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$\mathscr{I}$-Convergence and $\mathscr{I}$-Cauchy Sequence of Functions In 2-Normed Spaces

Mukaddes Arslan[1] and Erdinç Dündar[2*]

[1] *İhsaniye Anadolu İmam Hatip Lisesi, 03370 Afyonkarahisar, Turkey*
[2] *Afyon Kocatepe University, Faculty of Science, Department of Mathematics, Afyonkarahisar, Turkey.*
[*] *Corresponding author E-mail: erdincdundar79@gmail.com*

Abstract

In this paper, we study concepts of $\mathscr{I}$-convergence, $\mathscr{I}^*$-convergence, $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences of functions and investigate relationships between them and some properties in 2-normed spaces.

Keywords: *Ideal, Filter, Sequence of functions, $\mathscr{I}$-Convergence, $\mathscr{I}$-Cauchy, 2-normed spaces.*
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1. Introduction, Definitions and Notations

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [8] and Schoenberg [27].
The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [20] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of $\mathbb{N}$ [8, 9]. Nabiev et al. [23] studied $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequence, and then study their certain properties. Gökhan et al. [13] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. Gezer and Karakuş [12] investigated $\mathscr{I}$-pointwise and uniform convergence and $\mathscr{I}^*$-pointwise and uniform convergence of function sequences and they examined the relation between them. Baláz et al. [1] investigated $\mathscr{I}$-convergence and $\mathscr{I}$-continuity of real functions. Balcerzak et al. [2] studied statistical convergence and ideal convergence for sequences of functions Dündar and Altay [5, 6] studied the concepts of pointwise and uniformly $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [7] investigated some results of $\mathscr{I}_2$-convergence of double sequences of functions.
The concept of 2-normed spaces was initially introduced by Gähler [10, 11] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [17] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Şahiner et al. [29] and Gürdal [19] studied $\mathscr{I}$-convergence in 2-normed spaces. Gürdal and Açık [18] investigated $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [25] presented various kinds of statistical convergence and $\mathscr{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathscr{I}$-equistatistically convergence and study $\mathscr{I}$-equistatistically convergence of sequences of functions. Recently, Savaş and Gürdal [26] concerned with $\mathscr{I}$-convergence of sequences of functions in random 2-normed spaces and introduce the concepts of ideal uniform convergence and ideal pointwise convergence in the topology induced by random 2-normed spaces, and gave some basic properties of these concepts. Yegül and Dündar [31] studied statistical convergence of sequence of functions in 2-normed spaces. A lot of development have been made in this area after the works of [3, 4, 21, 22, 24, 28, 30].

2. Definitions and Notations

Now, we recall the concept of 2-normed space, ideal convergence and some fundamental definitions and notations [1, 2, 8, 9, 14–20, 25, 29].
If $K \subseteq \mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $\delta(K) = \lim_n \frac{1}{n}|K_n|$, if it exists. The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set

$$K = K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has natural density zero; in this case, we write $st - \lim x = L$.
Let $X \neq \emptyset$. A class $\mathscr{I}$ of subsets of X is said to be an ideal in X provided:

***Email addresses*:** , mukad.deu@gmail.com (Mukaddes Arslan), erdincdundar79@gmail.com (Erdinç Dündar)

(i) $\emptyset \in \mathscr{I}$,
(ii) $A,B \in \mathscr{I}$ implies $A \cup B \in \mathscr{I}$,
(iii) $A \in \mathscr{I}$, $B \subset A$ implies $B \in \mathscr{I}$.

$\mathscr{I}$ is called a nontrivial ideal if $X \notin \mathscr{I}$.
Let $X \neq \emptyset$. A non empty class $\mathscr{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathscr{F}$,
(ii) $A,B \in \mathscr{F}$ implies $A \cap B \in \mathscr{F}$,
(iii) $A \in \mathscr{F}$, $A \subset B$ implies $B \in \mathscr{F}$.

Lemma 2.1 ([20]). *If $\mathscr{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathscr{F}(\mathscr{I}) = \{M \subset X : (\exists A \in \mathscr{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathscr{I}$.

A nontrivial ideal $\mathscr{I}$ in X is called admissible if $\{x\} \in \mathscr{I}$, for each $x \in X$.

Example 2.1. Let $\mathscr{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then, $\mathscr{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathscr{I}_f$ convergence is the usual convergence.

Throughout the paper, we let $\mathscr{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.
A sequence (f_n) of functions is said to be $\mathscr{I}$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if for every $\varepsilon > 0$ and each $x \in D$,

$$\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathscr{I}.$$

In this case, we will write $f_n \xrightarrow{\mathscr{I}} f$ on D.
A sequence (f_n) of functions is said to be $\mathscr{I}^*$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if $\forall \varepsilon > 0$ and $\forall x \in D$, $\exists K_x \notin \mathscr{I}$ and $\exists n_0 = n_0(\varepsilon, x) \in K_x : \forall n \geq n_0$ and $n \in K_x$, $|f_n(x) - f(x)| < \varepsilon$.
Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x,y\| = \|y,x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x,y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y+z\| \leq \|x,y\| + \|x,z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1|; \;\; x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.
A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if

$$\lim_{n\to\infty} \|x_n - L, y\| = 0$$

for every $y \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).
A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathscr{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ and each nonzero $z \in X$,

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\} \in \mathscr{I}.$$

In this case, we write $\mathscr{I} - \lim\limits_{n\to\infty} \|x_n - L, z\| = 0$ or $\mathscr{I} - \lim\limits_{n\to\infty} \|x_n, z\| = \|L, z\|$.
A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathscr{I}^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathscr{F}(\mathscr{I})$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $\lim\limits_{n\to\infty} \|x_{m_k} - L, z\| = 0$, for each nonzero $z \in X$.
A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathscr{I}$-Cauchy sequence in X, if for each $\varepsilon > 0$ and nonzero $z \in X$ there exists a number $N = N(\varepsilon, z)$ such that

$$\{k \in \mathbb{N} : \|x_k - x_N, z\| \geq \varepsilon\} \in \mathscr{I}.$$

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathscr{I}^*$-Cauchy sequence in X, if there exists a set $M \in \mathscr{F}(\mathscr{I})$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that the subsequence $x_M = (x_{m_k})$ is a Cauchy sequence on X, i.e.,

$$\lim_{k,p\to\infty} \|x_{m_k} - x_{m_p}, z\| = 0, \text{ for each nonzero } z \in X.$$

Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. $\{f_n\}$ is said to be convergent to f if $f_n(x) \xrightarrow{\|.,.\|_Y} f(x)$ for each $x \in X$. We write $f_n \xrightarrow{\|.,.\|_Y} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0) \|f_n(x) - f(x), z\| < \varepsilon.$$

Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. $\{f_n\}$ is said to be $\mathscr{I}$-pointwise convergent to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \in \mathscr{I},$$

or $\mathscr{I}-\lim_{n\to\infty}\|f_n(x)-f(x),z\|_Y=0$ (in $(Y,\|.,.\|_Y)$, for each $x\in X$. In this case, we write $f_n \xrightarrow{\|.,.\|_Y}_{\mathscr{I}} f$. This can be expressed by the formula

$$(\forall z\in Y)(\forall\varepsilon>0)(\exists M\in\mathscr{I})(\forall n_0\in\mathbb{N}\backslash M)(\forall x\in X)(\forall n\geq n_0)\|f_n(x)-f(x),z\|\leq\varepsilon.$$

An admissible ideal $\mathscr{I}\subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1,A_2,...\}$ belonging to $\mathscr{I}$ there exists a countable family of sets $\{B_1,B_2,...\}$ such that $A_i\Delta B_i$ is a finite set for $j\in\mathbb{N}$ and $B=\bigcup_{i=1}^{\infty}B_i\in\mathscr{I}$.
Now we begin with quoting the lemma due to Nabiev et al. [23] which are needed throughout the paper.

Lemma 2.2 ([23]). *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i\in\mathscr{F}(\mathscr{I})$ for each i, where $\mathscr{F}(\mathscr{I})$ is a filter associated by an admissible ideal $\mathscr{I}$ with property (AP). Then, there is a set $P\subset\mathbb{N}$ such that $P\in\mathscr{F}(\mathscr{I})$ and the set $P\backslash P_i$ is finite for all i.*

3. Main Results

In this paper, we study concepts of $\mathscr{I}$-convergence, $\mathscr{I}^*$-convergence, $\mathscr{I}$-Cauchy and $\mathscr{I}^*$-Cauchy sequences of functions and investigate relationships between them and some properties in 2-normed spaces.
Throughout the paper, we let $\mathscr{I}\subset 2^{\mathbb{N}}$ be an admissible ideal, X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f,g be two functions from X to Y.

Definition 3.1. The sequence of functions $\{f_n\}$ is said to be (pointwise) $\mathscr{I}^*$-convergent to f, if there exists a set $M\in\mathscr{F}(\mathscr{I})$, (i.e., $\mathbb{N}\backslash M\in\mathscr{I}$), $M=\{m_1<m_2<\cdots<m_k<\cdots\}$, such that for each $x\in X$ and each nonzero $z\in Y$

$$\lim_{k\to\infty}\|f_{n_k}(x),z\|=\|f(x),z\|$$

and we write

$$\mathscr{I}^*-\lim_{n\to\infty}\|f_n(x),z\|=\|f(x),z\| \text{ or } f_n\xrightarrow{\mathscr{I}^*}f.$$

Theorem 3.1. *For each $x\in X$ and each nonzero $z\in Y$,*

$$\mathscr{I}^*-\lim_{n\to\infty}\|f_n(x),z\|=\|f(x),z\| \textit{ implies } \mathscr{I}-lim_{n\to\infty}\|f_n(x),z\|=\|f(x),z\|.$$

Proof. Since for each $x\in X$ and each nonzero $z\in Y$,

$$\mathscr{I}^*-\lim_{n\to\infty}\|f_n(x),z\|=\|f(x),z\|,$$

so there exists a set $H\in\mathscr{I}$ such that for $M=\mathbb{N}\backslash H=\{m_1<m_2<\cdots<m_k<\cdots\}$ we have

$$\lim_{k\to\infty}\|f_{n_k}(x),z\|=\|f(x),z\|.$$

Let $\varepsilon>0$. Then, for each $x\in X$ there exists a $k_0=k_0(\varepsilon,x)\in\mathbb{N}$ such that for each nonzero $z\in Y$, $\|f_n(x)-f(x),z\|<\varepsilon$, for all $n\in M$ such that $n\geq k_0$. Then, obviously we have

$$A(\varepsilon,z)=\{n\in\mathbb{N}:\|f_n(x)-f(x),z\|\geq\varepsilon\}\subset H\cup\{m_1<m_2<\cdots<m_{k_0}\},$$

for each $x\in X$ and each nonzero $z\in Y$. Since $\mathscr{I}\subset 2^{\mathbb{N}}$ be an admissible ideal, then

$$H\cup\{m_1<m_2<\cdots<m_{k_0}\}\in\mathscr{I}$$

and therefore, $A(\varepsilon,z)\in\mathscr{I}$. This implies that $\mathscr{I}-\lim_{n\to\infty}\|f_n(x),z\|=\|f(x),z\|$. □

Theorem 3.2. *Let $\mathscr{I}\subset 2^{\mathbb{N}}$ be an admissible ideal having the property (AP). Then,*

$$\mathscr{I}-\lim_{n\to\infty}\|f_n(x),z\|=\|f(x),z\| \textit{ implies } \mathscr{I}^*-\lim_{n\to\infty}\|f_n(x),z\|=\|f(x),z\|.$$

Proof. Let $\mathscr{I}\subset 2^{\mathbb{N}}$ satisfy the property (AP) and $\mathscr{I}-\lim_{n\to\infty}\|f_n(x),z\|=\|f(x),z\|$, for each $x\in X$ and each nonzero $z\in Y$. Then, for any $\varepsilon>0$

$$A(\varepsilon,z)=\{n\in\mathbb{N}:\|f_n(x)-f(x),z\|\geq\varepsilon\}\in\mathscr{I},$$

for each $x\in X$ and each nonzero $z\in Y$. Now put

$$A_1(\varepsilon,z)=\{n\in\mathbb{N}:\|f_n(x)-f(x),z\|\geq 1\}$$

and

$$A_k(\varepsilon,z)=\{n\in\mathbb{N}:\frac{1}{k}\leq\|f_n(x)-f(x),z\|<\frac{1}{k-1}\}$$

for $k\geq 2$. It is clear that $A_i\cap A_j=\emptyset$ for $i\neq j$ and $A_i\in\mathscr{I}$ for each $i\in\mathbb{N}$. By property (AP) there exists a sequence $\{B_k\}_{k\in\mathbb{N}}$ of sets such that $A_j\Delta B_j$ is finite and $B=\bigcup_{j=1}^{\infty}B_j\in\mathscr{I}$.
We shall prove that, for each $x\in X$ and each nonzero $z\in Y$,

$$\lim_{k\to\infty}\|f_{n_k}(x),z\|=\|f(x),z\|,\ \ k\in M,$$

for $M = \mathbb{N}\backslash B \in \mathscr{F}(\mathscr{I})$. Let $\delta > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k}$. Then we have

$$\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \delta\} \subset \bigcup_{j=1}^{k} A_j.$$

Since $A_j \Delta B_j$, $j = 1, 2, ..., k$, is finite set there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{k} B_j\right) \cap \{n \in \mathbb{N} : n \geq n_0\} = \left(\bigcup_{j=1}^{k} A_j\right) \cap \{n \in \mathbb{N} : n \geq n_0\}.$$

If $n \geq n_0$ and $n \notin B$ then

$$n \notin \bigcup_{j=1}^{k} B_j \text{ and so } n \notin \bigcup_{j=1}^{k} A_j.$$

Hence, we have $\|f_n(x) - f(x), z\| < \frac{1}{k} < \delta$, for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|, \ \ k \in M,$$

and so, we have

$$\mathscr{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$$

for each $x \in X$ and each nonzero $z \in Y$. □

Now we give the concepts of $\mathscr{I}$-Cauchy sequence and $\mathscr{I}^*$-Cauchy sequence and investigate some properties about them.

Definition 3.2. $\{f_n\}$ is said to be $\mathscr{I}$-Cauchy sequence, if for every $\varepsilon > 0$ and each $x \in X$ there exists $s = s(\varepsilon, x) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : \|f_n(x) - f_s(x), z\| \geq \varepsilon\} \in \mathscr{I},$$

for each nonzero $z \in Y$.

Theorem 3.3. *If $\{f_n\}$ is $\mathscr{I}$-convergent, then it is $\mathscr{I}$-Cauchy sequence.*

Proof. Suppose that $\{f_n\}$ is $\mathscr{I}$-convergent to f. Then, for $\varepsilon > 0$

$$A\left(\frac{\varepsilon}{2}, z\right) = \left\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathscr{I},$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$A^c\left(\frac{\varepsilon}{2}, z\right) = \left\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \frac{\varepsilon}{2}\right\} \in \mathscr{F}(\mathscr{I}),$$

for each $x \in X$ and each nonzero $z \in Y$ and therefore $A^c\left(\frac{\varepsilon}{2}, z\right)$ is non-empty. So, we can choose a positive integer k such that $k \notin A\left(\frac{\varepsilon}{2}, z\right)$ and $\|f_k(x) - f(x), z\| < \frac{\varepsilon}{2}$. Now, we define the set

$$B(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f_k(x), z\| \geq \varepsilon\},$$

for each $x \in X$ and each nonzero $z \in Y$, such that we show that $B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z)$. Let $n \in B(\varepsilon, z)$, then we have

$$\begin{aligned} \varepsilon \leq \|f_n(x) - f_k(x), z\| &\leq \|f_n(x) - f(x), z\| + \|f_k(x) - f(x), z\| \\ &< \|f_n(x) - f(x), z\| + \frac{\varepsilon}{2}, \end{aligned}$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$\frac{\varepsilon}{2} < \|f_n(x) - f(x), z\|$$

and so, $n \in A(\frac{\varepsilon}{2}, z)$. Hence, we have $B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z)$ and $\{f_n\}$ is $\mathscr{I}$-Cauchy sequence. □

Definition 3.3. The sequence $\{f_n\}$ is said to be $\mathscr{I}^*$-Cauchy sequence, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, such that the subsequence $\{f_M\} = \{f_{m_k}\}$ is a Cauchy sequence, i.e.,

$$\lim_{k,p\to\infty} \|f_{m_k}(x) - f_{m_p}(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$.

Theorem 3.4. *If $\{f_n\}$ is a $\mathscr{I}^*$-Cauchy sequence, then it is $\mathscr{I}$-Cauchy sequence in 2-normed spaces.*

Proof. Let (f_n) is a $\mathscr{I}^*$-Cauchy sequence in 2-normed spaces. Then, by definition there exists the set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$, $M \in \mathscr{F}(\mathscr{I})$ such that for every $\varepsilon > 0$,

$$\|f_{n_k}(x) - f_{n_p}(x), z\| < \varepsilon,$$

for every $\varepsilon > 0$, for each $x \in X$, each nonzero $z \in Y$ and $k, p > k_0 = k_0(\varepsilon, x)$. Let $N = N(\varepsilon, x) = m_{k_0} + 1$. Then, for every $\varepsilon > 0$ we have

$$\|f_{n_k}(x) - f_N(x), z\| < \varepsilon,$$

for each $x \in X$, each nonzero $z \in Y$ and $k > k_0$. Now put $H = \mathbb{N} \backslash M$. It is clear that $H \in \mathscr{I}$ and

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f_N(x), z\| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < ... < m_{k_0}\}.$$

Since $\mathscr{I}$ is an admissible ideal, then $H \cup \{m_1 < m_2 < ... < m_{k_0}\} \in \mathscr{I}$. Hence, for every $\varepsilon > 0$ we find $N = N(\varepsilon, x)$ such that $A(\varepsilon, z) \in \mathscr{I}$, i.e., (f_n) is a $\mathscr{I}$-Cauchy sequence. □

Theorem 3.5. *If* $\mathscr{I}^* - \lim_{n\to\infty} \|f_n(x) - f(x), z\| = 0$, *then* $\{f_n\}$ *is a* $\mathscr{I}$*-Cauchy sequence.*

Proof. By assumption there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$, $M \in \mathscr{F}(\mathscr{I})$ such that $\lim_{n\to\infty} \|f_n(x) - f(x), z\| = 0$ for each $x \in X$ and each nonzero $z \in Y$. It shows that there exists $k_0 = k_0(\varepsilon, x)$ such that

$$\|f_n(x) - f(x), z\| < \frac{\varepsilon}{2},$$

for every $\varepsilon > 0$, each $x \in X$, each nonzero $z \in Y$ and $k > k_0$. Since

$$\begin{aligned} \|f_{n_k}(x) - f_{n_p}(x), z\| &< \|f_{n_k}(x) - f(x), z\| + \|f_{n_p}(x) - f(x), z\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for each $x \in X$, each nonzero $z \in Y$ and $k > k_0$, $p > k_0$ so we have

$$\lim_{k,p\to\infty} \|f_{n_k}(x) - f_{n_p}(x), z\| = 0,$$

i.e., (f_n) is a $\mathscr{I}^*$-Cauchy sequence. Then by Theorem 3.4 (f_n) is a $\mathscr{I}$-Cauchy sequence. □

Theorem 3.6. *Let* $\mathscr{I}$ *be an admissible ideal with property* (AP). *Then the concepts* $\mathscr{I}$*-Cauchy sequence and* $\mathscr{I}^*$*-Cauchy sequence coincide.*

Proof. By Theorem 3.4, $\mathscr{I}^*$-Cauchy sequence implies $\mathscr{I}$-Cauchy sequence (in this case $\mathscr{I}$ need not to have (AP) condition). Then, under assumption that (f_n) is a $\mathscr{I}$-Cauchy sequence, it suffices to prove (f_n) is a $\mathscr{I}^*$-Cauchy sequence. Let (f_n) is a $\mathscr{I}$-Cauchy sequence. Then, for every $\varepsilon > 0$ and each $x \in X$ there exists $s = s(\varepsilon, x) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : \|f_n(x) - f_s(x), z\| \geq \varepsilon\} \in \mathscr{I},$$

for each nonzero $z \in Y$. Let

$$P_i = \left\{n \in \mathbb{N} : \|f_n(x) - f_{m_i}(x), z\| < \frac{1}{i}\right\},$$

$i = 1, 2, ...$ where $m_i = s\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathscr{F}(\mathscr{I})$, $i = 1, 2, ...$. Since $\mathscr{I}$ has (AP) property then by Lemma 2.2 there exists a set $P \subset \mathbb{N}$ such that $P \subset \mathscr{F}(\mathscr{I})$ and $P \backslash P_i$ is finite for all i.
Now, we show that

$$\lim_{m,n\to\infty} \|f_n(x) - f(x), z\| = 0$$

for each $x \in X$, each nonzero $z \in Y$. Let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > \frac{2}{\varepsilon}$. If $m, n \in P$ then $P \backslash P_j$ is a finite set, so there exists $k = k(j)$ such that $m \in P_j$ and $n \in P_j$, for all $m, n > k(j)$. Therefore,

$$\|f_n(x) - f_{m_j}(x), z\| < \frac{1}{j} \text{ and } \|f_m(x) - f_{m_j}(x), z\| < \frac{1}{j}$$

for all $m, n > k(j)$, each $x \in X$ and each nonzero $z \in Y$ and so, we get

$$\begin{aligned} \|f_n(x) - f_m(x), z\| &< \|f_n(x) - f_{m_j}(x), z\| + \|f_m(x) - f_{m_j}(x), z\| \\ &< \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon \end{aligned}$$

for $m, n > k(j)$, each $x \in X$ and each nonzero $z \in Y$. Thus, for any $\varepsilon > 0$ and each $x \in X$ there exists $k = k(\varepsilon, x)$ such that for any $m, n > k$ and $m, n \in P \in \mathscr{F}(\mathscr{I})$,

$$\|f_n(x) - f_m(x), z\| < \varepsilon$$

for every nonzero $z \in Y$ and so, the sequence (f_n) is a $\mathscr{I}^*$-Cauchy sequence in 2-normed spaces. □

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