

Asymptotically \mathcal{J}_2^σ -Statistical Equivalence of Double Sequences of Sets Defined by Modulus Functions

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Abstract

Statistical convergence and ideal convergence of real numbers, which are of great importance in the theory of summability, are studied by many mathematicians. Fast (1951) and Schoenberg (1959), independently, introduced the concept of statistical convergence. Mursaleen and Edely (2003) extended this concept to the double sequences. Recently, the statistical convergence has been extended to ideal convergence of real numbers and some important properties about ideal convergence have been investigated by many mathematicians. Kostyrko et al. (2000) defined the ideal \mathcal{I} of subset of \mathbb{N} and investigated some properties about \mathcal{I} -convergence. The idea of \mathcal{J}_2 -convergence and some properties of this convergence were studied by Das et al. (2008).

Nuray et al. (2011) defined the notions of invariant uniform density of subsets E of \mathbb{N} , \mathcal{J}_σ -convergence and investigated relations between \mathcal{J}_σ -convergence and σ -convergence also \mathcal{J}_σ -convergence and $[V_\sigma]_p$ -convergence. Tortop and Dündar (2018) introduced \mathcal{J}_2 -invariant convergence of double set sequences and investigated some properties. Akın studied Wijsman lacunary \mathcal{J}_2 -invariant convergence of double sequences of sets. Asymptotically equivalent and some properties of equivalence are studied by several authors. Modulus function was introduced by Nakano (1953). Several authors define some new concepts and give inclusion theorems using a modulus function f . Kişi et al. (2015) introduced f -asymptotically \mathcal{J}_θ -equivalent set sequences. Akın and Dündar (2018) and Akın et al. (2018) give definitions of f -asymptotically \mathcal{J}_σ -statistical equivalence and $\mathcal{J}_{\sigma\theta}$ -statistical equivalence of set sequences and study some properties.

In this study, we introduced asymptotically \mathcal{J}_2^σ -statistical equivalence for double sequences of sets. Also we investigate relationships between asymptotically \mathcal{J}_2^σ -statistical equivalence and strongly f -asymptotically \mathcal{J}_2^σ -equivalence.

Key Words: Asymptotic Equivalence; \mathcal{J}_2 -Convergence; Invariant Statistical Convergence; Modulus Function.

Özet

Toplanabilme teorisinde önemli bir yeri olan reel sayıların istatistiksel ve ideal yakınsaklığı kavramı birçok matematikçi tarafından çalışılmıştır. Reel sayı dizilerinin yakınsaklığı genişletilerek oluşturulan istatistiksel yakınsaklık kavramı ile ilgili Fast (1951) ve Schoenberg (1959) çalışma yapmıştır. Mursaleen ve Edely (2003) istatistiksel yakınsaklık kavramını çift dizilere taşımıştır. Son zamanlarda, istatistiksel yakınsaklık kavramı ideal yakınsaklık kavramına genişletilerek bazı matematikçiler tarafından birçok çalışmalar yapılmıştır. İstatistiksel yakınsaklığın bir genelleştirmesi olan \mathcal{I} -yakınsaklık Kostyrko vd. (2000) tarafından tanımlanmış olup, bu kavram \mathbb{N} doğal sayılar kümesinin alt kümelerinin sınıfı olan \mathcal{I} idealinin yapısına bağlıdır. \mathcal{J}_2 -yakınsaklık kavramı ve bu kavramın bazı özellikleri Das vd. (2008) tarafından incelendi.

Nuray vd. (2011) $E \subseteq \mathbb{N}$ 'nin düzgün invariant yoğunluğu kavramını tanımladı. Nuray vd. (2011) \mathcal{J}_σ -yakınsaklık ile σ -yakınsaklık ve \mathcal{J}_σ -yakınsaklık ile $[V_\sigma]_p$ -yakınsaklık aralarındaki ilişkileri inceledi. Tortop ve Dündar (2018) çift küme dizileri için \mathcal{J}_2 -invariant yakınsaklık kavramını tanımladı. Akın çift küme dizilerinin Wijsman lacunary \mathcal{J}_2 -invariant yakınsaklığı ile ilgili bir çalışma yaptı. Asimptotik denklik kavramı ve bu kavramın bazı özellikleri birçok yazar tarafından çalışılmıştır. Modülüs fonksiyonu ilk defa Nakano (1953) tarafından tanımlandı. Maddox (1986), Pehlivan (1995) ve birçok yazar tarafından f modülüs fonksiyonu kullanılarak bazı yeni kavramları ve sonuç teoremlerini içeren çalışmalar yapıldı. . Kişi et al. (2015) küme dizilerinin f -asimptotik \mathcal{J}_θ denkliğini tanımladı. Modülüs fonksiyonunu kullanılarak lacunary ideal denk diziler ile ilgili Kumar ve Sharma (2012) tarafından bir çalışma yapıldı. Akın and Dündar (2018) and Akın vd. (2018) tarafından küme dizilerinin f -asimptotik \mathcal{J}_σ ve $\mathcal{J}_{\sigma\theta}$ -istatistiksel denkliği tanımları yapıldı.

Bu çalışmada çift küme dizileri için asimptotik \mathcal{J}_2^σ istatistiksel denklik kavramı tanımlandı. Aynı zamanda asimptotik \mathcal{J}_2^σ istatistiksel denklik ile kuvvetli f -asimptotik \mathcal{J}_2^σ -denklik arasındaki ilişkiler incelendi.

Key Words: Asimptotik Denklik; \mathcal{J}_2 -Yakınsaklık; Invariant İstatistiksel Yakınsaklık; Modülüs Fonksiyonu.

Introduction and Definitions

Kostyrko et al. (2000) defined \mathcal{I} of subset of \mathbb{N} (natural numbers) and investigated \mathcal{I} -convergence with some properties and proved theorems about \mathcal{I} -convergence. The idea of \mathcal{I}_2 -convergence and some properties of this convergence were studied by Das et al. (2008). Nuray and Rhoades (2012) defined the idea of statistical convergence of set sequence and investigated some theorems about this notion and important properties. Kişi and Nuray (2013) defined Wijsman \mathcal{I} -convergence of sequence of sets and also examined some theorems about it. After, several authors extended the convergence of real numbers sequences to convergence of sequences of sets and investigated its characteristic in summability.

Several authors have studied invariant convergent sequences [see, Mursaleen (1983), Nuray and Savaş (1994), Pancaroğlu and Nuray (2013, 2013, 2014), Raimi (1963), Savaş (1989, 1989), Savaş and Nuray (1993), Schaefer (1972) and Ulusu et al. (2018)].

Asymptotically equivalent and some properties of equivalence are studied by several authors [see, Kişi et al. (2015), Pancaroğlu et al. (2013), Patterson (2003), Savaş (2013), Ulusu and Nuray (2013)]. Ulusu and Gülle introduced the concept of asymptotically \mathcal{I}_σ -equivalence of sequences of sets. Recently, Dündar et al. studied on asymptotically ideal invariant equivalence of double sequences.

Several authors define some new concepts and give inclusion theorems using a modulus function f [see, Khan and Khan (2013), Kılınc and Solak (2014), Maddox (1986), Nakano (1953), Pehlivan and Fisher(1995)]. Kumar and Sharma (2012) studied \mathcal{I}_θ -equivalent sequences using a modulus function f . Kişi et al. (2015) introduced f -asymptotically \mathcal{I}_θ -equivalent set sequences. Akin and Dündar (2018) and Akin et al. (2018) give definitions of f -asymptotically \mathcal{I}_σ and $\mathcal{I}_{\sigma\theta}$ -statistical equivalence of set sequences.

Now, we recall the basic concepts and some definitions and notations (see, [Baronti and Papini (1986), Beer (1985, 1994), Das et al. (2008), Dündar et al. (2016, 2017), Fast (1951), Kostyrko et al. (2000), Lorentz (1948), Marouf (1993), Mursaleen (1983), Nakano (1953), Nuray et al. (2011, 2016), Pancaroğlu and Nuray (2014), Akin and Dündar (2018), Pehlivan and Fisher (1995), Raimi (1963), Tortop and Dündar, Ulusu and Dündar (2014) and Wijsman (1964, 1966)]).

Let $u = (u_k)$ and $v = (v_k)$ be two non-negative sequences. If $\lim_k \frac{u_k}{v_k} = 1$, then they are said to be asymptotically equivalent (denoted by $u \sim v$).

Let (Y, ρ) be a metric space, $y \in Y$ and E be any non-empty subset of Y , we define the distance from y to E by

$$d(y, E) = \inf_{e \in E} \rho(y, e).$$

Let σ be a mapping of the positive integers into itself. A continuous linear functional ϕ on ℓ_∞ , the space of real bounded sequences, is said to be an invariant mean or a σ mean if and only if

1. $\phi(u) \geq 0$, when the sequence $u = (u_j)$ has $u_j \geq 0$ for all j ,
2. $\phi(i) = 1$, where $i = (1, 1, 1, \dots)$,
3. $\phi(u_{\sigma(j)}) = \phi(u)$, for all $u \in \ell_\infty$.

The mapping ϕ is supposed to be one-to-one and such that $\sigma^m(j) \neq j$ for all positive integers j and m , where $\sigma^m(j)$ denotes the m th iterate of the mapping σ at j . Hence, ϕ extends the limit functional on c , the space of convergent sequences, in the sense that $\phi(u) = \lim u$ for all $u \in c$. If σ is a translation mapping that is $\sigma(j) = j + 1$, the σ mean is often called a Banach limit.

Let (Y, ρ) be a metric space and E, F, E_i and F_i ($i = 1, 2, \dots$) be non-empty closed subsets of Y .

Let $L \in \mathbb{R}$. Then, we define $d(y; E_i, F_i)$ as follows:

$$d(y; E_i, F_i) = \begin{cases} \frac{d(y, E_i)}{d(y, F_i)}, & y \notin E_i \cup F_i, \\ L, & y \in E_i \cup F_i. \end{cases}$$

$\mathcal{J} \subseteq 2^{\mathbb{N}}$ which is a family of subsets of \mathbb{N} is called an ideal, if the followings hold:

- (i) $\emptyset \in \mathcal{J}$,
- (ii) For each $E, F \in \mathcal{J}$, $E \cup F \in \mathcal{J}$,
- (iii) For each $E \in \mathcal{J}$ and each $F \subseteq E$, we have $F \in \mathcal{J}$.

Let $\mathcal{J} \subseteq 2^{\mathbb{N}}$ be an ideal. $\mathcal{J} \subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N} \notin \mathcal{J}$. Also, for non-trivial ideal and for each $n \in \mathbb{N}$ if $\{n\} \in \mathcal{J}$, then $\mathcal{J} \subseteq 2^{\mathbb{N}}$ is admissible ideal. After that, we consider that \mathcal{J} is an admissible ideal.

Let $K \subseteq \mathbb{N}$ and

$$s_m = \min_n |K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|$$

and

$$S_m = \max_n |K \cap \{\sigma(n), \sigma^2(n), \dots, \sigma^m(n)\}|.$$

If the limits $\underline{V}(K) = \lim_{m \rightarrow \infty} \frac{s_m}{m}$ and $\overline{V}(K) = \lim_{m \rightarrow \infty} \frac{S_m}{m}$ exists then, they are called a lower σ -uniform density and an upper σ -uniform density of the set K , respectively. If $\underline{V}(K) = \overline{V}(K)$, then $V(K) = \underline{V}(K) = \overline{V}(K)$ is called the σ -uniform density of K .

Denote by \mathcal{J}_σ the class of all $K \subseteq \mathbb{N}$ with $V(K) = 0$. It is clearly that \mathcal{J}_σ is admissible ideal.

If for every $\gamma > 0$, $A_\gamma = \{i: |x_i - L| \geq \gamma\}$ belongs to \mathcal{J}_σ , i.e., $V(A_\gamma) = 0$ then, the sequence $u = (u_i)$ is said to be \mathcal{J}_σ -convergent to L . It is denoted by $\mathcal{J}_\sigma - \lim u_i = L$.

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$A_{\tilde{y}, \gamma} = \{i: |d(y; E_i, F_i) - L| \geq \gamma\}$$

belongs to \mathcal{J}_σ , that is, $V(A_{\tilde{y}, \gamma}) = 0$ then, the sequences $\{E_i\}$ and $\{F_i\}$ are asymptotically \mathcal{J} -invariant equivalent or asymptotically \mathcal{J}_σ -equivalent of multiple L . In this instance, we write $E_i \sim_{\mathcal{J}_\sigma}^w F_i$ and if $L = 1$, simply asymptotically \mathcal{J} -invariant equivalent.

If following conditions hold for the function $f: [0, \infty) \rightarrow [0, \infty)$, then it is called a modulus function:

1. $f(u) = 0$ if and only if $u = 0$,
2. $f(u + v) \leq f(u) + f(v)$,
3. f is nondecreasing,
4. f is continuous from the right at 0.

This after, we let f as a modulus function.

The modulus function f may be unbounded (for example $f(u) = u^q$, $0 < q < 1$) or bounded (for example $f(u) = \frac{u}{u+1}$).

If for every $\gamma > 0$ and for each $y \in Y$,

$$\{i \in \mathbb{N} : f(|d(y; E_i, F_i) - L|) \geq \gamma\} \in \mathcal{J}_\sigma$$

then, we say that the sequences $\{E_i\}$ and $\{F_i\}$ are said to be f -asymptotically \mathcal{J} -invariant equivalent of multiple L (denoted by $E_i \stackrel{w_{\sigma}^L(f)}{\sim} F_i$) and if $L = 1$ simply f -asymptotically \mathcal{J} -invariant equivalent.

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^n f(|d(y; E_i, F_i) - L|) \geq \gamma\right\} \in \mathcal{J}_\sigma$$

then, we say that the sequences $\{E_i\}$ and $\{F_i\}$ are said to be strongly f -asymptotically \mathcal{J} -invariant equivalent of multiple L (denoted by $E_i \stackrel{[w_{\sigma}^L(f)]}{\sim} F_i$) and if $L = 1$, simply strongly f -asymptotically \mathcal{J} -invariant equivalent.

Let $\{E_i\}$ and $\{F_i\}$ be two sequences. If for every $\gamma > 0$, $\gamma > 0$ and for each $y \in Y$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{i \leq n : |d(y; E_i, F_i) - L| \geq \gamma\}| \geq \gamma\right\} \in \mathcal{J}_\sigma$$

then, we say that the sequences $\{E_i\}$ and $\{F_i\}$ are asymptotically \mathcal{J} -invariant statistical equivalent of multiple L (denoted by $E_i \stackrel{w_{\sigma}^L(S)}{\sim} F_i$) and if $L = 1$, simply asymptotically \mathcal{J} -invariant statistical equivalent.

Let \mathcal{J}_2 be a nontrivial ideal of $\mathbb{N} \times \mathbb{N}$. It is called strongly admissible ideal if $\{k\} \times \mathbb{N}$ and $\mathbb{N} \times \{k\}$ belong to \mathcal{J}_2 for each $k \in \mathbb{N}$. This after, we let \mathcal{J}_2 as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

If we let a ideal as a strongly admissible ideal then, it is clear that it is admissible also.

Let

$$\mathcal{J}_2^0 = \{E \subset \mathbb{N} \times \mathbb{N} : (\exists i(E) \in \mathbb{N})(r, s \geq i(E) \Rightarrow (r, s) \notin E)\}.$$

It is clear that \mathcal{J}_2^0 is a strongly admissible ideal. Also, it is evidently \mathcal{J}_2 is strongly admissible if and only if $\mathcal{J}_2^0 \subset \mathcal{J}_2$.

Let (Y, ρ) be a metric space and $y = (y_{ij})$ be a sequence in Y . If for any $\gamma > 0$,

$$A(\gamma) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : \rho(y_{ij}, L) \geq \gamma\} \in \mathcal{J}_2$$

then, it is said to be \mathcal{J}_2 -convergent to L . In this instance, y is \mathcal{J}_2 -convergent and we write

$$\mathcal{J}_2 - \lim_{i, j \rightarrow \infty} y_{ij} = L.$$

Let $E \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mk} : \min_{i, j} |E \cap \{(\sigma(i), \sigma(j)), (\sigma^2(i), \sigma^2(j)), \dots, (\sigma^m(i), \sigma^k(j))\}|$$

and

$$S_{mk} : \max_{i, j} |E \cap \{(\sigma(i), \sigma(j)), (\sigma^2(i), \sigma^2(j)), \dots, (\sigma^m(i), \sigma^k(j))\}|.$$

If the limits

$$\underline{V}_2(E) := \lim_{m, k \rightarrow \infty} \frac{s_{mk}}{mk}, \quad \overline{V}_2(E) := \lim_{m, k \rightarrow \infty} \frac{S_{mk}}{mk}$$

exists then $\underline{V}_2(E)$ is called a lower and $\overline{V}_2(E)$ is called an upper σ -uniform density of the set E , respectively. If $\underline{V}_2(E) = \overline{V}_2(E)$ holds then, $V_2(E) = \underline{V}_2(E) = \overline{V}_2(E)$ is called the σ -uniform density of E .

Denote by \mathcal{J}_2^σ the class of all $E \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(E) = 0$.

This after, let (Y, ρ) be a separable metric space and E_{ij}, F_{ij}, E, F be any nonempty closed subsets of Y .

If for each $y \in Y$,

$$\lim_{m,k \rightarrow \infty} \frac{1}{mk} \sum_{i,j=1,1}^{m,k} d(y, E_{\sigma^i(s), \sigma^j(t)}) = d(y, E),$$

uniformly in s, t then, the double sequence $\{E_{ij}\}$ is said to be invariant convergent to E in Y .

If for every $\gamma > 0$,

$$A(\gamma, y) = \{(i, j): |d(y, E_{ij}) - d(y, E)| \geq \gamma\} \in \mathcal{J}_2^\sigma$$

that is, $V_2(A(\gamma, y)) = 0$, then, the double sequence $\{E_{ij}\}$ is said to be Wijsman \mathcal{J}_2 -invariant convergent or $\mathcal{J}_{W_2}^\sigma$ -convergent to E . In this instance, we write $E_{ij} \rightarrow E(\mathcal{J}_{W_2}^\sigma)$ and by $\mathcal{J}_{W_2}^\sigma$ we will denote the set of all Wijsman \mathcal{J}_2^σ -convergent double sequences of sets.

For non-empty closed subsets E_{ij}, F_{ij} of Y define $d(y; E_{ij}, F_{ij})$ as follows:

$$d(y; E_{ij}, F_{ij}) = \begin{cases} \frac{d(y, E_{ij})}{d(y, F_{ij})} & , \quad y \notin E_{ij} \cup F_{ij} \\ L & , \quad y \in E_{ij} \cup F_{ij}. \end{cases}$$

Method

In the proofs of the theorems obtained in this study, used frequently in mathematics,

- i. Direct proof method,
- ii. Reverse proof method,
- iii. Contrapositive method,
- iv. Induction method

methods were used as needed.

Main Results

Definition 2.1 If for every $\gamma > 0$ and each $y \in Y$,

$$\left\{ (m, k) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mk} \sum_{i,j=1,1}^{mk} f(|d(y; E_{ij}, F_{ij}) - L|) \geq \gamma \right\} \in \mathcal{J}_2^\sigma$$

then, the double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be strongly f -asymptotically \mathcal{J}_2^σ -equivalent of multiple L denoted by

$$E_{ij} \stackrel{[W_{\mathcal{J}_2^\sigma}^L(f)]}{\sim} F_{ij}$$

and if $L = 1$, simply strongly f -asymptotically \mathcal{J}_2^σ -equivalent.

Definition 2.2 If for every $\gamma > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{ (m, k) : \frac{1}{mk} |\{i \leq m, j \leq k : |d(y; E_{ij}, F_{ij}) - L| \geq \gamma\}| \geq \delta \right\} \in \mathcal{J}_2^\sigma$$

then, the double sequences $\{E_{ij}\}$ and $\{F_{ij}\}$ are said to be asymptotically \mathcal{J}_2 -invariant statistical equivalent of multiple L denoted by

$$E_{ij} \stackrel{W_{\mathcal{J}_2^\sigma}^L(S)}{\sim} F_{ij}$$

and if $L = 1$, simply asymptotically \mathcal{J}_2 -invariant statistical equivalent.

Theorem 2.1 For each $y \in Y$, following holds:

$$E_{kj} \stackrel{[W_{\mathcal{J}_2^\sigma}^L(f)]}{\sim} F_{kj} \Rightarrow E_{kj} \stackrel{W_{\mathcal{J}_2^\sigma}^L(S)}{\sim} F_{kj}.$$

Theorem 2.2 If f is bounded, then for each $y \in Y$,

$$E_{ij} \stackrel{[W_{\mathcal{J}_2^\sigma}^L(f)]}{\sim} F_{ij} \Leftrightarrow E_{ij} \stackrel{W_{\mathcal{J}_2^\sigma}^L(S)}{\sim} F_{ij}.$$

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