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METHODOLOGIES AND APPLICATION

Lacunary statistical convergence of double sequences of sets

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Abstract In this paper, we study the concepts of Wijsman statistical convergence, Wijsman lacunary statistical convergence, Wijsman lacunary convergence and Wijsman strongly lacunary convergence double sequences of sets and investigate the relationship among them.

Keywords Statistical convergence · Lacunary sequence · Double sequence of sets · Wijsman convergence

1 Introduction

Hill (1940) was the first who applied methods of functional analysis to double sequences. Also, Kull (1958) applied methods of functional analysis of matrix maps of double sequences. A lot of useful developments of double sequences in summability methods can be found in Altay and Başar (2005), Limayea and Zeltser (2009), Savaş (2010), Zeltser et al. (2009).

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast (1951) and Schoenberg (1959). This concept was extended to the double sequences by Mursaleen and Edely (2003). Çakan and Altay (2006) presented multidimensional analogues of the results presented by Fridy and Orhan (1997).

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, Baronti and Papini 1986; Beer 1985, 1994; Nuray and Rhoades 2012; Ulusu and Nuray 2013; Wijsman 1964, 1966). Nuray and Rhoades (2012) extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems. Ulusu and Nuray (2012) defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Ulusu and Nuray (2013) introduced the concept of Wijsman strongly lacunary summability for sequences of sets and discussed its relation with Wijsman strongly Cesàro summability. Nuray et al. (2015) studied the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigated the relationship between them. Talo and Sever (2015) examined the relationship between Kuratowski statistical convergence and Hausdorff statistical convergence.

In this paper, we study the concepts of Wijsman statistical convergence, Wijsman lacunary statistical convergence, Wijsman lacunary convergence and Wijsman strongly lacunary convergence double sequences of sets and investigate the relationship among them.

2 Definitions and notations

Now, we recall the basic definitions and concepts (see Altay and Başar 2005; Aubin and Frankowska 1990; Baronti and Papini 1986; Beer 1985, 1994; Nuray et al. 2015; Nuray and Rhoades 2012; Pringsheim 1900; Savaş 2010, 2012a, b;

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Sever et al. 2015; Talo and Sever 2015; Ulusu and Nuray 2013; Wijsman 1964, 1966).

For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the paper, we let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X.

We say that the sequence $\{A_k\}$ is Wijsman convergent to A, if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

We say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A, if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Let $\theta = \{k_r\}$ be a lacunary sequence. We say that the sequence $\{A_k\}$ is Wijsman lacunary summable to A, if for each $x \in X$,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} d(x, A_k) = d(x, A).$$

In this case we write $A_k \to A(WN_\theta)$.

Let $\theta = \{k_r\}$ be a lacunary sequence. We say that the sequence $\{A_k\}$ is Wijsman strongly lacunary summable to A, if for each $x \in X$,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \to A([WN_\theta])$.

We say that the sequence $\{A_k\}$ is Wijsman lacunary statistically convergent to A, if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r} \frac{1}{h_r} |k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon| = 0.$$

A double sequence $x = (x_{kj})_{k,j \in \mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case we write

$$P - \lim_{k,j \to \infty} x_{kj} = L \quad \text{or} \quad \lim_{k,j \to \infty} x_{kj} = L.$$

A double sequence $x = (x_{kj})$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{kj}| < M$, for all $k, j \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{k,j} |x_{kj}| < \infty.$$

Throughout the paper, A, A_{kj} be any non-empty closed subsets of X.

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A, if for each $x \in X$

$$P - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A) \quad \text{or}$$
$$\lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A).$$

In this case we write $W_2 - \lim A_{kj} = A$.

We say that the double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n \to \infty} \frac{1}{mn} |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0,$$

that is,

$$|d(x, A_{kj}) - d(x, A)| < \varepsilon, \quad \text{a.a. (k,j)}.$$

In this case we write $st_2 - \lim_W A_k = A$.

The set of Wijsman statistically convergent double sequences will be denoted by

$$W_2S := \left\{\{A_{kj}\} : st_2 - \lim_W A_{kj} = A\right\}.$$

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad \text{as} \quad r \to \infty$$

and

$$j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad \text{as} \quad u \to \infty.$$

We use following notations in the sequel:

$$k_{ru} = k_r j_u, \quad h_{ru} = h_r \bar{h}_u, \quad I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \text{ and } j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \text{ and } q_u = \frac{j_u}{j_{u-1}}.$$

Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman lacunary convergent to A, if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} d(x, A_{kj}) = d(x, A).$$

In this case we write $A_{kj} \overset{(W_2N_\theta)}{\longrightarrow} A$ Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A, if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)| = 0.$$

In this case we write $A_{kj} \overset{[W_2N_\theta]}{\longrightarrow} A$.

Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman strongly p-lacunary convergent to A, if for each p positive real number and for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)|^p = 0.$$

In this case we write $A_{kj} \overset{[W_2^pN_\theta]}{\longrightarrow} A$.

3 Main results

In this section, we will define the notions of Wijsman lacunary convergence, Wijsman strongly lacunary convergence and Wijsman lacunary statistical convergence of double sequences of sets and investigate the relationship among them. Also we will give the relationship between Wijsman statistical convergence and Wijsman lacunary statistical convergence of double sequences of sets.

Definition 3.1 We say that the double sequence $\{A_{kj}\}$ is Wijsman lacunary statistically convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} |\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st_2 - \lim_{W_\theta} A_{kj} = A$.

The set of Wijsman lacunary statistically convergent double sequences will be denoted by

$$W_2S_\theta := \left\{\{A_{kj}\} : st_2 - \lim_{W_\theta} A_{kj} = A\right\}.$$

Example 3.2 Let $X = \mathbb{R}$ and we define a double sequence $\{A_{kj}\}$ as follows:

$$A_{kj} := \begin{cases} \{(x, y) \in \mathbb{R}^2 : 2 \leq x \leq k_r - k_{r-1}, 2 \leq y \leq j_u - j_{u-1}\} \\ \quad \text{if } k, j \geq 2 \text{ and } k, j \text{ is square integer}, \\ \{(1, 1)\} \quad \text{otherwise}. \end{cases}$$

This sequence is not Wijsman lacunary summable. But since

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \left|\left\{(k, j) \in I_{ru} : \left|d(x, A_{kj}) - d(x, \{(1, 1)\})\right| \right.\right.$$
$$\left.\left. \geq \varepsilon\right\}\right| = \lim_{r,u\to\infty} \frac{\sqrt{(k_r - k_{r-1})(j_u - j_{u-1})}}{h_r\bar{h}_u} = 0$$

this sequence is Wijsman lacunary statistically convergent to the set $A = \{(1, 1)\}$.

Theorem 3.3 *(i) If $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A, then $\{A_{kj}\}$ is Wijsman lacunary statistical convergent to A*

(ii) $[W_2N_\theta]$ is a proper subset of (W_2S_θ).

Proof (i) if $\varepsilon > 0$ and $A_{kj} \to A([W_2N_\theta])$ we can write

$$\sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)|$$
$$\geq \sum_{\substack{(k,j)\in I_{ru} \\ |d(x,A_{kj})-d(x,A)|\geq\varepsilon}} |d(x, A_{kj}) - d(x, A)|$$
$$\geq \varepsilon. \left|\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}\right|$$

which gives the result.

(ii) In order to show that the inclusion $[W_2N_\theta] \subset W_2S_\theta$ in (i) is proper, let θ be given and we define a sequence $\{A_{kj}\}$ as follows:

$$A_{kj} = \begin{cases} \{(k, j)\}, \quad if\ k_{r-1} < k \leq k_{r-1} + \left[\sqrt{h_r}\right], \\ \quad j_{u-1} < j \leq j_{u-1} + \left[\sqrt{\bar{h}_u}\right], \ (r, u = 1, 2, \cdots) \\ \{(0, 0)\}, \quad \text{otherwise}. \end{cases}$$

Note that $\{A_{kj}\}$ is not bounded. We have, for every $\varepsilon > 0$ and for each $x \in X$,

$$\frac{1}{h_r\bar{h}_u} |\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, \{(0, 0)\})| \geq \varepsilon\}|$$
$$= \frac{\left[\sqrt{h_r}\right]\left[\sqrt{\bar{h}_u}\right]}{h_r\bar{h}_u} \to 0 \text{ as } r, u \to \infty,$$

i.e., $A_{kj} \to \{(0, 0)\}(W_2S_\theta)$. But,

$$\frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,\{(0,0)\}|$$
$$=\frac{1}{h_r\bar{h}_u}\frac{\left(\left[\sqrt{h_r}\right].\left(\left[\sqrt{h_r}\right]+1\right)\right)\left(\left[\sqrt{\bar{h}_u}\right].\left(\left[\sqrt{\bar{h}_u}\right]+1\right)\right)}{4}$$
$$\to\frac{1}{4}\neq 0.$$

Hence $A_{kj}\nrightarrow\{(0,0)\}([W_2N_\theta])$.

□

Theorem 3.4 *Let L_∞ be a set of bounded double sequences of sets. If $\{A_{kj}\}\in L_\infty$ and $\{A_{kj}\}$ is Wijsman lacunary statistical convergent to A, then $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A.*

Proof Suppose that $\{A_{kj}\}\in L_\infty$ and $A_{kj}\to A(W_2S_\theta)$, say $|d(x,A_{kj})-d(x,A)|\leq M$ for each $x\in X$ and all (k,j). Given $\varepsilon>0$, we get

$$\frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|$$
$$=\frac{1}{h_r\bar{h}_u}\sum_{\substack{(k,j)\in I_{ru}\\ |d(x,A_{kj})-d(x,A)|\geq\varepsilon}}|d(x,A_{kj})-d(x,A)|$$
$$+\frac{1}{h_r\bar{h}_u}\sum_{\substack{(k,j)\in I_{ru}\\ |d(x,A_{kj})-d(x,A)|<\varepsilon}}|d(x,A_{kj})-d(x,A)|$$
$$\leq\frac{M}{h_r\bar{h}_u}\left|\{(k,j)\in I_{ru}:|d(x,A_{kj})-d(x,A)|\geq\varepsilon\}\right|+\varepsilon,$$

for each $x\in X$. Hence we have the result. □

Theorem 3.5 $\{W_2S_\theta\}\cap L_\infty=\{[W_2N_\theta]\}\cap L_\infty$.

Proof This follows from consequences Theorems 3.3 and 3.4. □

Theorem 3.6 *For any double lacunary sequence $\theta=\{(k_r,j_s)\}$, if $\liminf_r q_r>1$ and $\liminf_u q_u>1$, then $st_2-\lim_W A_{kj}=A$ implies $st_2-\lim_{W_\theta}A_{kj}=A$.*

Proof Assume that $\liminf_r q_r>1$, and $\liminf_u q_u>1$, then there exist $\lambda,\mu>0$ such that $q_r\geq 1+\lambda$ and $q_u\geq 1+\mu$ for sufficiently large r,u which implies that

$$\frac{h_r\bar{h}_u}{k_{ru}}\geq\frac{\lambda\mu}{(1+\lambda)(1+\mu)}.$$

If $st_2-\lim_W A_{kj}=A$, then for every $\varepsilon>0$ and for sufficiently large r,u we have

$$\frac{1}{k_rj_u}|\{k\leq k_r,\ j\leq j_u:|d(x,A_{kj})-d(x,A)|\geq\varepsilon\}|$$
$$\geq\frac{1}{k_rj_u}|\{(k,j)\in I_{ru}:|d(x,A_{kj})-d(x,A)|\geq\varepsilon\}|$$
$$\geq\frac{\lambda\mu}{(1+\lambda)(1+\mu)}$$
$$.\left(\frac{1}{h_r\bar{h}_u}|\{(k,j)\in I_{ru}:|d(x,A_{kj})-d(x,A)|\geq\varepsilon\}|\right).$$

for each $x\in X$. Hence, $st_2-\lim_{W_\theta}A_{kj}=A$. □

Theorem 3.7 *For any double lacunary sequence $\theta=\{(k_r,j_s)\}$, if $\limsup_r q_r<\infty$ and $\limsup_u q_u<\infty$, then $st_2-\lim_{W_\theta}A_{kj}=A$ implies $st_2-\lim_W A_{kj}=A$.*

Proof If $\limsup_r q_r<\infty$ and $\limsup_u q_u<\infty$, then there is an $M,N>0$ such that $q_r<M$ and $q_u<N$, for all r,u. Suppose that $st_2-\lim_{W_\theta}A_{kj}=A$ and let

$$U_{ru}=U(r,u,x):=|\{(k,j)\in I_{ru}:|d(x,A_{kj})$$
$$-d(x,A)|\geq\varepsilon|.$$

Then, given $\varepsilon>0$, there is an $r_0,u_0\in\mathbb{N}$ such that

$$\frac{U_{ru}}{h_r\bar{h}_u}<\varepsilon,\ \text{for all } r>r_0,u>u_0.$$

Now let

$$K:=\max\{U_{ru}:1\leq r\leq r_0,1\leq u\leq u_0\}$$

and let t and v be any integers satisfying $k_{r-1}<t\leq k_r$ and $j_{u-1}<v\leq j_u$. Then we can write

$$\frac{1}{tv}|\{k\leq t,\ j\leq v:|d(x,A_{kj})-d(x,A)|\geq\varepsilon\}|$$
$$\leq\frac{1}{k_{r-1}j_{u-1}}|\{k\leq k_r,\ j\leq j_u:|d(x,A_{kj})-d(x,A)|\geq\varepsilon\}|$$
$$=\frac{1}{k_{r-1}j_{u-1}}\{U_{11}+U_{12}+U_{21}+U_{22}+\cdots+U_{r_0u_0}+\cdots+U_{ru}\}$$
$$\leq\frac{K}{k_{r-1}j_{u-1}}\cdot r_0u_0+\frac{1}{k_{r-1}j_{u-1}}$$
$$\times\left\{h_{r_0}\bar{h}_{u_0+1}\frac{U_{r_0,u_0+1}}{h_{r_0}\bar{h}_{u_0+1}}+h_{r_0+1}\bar{h}_{u_0}\frac{U_{r_0+1,u_0}}{h_{r_0+1}\bar{h}_{u_0}}+\cdots+h_r\bar{h}_u\frac{U_{ru}}{h_r\bar{h}_u}\right\}$$
$$\leq\frac{r_0u_0\cdot K}{k_{r-1}j_{u-1}}+\frac{1}{k_{r-1}j_{u-1}}\left(\sup_{\substack{r>r_0\\u>u_0}}\frac{U_{ru}}{h_r\bar{h}_u}\right)$$
$$\times\{h_{r_0}\bar{h}_{u_0+1}+h_{r_0+1}\bar{h}_{u_0}+\cdots+h_r\bar{h}_u\}$$
$$\leq\frac{r_0u_0\cdot K}{k_{r-1}j_{u-1}}+\varepsilon\cdot\frac{(k_r-k_{r_0})(j_u-j_{u_0})}{k_{r-1}j_{u-1}}$$
$$\leq\frac{r_0u_0\cdot K}{k_{r-1}j_{u-1}}+\varepsilon\cdot q_r\cdot q_u\leq\frac{r_0u_0\cdot K}{k_{r-1}j_{u-1}}+\varepsilon\cdot M\cdot N$$

and the sufficiently follows immediately. □

Theorem 3.8 *For any double lacunary sequence* $\theta = \{(k_r, j_s)\}$*, if* $1 < \liminf_r q_r \leq \limsup_r q_r < \infty$ *and* $1 < \liminf_u q_u \leq \limsup_u q_u < \infty$ *then* $W_2S = W_2S_\theta$.

Proof This follows from Theorems 3.6 and 3.7. □

Theorem 3.9 *If* $\{A_{kj}\} \in W_2S \cap W_2S_\theta$*, then* $st_2 - \lim_{W_\theta} A_{kj} = st_2 - \lim_W A_{kj}$.

Proof Suppose that $st - \lim_W A_{kj} = A$ and $st_2 - \lim_{W_\theta} A_{kj} = B$ and $A \neq B$. For

$$\frac{1}{2}|d(x, A) - d(x, B)| > \varepsilon$$

and each $x \in X$ we get

$$\lim_{m,n\to\infty} \frac{1}{mn}|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}| = 1.$$

Consider the $k_t j_v$th term of the statistical limit expression

$$\frac{1}{mn}|\{k \leq n, j \leq n : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}| :$$

$$\begin{aligned}
&\frac{1}{k_t j_v}|\{k \leq k_t, j \leq j_v : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}| \\
&\quad = \frac{1}{k_t j_v}\left|\{(k, j) \in \bigcup_{r,u=1,1}^{t,v} I_{ru} : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}\right| \\
&\quad = \frac{1}{k_t j_v} \cdot \sum_{r,u=1,1}^{t,v} |\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}| \\
&\quad = \frac{1}{\sum_{r,u=1,1}^{t,v} h_r \overline{h}_u} \cdot \sum_{r,u=1,1}^{t,v} |\{(k, j) \in I_{ru} : |d(x, A_{kj}) \\
&\quad\quad - d(x, B)| \geq \varepsilon\}|, \\
&\quad = \frac{1}{\sum_{r,u=1,1}^{t,v} h_r \overline{h}_u} \cdot \sum_{r,u=1,1}^{t,v} h_r \overline{h}_u s_{ru},
\end{aligned}$$

where $s_{ru} = \frac{1}{h_r \overline{h}_u}|\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}| \to 0$ because $A_{kj} \to B(W_2S_\theta)$. Since θ is double lacunary sequence,

$$\frac{1}{\sum_{r,u=1,1}^{t,v} h_r \overline{h}_u} \cdot \sum_{r,u=1,1}^{t,v} h_r \overline{h}_u s_{ru}$$

is a regular weighted mean transform of s_{ru}, (see Fridy and Orhan 1993; Ulusu and Nuray 2012), and therefore it too tends to zero as $t, v \to \infty$. Also, since this is a subsequence of

$$\left\{\frac{1}{mn}|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}\right\}_{m,n=1,1}^{\infty,\infty},$$

we infer that

$$\lim_{m,n\to\infty} \frac{1}{mn}|\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon\}| \neq 1,$$

and this contradiction shows that we cannot have $A \neq B$. □

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