonksiyonlar ve \( u \neq 0 \) iken \( uf_i(u) > 0 \) olmak üzere

\[
\left[ y(t) + \sum_{i=1}^{m} P_i(t) y(\tau_i(t)) \right]^{(n)} + \sum_{i=1}^{m} Q_i(t) f_i(y(\sigma_i(t))) = 0
\]

tipindeki yüksek mertebeden lineer olmayan diferensiyel denklem in çözümlerinin sınırlılığı üzerine yeter şartlı iki kriter elde edilmektedir.

1. INTRODUCTION.

We consider the higher order nonlinear differential equation of the form

\[
\left[ y(t) + \sum_{i=1}^{m} P_i(t) y(\tau_i(t)) \right]^{(n)} + \sum_{i=1}^{m} Q_i(t) f_i(y(\sigma_i(t))) = 0 \quad (1.1)
\]

where \( n \geq 2 \); \( P_i(t), Q_i(t), \tau_i(t) \in C[t_0, +\infty) \) for \( i = 1, 2, \cdots, m \); \( P_i(t) \) is an oscillatory function for \( i = 1, 2, \cdots, m \); \( Q_i(t) \) is positive valued for \( i = 1, 2, \cdots, m \). \( \sigma_i(t) \in C'[t_0, +\infty) \), \( \sigma_i'(t) > 0 \), \( \sigma_i(t) \leq t \); \( \sigma_i(t) \to +\infty \) as \( t \to \infty \) for \( i = 1, 2, \cdots, m \); \( \tau_i(t) \to +\infty \) as \( t \to \infty \) for \( i = 1, 2, \cdots, m \); \( f_i(u) \in C(R, R) \) is a nondecreasing function, \( uf_i(u) > 0 \) for \( u \neq 0 \) and \( i = 1, 2, \cdots, m \).

As is customary, a solution of Eq. (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.
For the sake of convenience, the function \( z(t) \) is defined by,

\[
z(t) = y(t) + \sum_{i=1}^{m} P_i(t) y(\tau_i(t)).
\] (1.2)

2. SOME AUXILIARY LEMMAS

**Lemma 2.1:** Let \( y(t) \) be a positive and \( n \)-times differentiable function on \([t_0, +\infty)\). If \( y^{(n)}(t) \) is of constant sign and not identically zero in any interval \([b, +\infty)\), then there exist a \( t_i \geq t_0 \) and an integer \( l, 0 \leq l \leq n \) such that \( n + l \) is even, if \( y^{(n)}(t) \) is nonnegative, or \( n + l \) odd, if \( y^{(n)}(t) \) is nonpositive, and that, as \( t \to t_i \), if \( l > 0 \), \( y^{(k)}(t) > 0 \) for \( k = 0, 1, 2, \ldots, l - 1 \), and if \( l \leq n - 1 \), \( (-1)^{k+1} y^{(k)}(t) > 0 \) for \( k = l, l + 1, \ldots, n - 1 \) [1].

**Lemma 2.2:** Let \( y(t) \) defined Lemma 2.1. Let \( y^{(n-1)}(t) y^{(n)}(t) \leq 0 \) \((t \geq t_0)\) and there exists a constant \( M > 0 \) for every \( \lambda (0 < \lambda < 1) \), such that \( y(\lambda t) \geq Mt^{n-1} |y^{(n-1)}(t)| \) for sufficiently large \( t \) [1].

3. THE MAIN RESULTS

**Theorem 3.1:** Assume that \( n \) is odd and

\[
C_1) \quad \lim_{t \to +\infty} \sum_{i=1}^{m} P_i(t) = 0,
\]

\[
C_2) \quad \int_{t_0}^{+\infty} s^{n-1} \sum_{i=1}^{m} Q_i(s) ds = +\infty.
\]

Then every bounded solution of Eq. (1.1) is either oscillatory or tends to zero as \( t \to +\infty \).

**Proof:** Assume that Eq. (1.1) has a bounded nonoscillatory solution \( y(t) \).
Without loss of generality, assume that \( y(t) \) is eventually positive (the proof is similar when \( y(t) \) is eventually negative). That is, \( y(t) > 0 \),
$y(t_i(t)) > 0$ and $y(\sigma_i(t)) > 0$ for $t \geq t_i \geq t_0$ and $i = 1, 2, \cdots, m$. Furthermore suppose that $y(t)$ does not tend to zero as $t \to +\infty$. By (1.1) and (1.2), we have for $t \geq t_i$

$$z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i(y(\sigma_i(t))) < 0 \quad (3.1)$$

That is, $z^{(n)}(t) < 0$. It follows that, $z^{(j)}(t)$ ($j = 0, 1, 2, \cdots, n-1$) is strictly monotone and of constant sign eventually. Since $y(t)$ is a bounded function and $\lim_{t \to \infty} \sum_{i=1}^{m} P_i(t) = 0$ for $i = 1, 2, \cdots, m$, there exists a $t_2 \geq t_1$ such that as $t \geq t_2$ $z(t) > 0$ eventually and there is a $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Because of $n$ is odd and $z(t)$ is bounded, by Lemma 2.1, when $l = 0$ (otherwise $z(t)$ is not bounded) there exists a $t_4 \geq t_3$ such that $(-1)^k z^{(k)}(t) > 0$ ($k = 0, 1, 2, \cdots, n-1$) as $t \geq t_4$. In particular, since $z'(t) < 0$ for $t \geq t_4$, $z(t)$ is decreasing. Since $z(t)$ is bounded, we may write $\lim_{t \to \infty} z(t) = L$ ($-\infty < L < +\infty$). Assume that $0 \leq L \leq +\infty$. Let be $L > 0$. Then there exists a constant $c > 0$ and a $t_5 \geq t_4$ such that $z(t) > c > 0$ for $t \geq t_5$. Since $y(t)$ is bounded, $\lim_{t \to \infty} \sum_{i=1}^{m} P_i(t) y(t_i(t)) = 0$ by (C.i). Therefore, there exist a constant $c_1 > 0$ and a $t_6 \geq t_5$ such that $y(t) = z(t) - \sum_{i=1}^{m} P_i(t) y(t_i(t)) > c_1 > 0$ for $t \geq t_6$. So, we may take a $t_7$ with the property of $t_7 \geq t_6$ such that $y(\sigma_i(t)) > c_1 > 0$ for $t \geq t_7$. From (3.1), we have

$$z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i(c_1) < 0, \ t \geq t_7 \quad (3.2)$$

If we multiply (3.2) by $t^{n-1}$ and integrate from $t_7$ to $t$ then we obtain

$$F(t) - F(t_7) = -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1} ds \quad (3.3)$$

where
\[ F(t) = \]
\[ t^{n-1}z^{(n-1)}(t) - (n-1)t^{n-2}z^{(n-2)}(t) + (n-1)(n-2)t^{n-3}z^{(n-3)}(t)\]
\[ \cdots - (n-1)(n-2)(n-3) \cdots 3.2tz'(t) \]
\[ + (n-1)(n-2)(n-3) \cdots 3.2lz(t) \]

Since \((-1)^k z^{(k)}(t) > 0\) for \(k = 0, 1, 2, \cdots, n-1\) and \(t \geq t_4\), \(F(t) > 0\) for \(t \geq t_7\). From (3.3), we have
\[ -F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1} ds \]

From (C_2), we obtain
\[ -F(t_7) \leq -f(c_1) \int_{t_7}^{t} \sum_{i=1}^{m} Q_i(s) s^{n-1} ds = -\infty \]
at \(t \to \infty\). This is a contradiction. So, \(L > 0\) is impossible. Therefore, \(L = 0\) is the only possible case. That is, \(\lim_{t \to \infty} z(t) = 0\). Since \(y(t)\) is bounded, we obtain
\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} z(t) - \lim_{t \to \infty} \sum_{i=1}^{m} P_i(t) y(\tau_i(t)) = 0 \]
by (C_1) and (1.2). Now let us consider the case of \(y(t) < 0\) for \(t \geq t_1\). By (1.1) an (1.2),
\[ z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i\left(y(\sigma_i(t))\right) < 0, \quad t \geq t_1. \]
That is, \(z^{(n)}(t) > 0\). It follows that, \(z^{(j)}(t) \quad (j = 0, 1, 2, \cdots, n-1)\) is strictly monotone and of constant sign eventually. Since \(y(t)\) is a bounded function and \(\lim_{t \to \infty} \sum_{i=1}^{m} P_i(t) = 0\) for \(i = 1, 2, \cdots, m\), there exists a \(t_2 \geq t_1\) such that as \(t \geq t_2\), \(z(t) > 0\) eventually and there is a \(t_3 \geq t_2\) such that \(z(t)\) is also bounded for \(t \geq t_3\). Assume that \(x(t) = -z(t)\). Then \(x^{(n)}(t) = -z^{(n)}(t)\). Therefore, \(x(t) > 0\) and \(x^{(n)}(t) < 0\) for \(t \geq t_3\).

Hence, we observe that \(x(t)\) is bounded. Since \(n\) is odd. By Lemma 2.1, there is a \(t_4 \geq t_3\) and \(l = 0\) (otherwise, \(x(t)\) is not bounded) such that
\((-1)^k x^{(k)}(t) > 0\) for \(k = 0, 1, 2, \cdots, n - 1\) and \(t \geq t_4\). That is \((-1)^k z^{(k)}(t) < 0\) for \(k = 0, 1, 2, \cdots, n - 1\) and \(t \geq t_4\). In particular, as \(t \geq t_4\), \(z'(t) > 0\). Therefore, \(z(t)\) is increasing. So, we can assume that 
\[
\lim_{t \to \infty} z(t) = L ~ (-\infty < L < +\infty).
\]
As in the proof of \(y(t) > 0\), we may prove that \(L = 0\). As for the rest of proof, it similar to the case of \(y(t) > 0\). That is, \(\lim_{t \to \infty} y(t) = 0\). Hence, the proof is completed □

**Theorem 3.2:** Assume that \(n\) is even and (C₁) is held. If (C₃) there is a function \(\varphi(t) \in C'[t_0, +\infty)\). Moreover

\[
\lim_{t \to \infty} \sup_{t_0} \int_t^{t_0} \varphi(s) \sum_{i=1}^{m} Q_i(s) ds = +\infty
\]

and

\[
\lim_{t \to \infty} \sup_{t_0} \int_t^{t_0} \left[ \frac{[\varphi'(s)]^2}{\varphi(s) \sigma_i'(s) \sigma_i^{m-2}(s)} \right] ds < +\infty
\]

for \(i = 1, 2, \cdots, m\) is satisfied, then every bounded solution of Eq. (1.1) is oscillatory.

**Proof:** Assume that Eq. (1.1) has a bounded nonoscillatory solution \(y(t)\).

Without loss of generality, assume that \(y(t)\) is eventually positive (the proof is similar when \(y(t)\) is eventually negative). That is, \(y(t) > 0\), \(y(\tau_i(t)) > 0\) and \(y(\sigma_i(t)) > 0\) for \(t \geq t_i \geq t_0\) and \(i = 1, 2, \cdots, m\). By (1.1) and (1.2), we have for \(t \geq t_1\)

\[
z^{(\nu)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i\left(y(\sigma_i(t))\right) < 0 \quad (3.4)
\]

That is, \(z^{(\nu)}(t) < 0\). It follows that, \(z^{(j)}(t) \quad (j = 0, 1, 2, \cdots, n - 1)\) is strictly monotone and of constant sign eventually. Since \(y(t)\) is a bounded function and 
\[
\lim_{t \to \infty} \sum_{i=1}^{m} P_i(t) = 0 \quad \text{for} \quad i = 1, 2, \cdots, m,
\]
there exists a \(t_2 \geq t_1\) such that as \(t \geq t_2\), \(z(t) > 0\) eventually and there is a \(t_3 \geq t_2\) such that \(z(t)\) is
also bounded for $t \geq t_3$. Because of $n$ is even, by Lemma 2.1, when $l = 1$ (otherwise $z(t)$ is not bounded) there exists a $t_4 \geq t_3$ such that as $t \geq t_4$
\[ (-1)^{k+1} z^{(k)}(t) > 0 \quad (k = 0, 1, 2, \cdots, n-1) \] (3.5)
In particular, since $z'(t) > 0$ for $t \geq t_4$, $z(t)$ is increasing. Since $y(t)$ is bounded, $\lim_{t \to \infty} \sum_{i=1}^{m} P_i(t) y(t_i(t)) = 0$ by (C_i). Then, there exists a $t_5 \geq t_4$ and $\delta$ positive integer,
\[ y(t) = z(t) - \sum_{i=1}^{m} P_i(t) y(t_i(t)) > \frac{1}{\delta} z(t) > 0 \]
for $t \geq t_5$ by (1.2). We may get $t_6 \geq t_5$ such that for $t \geq t_6$ and $i=1, 2, \cdots, m$
\[ y(\sigma_i(t)) > \frac{1}{\delta} z(\sigma_i(t)) > 0 \] (3.6)
From (3.4), (3.6) and the properties of $f$, we have
\[ z^{(n)}(t) \leq -\sum_{i=1}^{m} Q_i(t) f_i \left( \frac{1}{\delta} z(\sigma_i(t)) \right) \]
\[ = -\sum_{i=1}^{m} Q_i(t) \frac{f_i \left( \frac{1}{\delta} z(\sigma_i(t)) \right)}{z(\sigma_i(t))} z(\sigma_i(t)) \] (3.7)
for $t \geq t_6$. Since $z(t) > 0$ is bounded and increasing, $\lim_{t \to \infty} z(t) = L$ ($-\infty < L < +\infty$). By the continuity of $f$, we have
\[ \lim_{t \to \infty} \frac{f_i \left( \frac{1}{\delta} z(\sigma_i(t)) \right)}{z(\sigma_i(t))} = \frac{f_i \left( \frac{L}{\delta} \right)}{L} > 0. \]
Then, there is a $t_7 \geq t_6$ such that as $t \geq t_7$ for $i=1, 2, \cdots, m$
\[ \lim_{t \to \infty} \frac{f_i \left( \frac{1}{\delta} z(\sigma_i(t)) \right)}{z(\sigma_i(t))} = \frac{f_i \left( \frac{L}{\delta} \right)}{2L} = \alpha > 0. \] (3.8)
By (3.7) and (3.8), we obtain
\[ z^{(n)}(t) \leq -\alpha \sum_{i=1}^{m} Q_i(t) z(\sigma_i(t)) \quad \text{for } t \geq t_7. \] (3.9)
Define
\[ w(t) = \frac{z^{(n-1)}(t)}{z\left(\frac{1}{\delta} \sigma_i(t)\right)}. \]

We know from (3.5) that, there is a \( t_9 \geq t_7 \) such that \( w(t) > 0 \) for sufficiently large \( t \geq t_8 \). Since \( z(t) > 0 \) is increasing, there exists a \( t_9 \geq t_8 \) such that \( z(\sigma_i(t)) \geq z\left(\frac{1}{\delta} \sigma_i(t)\right) > 0 \) for sufficiently large \( t \geq t_9 \). We may get a result together with (3.9) such that
\[ w'(t) = \frac{z\left(\frac{1}{\delta} \sigma_i(t)\right) z^{(n)}(t) - z\left(\frac{1}{\delta} \sigma_i(t)\right) z^{(n-1)}(t) \frac{\sigma'_i(t)}{\delta}}{z^2\left(\frac{1}{\delta} \sigma_i(t)\right)} \tag{3.10} \]
\[ = \frac{z^{(n)}(t)}{z\left(\frac{1}{\delta} \sigma_i(t)\right)} - \frac{1}{\delta} w(t) \frac{z\left(\frac{1}{\delta} \sigma_i(t)\right)}{z\left(\frac{1}{\delta} \sigma_i(t)\right)} \sigma'_i(t). \]

We know from (3.5) that, \( z'(t) > 0 \) and \( z^{(n-1)}(t) > 0 \) for \( t \geq t_9 \). Since \( \sigma_i(t) \leq t \) and \( \sigma'_i(t) > 0 \), there exists a constant \( M > 0 \) and a \( t_{10} \geq t_9 \) such that
\[ z\left(\frac{1}{\delta} \sigma_i(t)\right) \geq M \sigma_i^{n-2}(t) \sigma'_i(t) z^{(n-1)}(\sigma_i(t)) \geq M \sigma_i^{n-2}(t) \sigma'_i(t) z^{(n-1)}(t) \]
for \( \lambda = \frac{1}{\delta} \) and \( z'(t) \) and \( t \geq t_{10} \) by Lemma 2.2. Therefore, we may get a result together with (3.10)
\[ w'(t) \leq -\alpha \sum_{i=1}^{m} Q_i(t) - \frac{M}{\delta} w^2(t) \sigma_i^{n-2}(t) \sigma'_i(t) \]
\[ \tag{3.11} \]
From (3.11), we have
\[ \sum_{i=1}^{m} Q_i(t) \leq -w'(t) - \frac{M}{\delta} w^2(t) \sigma_i^{n-2}(t) \sigma'_i(t), \quad (t \geq t_{10}) \tag{3.12} \]
If we multiply (3.12) by $\varphi(t)$ and integrate it from $t_{10}$ to $t$, we obtain

$$\alpha \int_{t_{10}}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s) ds \leq -\int_{t_{10}}^{t} \varphi(s) w'(s) ds$$

$$-\frac{M}{\delta} \int_{t_{10}}^{t} \varphi(s) w^2(s) \sigma_i^{n-2}(s) \sigma'_i(s) ds$$

$$= -\varphi(t) w(t) + \varphi(t_{10}) w(t_{10}) + \int_{t_{10}}^{t} \varphi'(s) w(s) ds$$

$$-\frac{M}{\delta} \int_{t_{10}}^{t} \varphi(s) w^2(s) \sigma_i^{n-2}(s) \sigma'_i(s) ds$$

$$\leq \varphi(t_{10}) w(t_{10}) - \frac{M}{\delta} \int_{t_{10}}^{t} \varphi(s) w^2(s) \sigma_i^{n-2}(s) \sigma'_i(s) ds$$

$$\times \left[ w(s) - \frac{\delta \varphi'(s)}{2M \varphi(s) \sigma_i^{n-2}(s) \sigma'_i(s)} \right]^2 ds$$

$$+ \int_{t_{10}}^{t} \frac{\delta \left[ \varphi'(s) \right]^2}{4M \varphi(s) \sigma_i^{n-2}(s) \sigma'_i(s)} ds$$

$$\leq \varphi(t_{10}) w(t_{10}) + \int_{t_{10}}^{t} \frac{\delta \left[ \varphi'(s) \right]^2}{4M \varphi(s) \sigma_i^{n-2}(s) \sigma'_i(s)} ds < +\infty.$$ 

Therefore, we have

$$+\infty = \alpha \limsup_{t \to \infty} \int_{t_{10}}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s) ds$$

$$\leq \varphi(t_{10}) w(t_{10}) + \int_{t_{10}}^{t} \frac{\delta \left[ \varphi'(s) \right]^2}{4M \varphi(s) \sigma_i^{n-2}(s) \sigma'_i(s)} ds < +\infty$$

for $i = 1, 2, \cdots, m$ by (C3). This is a contradiction. If we assume that $y(t) < 0$ then we may prove when $y(t) < 0$ as in Theorem 3.1. Hence, the proof is completed.
REFERENCES


