Asymptotically Lacunary $\mathcal{I}_2^{\sigma}$-Statistical Equivalence for Double Set Sequences Defined by Modulus Functions

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Öz

Son zamanlarda reel sayıların istatistiksel yakınsaklık kavramı ideal yakınsaklık kavramına genişletilerek birçok matematikçi tarafından çalışılmıştır. Fast (1951) ve Schoenberg (1959) istatistiksel yakınsaklık kavramını tanımladı. Daha sonra birçok yazar bu kavram ve özellikleri ile ilgili çalışmalar yaptı. Kostyrko vd. (2000) ℕ'ın alt kümesi $\mathcal{I}$ ideal kavramını tanımladı. Sonra yeni tanımlanan $\mathcal{I}$-yakınsaklık kavramı ile ilgili bazı özellikleri inceleyip teoremler ile ispatladı. $\mathcal{I}_2$-yakınsaklık kavramı ve bu kavramın bazı özellikleri Das vd. (2008) tarafından incelendi. Nuray ve Rhoades (2012) küme dizileri için istatistiksel yakınsaklık kavramını tanımlayıp bu kavram ile ilgili bazı özellikleri ve teoremleri inceledi. Daha sonra birçok yazar tarafından reel sayı dizilerinin yakınsaklığı küme dizilerinin yakınsaklığına genişletilerek çalışmalar yapılmıştır.

Birçok yazar invariant yakınsaklık ile ilgili çalışmalar yapmıştır. Nuray vd. (2011) E ℕ'nin düzgün invariant yoğunluğu kavramını tanımladı. Nuray vd. (2011) $\mathcal{I}_\sigma$-yakınsaklık ile σ-yakınsaklık ve $\mathcal{I}_\sigma$-yakınsaklık ile $[V_\sigma]_p$-yakınsaklık kavramları arasındaki ilişkileri inceledi. Tortop ve Dündar (2018) çift küme dizileri için $\mathcal{I}_2$-invariant yakınsaklık kavramını tanımladı. Akın (incelemede) çift küme dizileri için tanımlanan Wijsman lacunary $\mathcal{I}_2$-invariant yakınsaklık ile ilgili bir çalışma yaptı. Modülüs fonksiyonu ilk defa Nakano (1953) tarafından tanımlandı. Bazı yazarlar f modülüs fonksiyonunu kullanarak bazı yeni kavramlar tanımlamıştır. Asimptotik denklik ve bu denkliğin özellikleri bazı yazarlar tarafından çalışılmıştır. Kişi et al. (2015) küme dizilerinin f-asimptotik $\mathcal{I}_\theta$ denkliği kavramını tanımladı. Modülüs fonksiyonunu kullanılarak lacunary ideal denk diziler ile ilgili Kumar ve Sharma (2012) tarafından bir çalışma yapıldı.

Bu çalışmada çift küme dizileri için asimptotik $\mathcal{I}_2^{\sigma\theta}$-istatistiksel denklik kavramı tanımlandı. Aynı zamanda asimptotik $\mathcal{I}_2^{\sigma\theta}$-istatistiksel denklik ile kuvvetli f-asimptotik $\mathcal{I}_2^{\sigma\theta}$-denklik kavramları arasındaki ilişkiler incelendi.

Key Words: Asimptotik Denklik, Lacunary Invariant Yakınsaklık, $\mathcal{I}_2$-Yakınsaklık, Modülüs Fonksiyonu.

Abstract

Recently, the statistical convergence extended to ideal convergence of real numbers and some important properties about ideal convergence investigated by many mathematicians. Fast (1951) and Schoenberg (1959), independently, introduced concept of statistical convergence and many authors studied this concepts. Kostyrko et al. (2000) defined thr ideal $\mathcal{I}$ of subset of ℕ and investigated $\mathcal{I}$-convergence with some properties and proved theorems about $\mathcal{I}$-convergence. The idea of $\mathcal{I}_2$-convergence and some properties of this convergence were studied by Das et al. (2008). Nuray and Rhoades (2012) defined the idea of statistical convergence of set sequence and investigated some theorems about this notion and importance properties. After, several authors extended the convergence of real numbers sequences to convergence of sequences of sets and investigated it's characteristic in summability.

Several authors have studied invariant convergent sequences. Nuray et al. (2011) defined invariant uniform density of subsets E of ℕ, $\mathcal{I}_\sigma$-convergence and investigated relationships between $\mathcal{I}_\sigma$-convergence and σ-convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Tortop and Dündar (2018) introduced $\mathcal{I}_2$-invariant convergence of double set sequences. Akın (in review) studied Wijsman lacunary $\mathcal{I}_2$-invariant convergence of double sequences of sets. Several authors define some new concepts and give inclusion theorems using a modulus function f. Asymptotically equivalent and some properties of equivalence studied by several authors. Kumar and Sharma (2012) studied $\mathcal{I}_\theta$-equivalent sequences using a modulus function f. Kişi et al. (2015) introduced f-asymptotically $\mathcal{I}_\theta$-equivalent set sequences.

In this paper, we define asymptotically $\mathcal{I}_2^{\sigma\theta}$-statistical equivalence for double sequences of sets. Also we investigate relationships between asymptotically $\mathcal{I}_2^{\sigma\theta}$-statistical equivalence and strongly f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalence.

Key Words: Asymptotic Equivalence, Lacunary Invariant Convergence, $\mathcal{I}_2$-Convergence, Modulus Function.

Introduction and Definitions

Statistical convergence and ideal convergence of real numbers, which are of great importance in the theory of summability, are studied by many mathematicians. Fast (1951) and Schoenberg (1959), independently, introduced concept of statistical convergence and many authors studied this concepts. Mursaleen and Edely (2003) extended this concept to the double sequences.

Asymptotically equivalent and some properties of equivalence studied by several authors (see, [Kişi et al., 2015; Pancaroğlu et al., 2013; Patterson, 2003; Patterson and Savaş, 2006; Savaş, 2013; Ulusu and Nuray, 2013]). Ulusu and Gülle (2019) introduced the concept of asymptotically $\mathcal{I}_\sigma$-equivalence of sequences of sets. Recently, Dündar et al. (in review) studied asymptotically ideal invariant equivalence of double sequences.

Several authors define some new concepts and give inclusion theorems using a modulus function f (see, [Khan and Kan, 2013; Kılınç and Solak, 2014; Maddox, 1986; Nakano, 1953; Pancaroğlu and Nuray 2015; Pehlivan and Fisher 1995]). Kumar and Sharma (2012) studied $\mathcal{I}_\theta$-equivalent sequences using a modulus function f. Kişi et al. (2015) introduced f-asymptotically $\mathcal{I}_\theta$-equivalent set sequences. Akın and Dündar (2018) and Akın et al. (2018) give definitions of f-asymptotically $\mathcal{I}_\sigma$ and $\mathcal{I}_{\sigma\theta}$-statistical equivalence of set sequences. Dündar and Akın (in review) studied f-asymptotically $\mathcal{I}_2^\sigma$-equivalence for double set sequences.

Now, we recall the basic concepts and some definitions and notations (See [Baronti and Papini, 1986; Beer, 2985,1994; Kişi et al., 2015; Kostyrko et al., 2000; Lorentz, 1948; Maddox, 1986; Marouf, 1993; Nuray et al., 2011; Pancaroğlu and Nuray, 2013b, Pancaroğlu et al., 2013, Patterson, 2003; Pehlivan and Fisher,1995; Savaş, 1989b, Ulusu and Nuray, 2013; Ulusu and Dündar, 2014; Ulusu and Nuray (in review); Ulusu Gülle, 2019, Wijsman, 1964]).

Let two non-negative sequences $u = (u_k)$ and $v = (v_k)$. If $\lim_{k\to\infty} \frac{u_k}{v_k} = 1$, then $u = (u_k)$ and $v = (v_k)$ are said to be asymptotically equivalent (denoted by $u \sim v$).

Let (Y, ρ) be a metric space, $y \in Y$ and any non-empty subset C of Y, then we define the distance from y to C by

$$d(y, C) = \inf_{c\in C} \rho(y, c).$$

This after, we let (Y, ρ) be a metric space and C, D, C_k and D_k $(k = 1,2,\ldots)$ be non-empty closed subsets of Y.

A sequence $\{C_k\}$ is Wijsman convergent to C if $\lim_{k\to\infty} d(y, C_k) = d(y, C)$ for each $y \in Y$. In this instance, it is showed by $W - \lim_{k\to\infty} C_k = C$.

If $\sup_k d(y, C_k) < \infty$ for each $y \in Y$, then $\{C_k\}$ is bounded and we write $\{C_k\} \in L_\infty$.

Let C_k, $D_k \subseteq Y$ such that $d(y, C_k) > 0$ and $d(y, D_k) > 0$ for each $y \in Y$. The sequences $\{C_k\}$ and $\{D_k\}$ are asymptotically equivalent if for each $y \in Y$, $\lim_{k\to\infty} \frac{d(y,C_k)}{d(y,D_k)} = 1$ (denoted by $C_k \sim D_k$).

Let C_k, $D_k \subseteq Y$ such that $d(y, C_k) > 0$ and $d(y, D_k) > 0$ for each $y \in Y$. If for every $\varepsilon > 0$ and each $y \in Y$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{d(y, C_k)}{d(y, D_k)} - L \right| \ge \varepsilon \right\} \right| = 0,$$

then $\{C_k\}$ and $\{D_k\}$ are asymptotically statistical equivalent of multiple L (denoted by $C_k \overset{WS_L}{\sim} D_k$) and if $L = 1$, then $\{C_k\}$ and $\{D_k\}$ are asymptotically statistical equivalent.

Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a mapping and ϕ be a continuous linear functional on the space of real bounded sequences (ℓ_∞). ϕ is an invariant mean or a σ-mean, if the following conditions hold:

1. $\phi(u) \ge 0$, when the sequence $u = (u_n)$ has $u_n \ge 0$, for all n,
2. $\phi(e) = 1$, where $e = (1,1,1\ldots)$,
3. $\phi(u_{\sigma(n)}) = \phi(u)$ for all $u \in \ell_\infty$.

Suppose that the mappings ϕ are injective and such that $\sigma^m(j) \ne j$, for all $j, m \in \mathbb{N}$, where $\sigma^m(j)$ is the mth iterate of σ at j. Therefore, for all $u \in c$ $\phi(u)$ equals to $\lim u$ which is the extension of the limit functional on c, where $c = \{x = (x_k) : \lim_k x_k exists\}$.

If the equality $\sigma(j) \ne j + 1$ exists, then σ-mean is called a Banach limit, generally.

Now, we give the definition of ideal. $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal, provided that the followings hold:

(i) $\emptyset \in \mathcal{I}$, (ii) For each $E, F \in \mathcal{I}$ we have $E \cup F \in \mathcal{I}$, (iii) For each $E \in \mathcal{I}$ and each $F \subseteq E$ we have $F \in \mathcal{I}$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a ideal. $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial provided that $\mathbb{N} \notin \mathcal{I}$. Also, for non-trivial ideal and for each $n \in \mathbb{N}$ provided that $\{n\} \in \mathcal{I}$, then $\mathcal{I}$ is admissible ideal. After that, we consider that $\mathcal{I}$ is an admissible ideal.

Let $C \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mk} := \min_{i,j} |C \cap \{(\sigma(i), \sigma(j)), (\sigma^2(i), \sigma^2(j)), \ldots, (\sigma^m(i), \sigma^k(j))\}|$$

and

$$S_{mk} := \max_{i,j} |C \cap \{(\sigma(i), \sigma(j)), (\sigma^2(i), \sigma^2(j)), \ldots, (\sigma^m(i), \sigma^k(j))\}|.$$

If the limits $\underline{V_2}(C) := \lim_{m,k\to\infty} \frac{s_{mk}}{mk}$ and $\overline{V_2}(C) := \lim_{m,k\to\infty} \frac{S_{mk}}{mk}$ exists then $\underline{V_2}(C)$ is called a lower and $\overline{V_2}(C)$ is called an upper σ-uniform density of the set C, respectively. If $\underline{V_2}(C) = \overline{V_2}(C)$, then $V_2(C) = \underline{V_2}(C) = \overline{V_2}(C)$ is called the σ-uniform density of C. Denote by $\mathcal{I}_2^\sigma$ the class of all $C \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(C) = 0$.

This after, let C_{ij}, D_{ij}, C, D be any nonempty closed subsets of Y.

If for each $y \in Y$,

$$\lim_{m,k\to\infty} \frac{1}{mk} \sum_{i,j=1,1}^{m,k} d(y, C_{\sigma^i(s),\sigma^j(t)}) = d(y, C), \qquad \text{uniformly in } s, t$$

then, the double sequence $\{C_{ij}\}$ is said to be invariant convergent to C in Y.

If for every $\gamma > 0$,

$$A(\gamma, y) = \{(i,j) : |d(y, C_{ij}) - d(y, C)| \geq \gamma\} \in \mathcal{I}_2^\sigma$$

that is, $V_2(A(\gamma, y)) = 0$ then, the double sequence $\{C_{ij}\}$ is said to be Wijsman $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_{W_2}^\sigma$-convergent to C, In this instance, we write $C_{ij} \to C(\mathcal{I}_{W_2}^\sigma)$ and by $\mathcal{I}_{W_2}^\sigma$ we will denote the set of all Wijsman $\mathcal{I}_2^\sigma$-convergent double sequences of sets.

For non-empty closed subsets C_{ij}, D_{ij} of Y define $d(y; C_{ij}, D_{ij})$ as follows:

$$d(y; C_{ij}, D_{ij}) = \begin{cases} \frac{d(y, C_{ij})}{d(y, D_{ij})}, & y \notin C_{ij} \cup D_{ij} \\ L, & y \in C_{ij} \cup D_{ij}. \end{cases}$$

If following conditions hold for $f : [0, \infty) \to [0, \infty)$ function, then it is called a modulus function:

1. $f(u) = 0$ if and if only if $u = 0$,
2. $f(u + v) \leq f(u) + f(v)$
3. f is increasing
4. f is continuous from the right at 0.

This after, we let f as a modulus function.

The modulus function f may be unbounded (for instance $f(u) = u^q$, $0 < q < 1$) or bounded (for example $f(u) = \frac{u}{u+1}$).

If for every $\gamma > 0$ and for each $y \in Y$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : f(|d(y; C_{ij}, D_{ij}) - L|) \geq \gamma\} \in \mathcal{I}_2^\sigma,$$

then the double sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be f-asymptotically $\mathcal{I}_2^\sigma$-equivalent of multiple L (denoted by $C_{ij} \overset{W_{\mathcal{I}_2^\sigma}^L(f)}{\sim} D_{ij}$) and if $L = 1$, then $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be f-asymptotically $\mathcal{I}_2^\sigma$-equivalent.

If for every $\gamma > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(m,k) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mk} |\{i \leq m, j \leq k : |d(y; C_{ij}, D_{ij}) - L| \geq \gamma\}| \geq \delta\right\} \in \mathcal{I}_2^\sigma,$$

then $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be asymptotically $\mathcal{I}_2^\sigma$-statistical equivalent of multiple L (denoted by $C_{ij} \overset{W^L_{\mathcal{I}_2^\sigma}(S)}{\sim} D_{ij}$) and if $L = 1$, then asymptotically $\mathcal{I}_2^\sigma$-statistical equivalent.

A double sequence $\theta_2 = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \quad and \quad j_0 = 0, \bar{h}_u = j_u - j_{u-1} \to \infty, \qquad as\ r, u \to \infty.$$

We use the following notations afterwards:

$$k_{ru} = k_r j_u, \quad h_{ru} = h_r \bar{h}_u, \quad I_{ru} = \{(k,j): k_{r-1} < k \le k_r \quad and \quad j_{u-1} < j \le j_u\}.$$

After this, we take $\theta_2 = \{(k_r, j_u)\}$ as a double lacunary sequence.

Let $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence, $C \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{ru} := \min_{m,n} \left| C \cap \left\{ \left(\sigma^k(m), \sigma^j(n) \right) : (k,j) \in I_{ru} \right\} \right|$$

and

$$S_{ru} := \max_{m,n} \left| C \cap \left\{ \left(\sigma^k(m), \sigma^j(n) \right) : (k,j) \in I_{ru} \right\} \right|.$$

If the limits $\underline{V_2^\theta}(C) := \lim_{r,u\to\infty} \frac{s_{ru}}{h_{ru}}$ and $\overline{V_2^\theta}(C) := \lim_{r,u\to\infty} \frac{S_{ru}}{h_{ru}}$ exist, then they are called a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set C, respectively. If $\underline{V_2^\theta}(C) = \overline{V_2^\theta}(C)$, then $V_2^\theta(C) = \underline{V_2^\theta}(C) = \overline{V_2^\theta}(C)$ is called the lacunary σ-uniform density of C. Denote by $\mathcal{I}_2^{\sigma\theta}$ the class of all $C \subseteq \mathbb{N} \times \mathbb{N}$ with $\overline{V_2^\theta}(C) = 0$.

After this, we take $\mathcal{I}_2^{\sigma\theta}$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Method

In the proofs of the theorems obtained in this study, used frequently in mathematics,

i. Direct proof method,

ii. Reverse proof method,

iii. Contrapositive method,

iv. Induction method

methods were used as needed.

Main Results

Definition 1. *If for every* $\gamma > 0$ *and each* $y \in Y$,

$$\left\{ (k,j) \in \mathbb{N} \times \mathbb{N} : f(|d(y; C_{kj}, D_{kj}) - L|) \ge \gamma \right\} \in \mathcal{I}_2^{\sigma\theta},$$

then the double sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalent of multiple L denoted by $C_{kj} \overset{W^L_{\mathcal{I}_2^{\sigma\theta}}(f)}{\sim} D_{kj}$

and if $L = 1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalent.

Definition 2. *If for every* $\gamma > 0$ *and each* $y \in Y$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f(|d(y; C_{kj}, D_{kj}) - L|) \geq \gamma \right\} \in \mathcal{I}_2^{\sigma\theta},$$

then the double sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be strongly f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalent of multiple L denoted by

$$C_{kj} \overset{[W^L_{\mathcal{I}_2^{\sigma\theta}}(f)]}{\sim} D_{kj}$$

and if $L = 1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be strongly f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalent.

Definition 3. *If for every $\varepsilon > 0$, each $\gamma > 0$ and each $y \in Y$,*

$$\{(r,u) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{ru}} |\{(k,j) \in I_{ru}: |d(y; C_{kj}, D_{kj}) - L| \geq \varepsilon\}| \geq \gamma\} \in \mathcal{I}_2^{\sigma\theta},$$

then the sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be asymptotically $\mathcal{I}_2^{\sigma\theta}$-statistical equivalent of multiple L denoted by $C_{kj} \overset{W^L_{\mathcal{I}_2^{\sigma}}(S)}{\sim} D_{kj}$

and if $L = 1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be asymptotically $\mathcal{I}_2^{\sigma\theta}$-statistical equivalent.

Theorem 1. *For each $y \in Y$, we have*

$$C_{kj} \overset{[W^L_{\mathcal{I}_2^{\sigma\theta}}(f)]}{\sim} D_{kj} \Rightarrow C_{kj} \overset{W^L_{\mathcal{I}_2^{\sigma\theta}}(S)}{\sim} D_{kj}.$$

Theorem 2. *If f is bounded, then for each $y \in Y$,*

$$C_{kj} \overset{[W^L_{\mathcal{I}_2^{\sigma\theta}}(f)]}{\sim} D_{kj} \Leftrightarrow C_{kj} \overset{W^L_{\mathcal{I}_2^{\sigma\theta}}(S)}{\sim} D_{kj}.$$

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