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ASYMPTOTICALLY $\mathcal{I}_2$-CESÀRO EQUIVALENCE OF DOUBLE SEQUENCES OF SETS

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Abstract. In this paper, we defined concept of asymptotically $\mathcal{I}_2$-Cesàro equivalence and investigate the relationships between the concepts of asymptotically strongly $\mathcal{I}_2$-Cesàro equivalence, asymptotically strongly $\mathcal{I}_2$-lacunary equivalence, asymptotically p-strongly $\mathcal{I}_2$-Cesàro equivalence and asymptotically $\mathcal{I}_2$-statistical equivalence of double sequences of sets.

1. INTRODUCTION

The concept of convergence of real number sequences has been extended to statistical convergence independently by Fast [8] and Schoenberg [23]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [12] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence.

Freedman et al. [9] established the connection between the strongly Cesàro summable sequences space and the strongly lacunary summable sequences space. Connor [4] gave the relationships between the concepts of statistical and strongly p-Cesàro convergence of sequences.

There are different convergence notions for sequence of sets. One of them handled in this paper is the concept of Wijsman convergence (see, [1, 3, 10, 14, 27, 32]). The concepts of statistical convergence and lacunary statistical convergence of sequences of sets were studied in [14, 27]. Also, new convergence notions, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence, Wijsman $\mathcal{I}$-statistical convergence and Wijsman $\mathcal{I}$-Cesàro summability by using ideal were introduced in [10, 11, 30].

Nuray et al. [17] studied the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigate the relationship between them. Also, Nuray et al. [15] studied the concepts of Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-convergence and Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-Cauchy double sequences of sets. Ulusu et al. [26] studied $\mathcal{I}_2$-Cesàro summability of double sequences of sets. Dündar et al. [7] investigated $\mathcal{I}_2$-lacunary statistical convergence of double sequences of sets.

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Marouf [13] peresented definitions for asymptotically equivalent and asymptotic regular matrices. This concepts was investigated in [19–21].

The concept of asymptotically equivalence of real numbers sequences which is defined by Marouf [13] has been extended by Ulusu and Nuray [28] to concepts of Wijsman asymptotically equivalence of set sequences. Moreover, natural inclusion theorems are presented. Kişi et al. [11] introduced the concepts of Wijsman asymptotically $\mathcal{I}$-equivalence of sequences of sets. Ulusu [24] investigated asymptotically $\mathcal{I}$-Cesàro equivalence of sequences of sets.

2. Definitions and Notations

Now, we recall the basic definitions and concepts (See [**?**, 1, 2, 5–7, 12, 15–17, 22, 25, 26, 29, 31]).

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the paper we take (X, ρ) be a separable metric space and A, A_{kj} be non-empty closed subsets of X.

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A) \quad or \quad \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A)$$

for each $x \in X$ and we write $W_2 - \lim A_{kj} = A$.

The double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n \to \infty} \frac{1}{mn} \Big| \big\{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \big\} \Big| = 0,$$

that is,

$$|d(x, A_{kj}) - d(x, A)| < \varepsilon, \quad \text{a.a. } (k, j)$$

and we write $st_2 - \lim_W A_k = A$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

$i)$ $\emptyset \in \mathcal{I}$, $ii)$ $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, $iii)$ $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a non-trivial ideal if $X \notin \mathcal{I}$.

A non-trivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as an admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

$i)$ $\emptyset \notin \mathcal{F}$, $ii)$ $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, $iii)$ $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

If $\mathcal{I}$ is a non-trivial ideal in X, $X \neq \emptyset$, then the class

$$\mathcal{F}(\mathcal{I}) = \big\{ M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A) \big\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

The double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\big\{ (k, j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \big\} \in \mathcal{I}_2$$

and we write $\mathcal{I}_{W_2} - \lim A_{kj} = A$.

The double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left|\frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{kj}) - d(x,A)\right| \geq \varepsilon\right\} \in \mathcal{I}_2$$

and we write $A_{kj} \stackrel{C_1(\mathcal{I}_{W_2})}{\longrightarrow} A$.

The double sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x,A)| \geq \varepsilon\right\} \in \mathcal{I}_2$$

and we write $A_{kj} \stackrel{C_1[\mathcal{I}_{W_2}]}{\longrightarrow} A$.

The double sequences $\{A_{kj}\}$ is Wijsman p-strongly $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$, for each p positive real number and for each $x \in X$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x,A)|^p \geq \varepsilon\right\} \in \mathcal{I}_2$$

and we write $A_{kj} \stackrel{C_p[\mathcal{I}_{W_2}]}{\longrightarrow} A$.

The double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A or $S(\mathcal{I}_{W_2})$-convergent to A if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\Big|\big\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x,A)| \geq \varepsilon\big\}\Big| \geq \delta\right\} \in \mathcal{I}_2$$

and we write $A_{kj} \to A\big(S(\mathcal{I}_{W_2})\big)$.

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad \text{as} \quad r \to \infty$$

and

$$j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad \text{as} \quad u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u, \;\; h_{ru} = h_r \bar{h}_u, \;\; I_{ru} = \big\{(k,j) : k_{r-1} < k \leq k_r \;\text{ and }\; j_{u-1} < j \leq j_u\big\},$$

$$q_r = \frac{k_r}{k_{r-1}} \quad \text{and} \quad q_u = \frac{j_u}{j_{u-1}}.$$

The double sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta[\mathcal{I}_{W_2}]$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$A(\varepsilon, x) = \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x,A)| \geq \varepsilon\right\} \in \mathcal{I}_2$$

and we write $A_{kj} \to A\big(N_\theta[\mathcal{I}_{W_2}]\big)$.

We define $d(x; A_{kj}, B_{kj})$ as follows:

$$d(x; A_{kj}, B_{kj}) = \begin{cases} \dfrac{d(x, A_{kj})}{d(x, B_{kj})} & , \quad x \notin A_{kj} \cup B_{kj} \\[2ex] L & , \quad x \in A_{kj} \cup B_{kj}. \end{cases}$$

The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically equivalent of multiple L if for each $x \in X$,

$$\lim_{k,j\to\infty} d(x; A_{kj}, B_{kj}) = L$$

and we write $A_{kj} \overset{W_2^L}{\sim} B_{kj}$ and simply Wijsman asymptotically equivalent if $L = 1$.

The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$ and each $x \in X$

$$\big\{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\} \in \mathcal{I}_2$$

and we write $A_{kj} \overset{\mathcal{I}_{W_2}^L}{\sim} B_{kj}$ and simply Wijsman asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent of multiple L if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\Big|\big\{k \leq m, j \leq n : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\} \in \mathcal{I}_2$$

and we write $A_{kj} \overset{S(\mathcal{I}_{W_2}^L)}{\sim} B_{kj}$ and simply Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent if $L = 1$.

Let θ be a double lacunary sequence. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are said to be Wijsman asymptotically strongly $\mathcal{I}_2$-lacunary equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\right\} \in \mathcal{I}_2$$

and we write $A_{kj} \overset{N_\theta[\mathcal{I}_{W_2}^L]}{\sim} B_{kj}$ and simply Wijsman asymptotically strongly $\mathcal{I}_2$-lacunary equivalent if $L = 1$.

$X \subset \mathbb{R}$, $f, g : X \to \mathbb{R}$ functions and a point $a \in X'$ are given. If $f(x) = \alpha(x)g(x)$ for $\forall x \in \overset{o}{U}_\delta(a) \cap X$, then for $x \in X$ we write $f = \mathcal{O}(g)$ as $x \to a$, where for any $\delta > 0$, $\alpha : X \to \mathbb{R}$ is bounded function on $\overset{o}{U}_\delta(a) \cap X$. In this case, if there exists a $c \geq 0$ such that $|f(x)| \leq c|g(x)|$ for $\forall x \in \overset{o}{U}_\delta(a) \cap X$, then for $x \in X$, $f = \mathcal{O}(g)$ as $x \to a$.

3. Main Results

In this section, we defined concepts of asymptotically $\mathcal{I}_2$-Cesàro equivalence, asymptotically strongly $\mathcal{I}_2$-Cesàro equivalence and asymptotically p-strongly $\mathcal{I}_2$-Cesàro equivalence of double sequences of sets. Also, we investigate the relationship between the concepts of asymptotically strongly $\mathcal{I}_2$-Cesàro equivalence, asymptotically strongly $\mathcal{I}_2$-lacunary equivalence, asymptotically p-strongly $\mathcal{I}_2$-Cesàro equivalence and asymptotically $\mathcal{I}_2$-statistical equivalence of double sequences of sets.

Definition 3.1. *The double sequence $\{A_{kj}\}$ and $\{B_{kj}\}$ are asymptotically $\mathcal{I}_2$-Cesàro equivalence of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \Big|\frac{1}{mn}\sum_{k,j=1,1}^{m,n} d(x; A_{kj}, B_{kj}) - L\Big| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{C_1^L(\mathcal{I}_{W_2})}{\sim} B_{kj}$ *and simply asymptotically* $\mathcal{I}_2$*-Cesàro equivalent if* $L = 1$.

Definition 3.2. *The double sequence* $\{A_{kj}\}$ *and* $\{B_{kj}\}$ *are asymptotically strongly* $\mathcal{I}_2$*-Cesàro equivalence of multiple* L *if for every* $\varepsilon > 0$ *and for each* $x \in X$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{C_1^L[\mathcal{I}_{W_2}]}{\sim} B_{kj}$ *and simply asymptotically strongly* $\mathcal{I}_2$*-Cesàro equivalent if* $L = 1$.

Theorem 3.3. *Let* θ *be a double lacunary sequence. If* $\liminf_r q_r > 1$, $\liminf_u q_u > 1$, *then*

$$A_{kj} \overset{C_1^L[\mathcal{I}_{W_2}]}{\sim} B_{kj} \Rightarrow A_{kj} \overset{N_\theta^L[\mathcal{I}_{W_2}]}{\sim} B_{kj}.$$

Proof. Let $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$. Then, there exist $\lambda, \mu > 0$ such that $q_r \geq 1 + \lambda$ and $q_u \geq 1 + \mu$ for all $r, u \geq 1$, which implies that

$$\frac{k_r j_u}{h_r \overline{h_u}} \leq \frac{(1+\lambda)(1+\mu)}{\lambda\mu} \quad \text{and} \quad \frac{k_{r-1} j_{u-1}}{h_r \overline{h_u}} \leq \frac{1}{\lambda\mu}.$$

Let $\varepsilon > 0$ and for each $x \in X$ we define the set

$$S = \left\{(k_r, j_u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k_r j_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x; A_{is}, B_{is}) - L| < \varepsilon \right\}.$$

We can easily say that $S \in \mathcal{F}(\mathcal{I}_2)$, which is a filter of the ideal $\mathcal{I}_2$, so we have

$$\begin{aligned}
\frac{1}{h_r \overline{h_u}} \sum_{(i,s)\in I_{ru}} &|d(x; A_{is}, B_{is}) - L| \\
&= \frac{1}{h_r \overline{h_u}} \sum_{i,s=1,1}^{k_r, j_u} |d(x; A_{is}, B_{is}) - L| \\
&\quad - \frac{1}{h_r \overline{h_u}} \sum_{i,s=1,1}^{k_{r-1}, j_{u-1}} |d(x; A_{is}, B_{is}) - L| \\
&= \frac{k_r j_u}{h_r \overline{h_u}} \left(\frac{1}{k_r j_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x; A_{is}, B_{is}) - L| \right) \\
&\quad - \frac{k_{r-1} j_{u-1}}{h_r \overline{h_u}} \left(\frac{1}{k_{r-1} j_{u-1}} \sum_{i,s=1,1}^{k_{r-1}, j_{u-1}} |d(x; A_{is}, B_{is}) - L| \right) \\
&\leq \left(\frac{(1+\lambda)(1+\mu)}{\lambda\mu} \right) \varepsilon - \left(\frac{1}{\lambda\mu} \right) \varepsilon'
\end{aligned}$$

for every $\varepsilon' > 0$, for each $x \in X$ and $(k_r, j_u) \in S$. Choose $\eta = \left(\frac{(1+\lambda)(1+\mu)}{\lambda\mu}\right)\varepsilon + \left(\frac{1}{\lambda\mu}\right)\varepsilon'$. Therefore,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \ < \eta \right\} \in \mathcal{F}(\mathcal{I}_2)$$

and it completes the proof. $\square$

Theorem 3.4. *Let θ be a double lacunary sequence. If $\limsup_r q_r < \infty$, $\limsup_u q_u < \infty$, then*

$$A_{kj} \overset{N_\theta^L[\mathcal{I}_{W_2}]}{\sim} B_{kj} \Rightarrow A_{kj} \overset{C_1^L[\mathcal{I}_{W_2}]}{\sim} B_{kj}.$$

Proof. Let $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$. Then, there exist $M, N > 0$ such that $q_r < M$ and $q_u < N$ for all $r, u \geq 1$. Let $A_{kj} \overset{N_\theta^L[\mathcal{I}_{W_2}]}{\sim} B_{kj}$ and for $\varepsilon_1, \varepsilon_2 > 0$ define the sets T and R such that

$$T = \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| < \varepsilon_1 \right\}$$

and

$$R = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x; A_{kj}, B_{kj}) - L)| \ < \varepsilon_2 \right\},$$

for each $x \in X$. Let

$$a_{tv} = \frac{1}{h_t \overline{h}_v} \sum_{(i,s)\in I_{tv}} |d(x; A_{is}, B_{is}) - L| < \varepsilon_1,$$

for each $x \in X$ and for all $(t, v) \in T$. It is obvious that $T \in \mathcal{F}(\mathcal{I}_2)$.

Choose m, n is any integer with $k_{r-1} < m < k_r$ and $j_{u-1} < n < j_u$, where $(r, u) \in T$. Then, for each $x \in X$ we have

$$\begin{aligned}
\frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x; A_{kj}, B_{kj}) - L| &\leq \frac{1}{k_{r-1} j_{u-1}} \sum_{k,j=1,1}^{k_r, j_u} |d(x; A_{kj}, B_{kj}) - L| \\
&= \frac{1}{k_{r-1} j_{u-1}} \Bigg(\sum_{(k,j)\in I_{11}} |d(x; A_{kj}, B_{kj}) - L| \\
&\quad + \sum_{(k,j)\in I_{12}} |d(x; A_{kj}, B_{kj}) - L| \\
&\quad + \sum_{(k,j)\in I_{21}} |d(x; A_{kj}, B_{kj}) - L| \\
&\quad + \sum_{(k,j)\in I_{22}} |d(x; A_{kj}, B_{kj}) - L| \\
&\quad + \cdots + \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \Bigg)
\end{aligned}$$

$$
\begin{aligned}
= \ & \tfrac{k_1 j_1}{k_{r-1} j_{u-1}} \left(\tfrac{1}{h_1 \overline{h_1}} \sum_{(k,j)\in I_{11}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
& + \tfrac{k_1 (j_2 - j_1)}{k_{r-1} j_{u-1}} \left(\tfrac{1}{h_1 \overline{h_2}} \sum_{(k,j)\in I_{12}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
& + \tfrac{(k_2 - k_1) j_1}{k_{r-1} j_{u-1}} \left(\tfrac{1}{h_1 \overline{h_2}} \sum_{(k,j)\in I_{21}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
& + \tfrac{(k_2 - k_1)(j_2 - j_1)}{k_{r-1} j_{u-1}} \left(\tfrac{1}{h_1 \overline{h_2}} \sum_{(k,j)\in I_{22}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
& + \ldots + \tfrac{(k_r - k_{r-1})(j_u - j_{u-1})}{k_{r-1} j_{u-1}} \left(\tfrac{1}{h_r \overline{h_u}} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \right) \\
= \ & \tfrac{k_1 j_1}{k_{r-1} j_{u-1}} a_{11} + \tfrac{k_1 (j_2 - j_1)}{k_{r-1} j_{u-1}} a_{12} + \tfrac{(k_2 - k_1) j_1}{k_{r-1} j_{u-1}} a_{21} \\
& + \tfrac{(k_2 - k_1)(j_2 - j_1)}{k_{r-1} j_{u-1}} a_{22} + \ldots + \tfrac{(k_r - k_{r-1})(j_u - j_{u-1})}{k_{r-1} j_{u-1}} a_{ru} \\
\leq \ & \left(\sup_{(t,v)\in T} a_{tv} \right) \tfrac{k_r j_u}{k_{r-1} j_{u-1}} \\
< \ & \varepsilon_1 \cdot M \cdot N.
\end{aligned}
$$

Choose $\varepsilon_2 = \frac{\varepsilon_1}{M \cdot N}$ and in view of the fact that

$$
\bigcup_{(r,u)\in T} \left\{ (m,n) : k_{r-1} < m < k_r, \ j_{u-1} < n < j_u \right\} \subset R,
$$

where $T \in \mathcal{F}(\mathcal{I}_2)$, it follows from our assumption on θ that the set R also belongs to $\mathcal{F}(\mathcal{I}_2)$ and this completes the proof of the theorem. □

We have the following Theorem by Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *Let θ be a double lacunary sequence. If $1 < \liminf_r q_r < \limsup_r q_r < \infty$ and $1 < \liminf_u q_u < \limsup_u q_u < \infty$, then*

$$
A_{kj} \overset{C_1^L[\mathcal{I}_{W_2}]}{\sim} B_{kj} \Leftrightarrow A_{kj} \overset{N_\theta^L[\mathcal{I}_{W_2}]}{\sim} B_{kj}.
$$

Definition 3.6. *The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are asymptotically p-strongly $\mathcal{I}_2$-Cesàro equivalence of multiple L if for every $\varepsilon > 0$, for each p positive real number and for each $x \in X$,*

$$
\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x; A_{kj}, B_{kj}) - L|^p \geq \varepsilon \right\} \in \mathcal{I}_2.
$$

In this case, we write $A_{kj} \overset{C_p^L[\mathcal{I}_{W_2}]}{\sim} B_{kj}$ and simply asymptotically p-strongly $\mathcal{I}_2$-Cesàro equivalent if $L = 1$.

Theorem 3.7. *If the sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are asymptotically p-strongly $\mathcal{I}_2$-Cesàro equivalence of multiple L, then $\{A_{kj}\}$ and $\{B_{kj}\}$ are asymptotically $\mathcal{I}_2$-statistical equivalence of multiple L.*

Proof. Let $A_{kj} \overset{C_p^L[\mathcal{I}_{W_2}]}{\sim} B_{kj}$ and $\varepsilon > 0$ given. Then, for each $x \in X$ we have

$$\begin{aligned}\sum_{k,j=1,1}^{m,n} |d(x;A_{kj},B_{kj}) - L|^p &\geq \sum_{\substack{k,j=1,1 \\ \left|d(x;A_{kj},B_{kj})-L\right| \geq \varepsilon}}^{m,n} |d(x;A_{kj},B_{kj}) - L|^p \\ &\geq \varepsilon^p \cdot \left|\left\{k \leq m, j \leq n : |d(x;A_{kj},B_{kj}) - L| \geq \varepsilon\right\}\right|\end{aligned}$$

and so

$$\frac{1}{\varepsilon^p mn} \sum_{k,j=1,1}^{m,n} |d(x;A_{kj},B_{kj}) - L|^p \geq \frac{1}{mn}\left|\left\{k \leq m, j \leq n : |d(x;A_{kj},B_{kj}) - L| \geq \varepsilon\right\}\right|.$$

So for a given $\delta > 0$ and for each $x \in X$

$$\begin{aligned}&\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\left|\left\{k \leq m, j \leq n : |d(x;A_{kj},B_{kj}) - L| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\qquad \subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x;A_{kj},B_{kj}) - L|^p \geq \varepsilon^p \cdot \delta\right\} \in \mathcal{I}_2.\end{aligned}$$

Therefore, $A_{kj} \overset{S(\mathcal{I}_{W_2})}{\sim} B_{kj}$. □

Theorem 3.8. *Let $d(x, A_{kj}) = \mathcal{O}\big(d(x, B_{kj})\big)$. If $\{A_{kj}\}$ and $\{B_{kj}\}$ are asymptotically $\mathcal{I}_2$-statistical equivalence of multiple L, then $\{A_{kj}\}$ and $\{B_{kj}\}$ are asymptotically p-strongly $\mathcal{I}_2$-Cesàro equivalence of multiple L.*

Proof. Suppose that $d(x, A_{kj}) = \mathcal{O}\big(d(x, B_{kj})\big)$ and $A_{kj} \overset{S(\mathcal{I}_{W_2})}{\sim} B_{kj}$. Then, there is an $M > 0$ such that

$$|d(x;A_{kj},B_{kj}) - L| \leq M,$$

for all k, j and for each $x \in X$. Given $\varepsilon > 0$ and for each $x \in X$, we have

$$\begin{aligned}&\frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x;A_{kj},B_{kj}) - L|^p \\ &\qquad = \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ \left|d(x;A_{kj},B_{kj})-L\right| \geq \varepsilon}}^{m,n} |d(x;A_{kj},B_{kj}) - L|^p \\ &\qquad\quad + \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ \left|d(x;A_{kj},B_{kj})-L\right| < \varepsilon}}^{m,n} |d(x;A_{kj},B_{kj}) - L|^p \\ &\qquad \leq \frac{1}{mn} M^p \cdot \left|\left\{k \leq m, j \leq n : |d(x;A_{kj},B_{kj}) - L| \geq \varepsilon\right\}\right| \\ &\qquad\quad + \frac{1}{mn} \varepsilon^p \cdot \left|\left\{k \leq m, j \leq n : |d(x;A_{kj},B_{kj}) - L| < \varepsilon\right\}\right| \\ &\qquad \leq \frac{M^p}{mn} \cdot \left|\left\{k \leq m, j \leq n : |d(x;A_{kj},B_{kj}) - L| \geq \varepsilon\right\}\right| + \varepsilon^p.\end{aligned}$$

Then, for any $\delta > 0$ and for each $x \in X$,

$$\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n} |d(x;A_{kj},B_{kj}) - L|^p \geq \delta\Big\}$$

$$\subseteq \Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\Big|\big\{k \leq m, j \leq n : |d(x;A_{kj},B_{kj}) - L| \geq \varepsilon\big\}\Big| \geq \frac{\delta^p}{M^p}\Big\} \in \mathcal{I}_2.$$

Therefore, $A_{kj} \overset{C_p^L[\mathcal{I}_{W_2}]}{\sim} B_{kj}$. □

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