Basic Properties of Statistical Epi-Convergence

Abstract

In this paper, we give some basic properties in order to use statistical epi-convergence more efficiently in future studies. Such situations are studied: Uniform statistical convergence of sequence of functions, statistical epi-limit of compound of sequence of functions, statistical epi-limit of the sum of sequence of functions, the property of epi-limit function if the sequence of functions are lower semi-continuous and the convexity of epi-limit function if each function in the sequence is convex.

1. Introduction


Zygmund (1979) studied statistical convergence in 1935 for the first time. Then it is investigated by other mathematicians including Fast (1951), Steinhaus (1951) and Schoenberg (1959). The definitions of pointwise and uniform statistical convergence of real-valued functions were given by Gökhlan and Güngör (2002, 2005) and by Duman and Orhan (2004) independently. Statistical limit inferior and superior were studied by Fridy and Orhan (1997). Statistical limit points and cluster points were defined by Fridy (1993). Furthermore statistical lower and upper limits of closed sets were defined and characterized by Talo et al. (2016).

2. Preliminaries

In this part, fundamental definitions and theorems will be given. First of all, let \((X, d)\) be a metric space and \(f, (f_n)\) are functions defined on \(X\) with \(n \in \mathbb{N}\). If it is not mentioned explicitly the symbol \(d\) stands for the metric on \(X\).

Let \(K \subseteq \mathbb{N}\) and if the limit

\[
\delta(K) = \lim_{n \to \infty} \frac{1}{n} \{k \leq n : k \in K\}
\]
exists then it is called asymptotic density of $K$. 
$\lfloor k \leq n : k \in K \rfloor$ tells the number of elements of $K$ less than or equal to $n$ (Anastassiou and Duman 2011).

If $\delta(K_1) = \delta(K_2) = 1$ then,
$$
\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 1.
$$

If $\delta(K_1) = \delta(K_2) = 0$, then,
$$
\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 0.
$$

Let $(x_n)$ be a sequence of real numbers. If $\forall \varepsilon > 0, \exists x_0$ such that
$$
\lim_{k \to \infty} \left\lfloor n \leq k : |x_n - x_0| \geq \varepsilon \right\rfloor = 0,
$$
then $(x_n)$ is statistically convergent to $x_0$.

Let $n$ be a positive integer and $x = (x_n)$ be a sequence of real numbers. Define the sets $B_x$ and $A_x$ as
$$
B_x = \{ b \in \mathbb{R} : \delta(\{ n : x_n > b \}) \neq 0 \},
$$
$$
A_x = \{ a \in \mathbb{R} : \delta(\{ n : x_n < a \}) \neq 0 \}.
$$

Then statistical limit inferior and superior of $x = (x_n)$ is given by
$$
\text{st-lim inf } x = \begin{cases} 
\inf A_x & \text{if } A_x \neq \emptyset, \\
+\infty & \text{if } A_x = \emptyset.
\end{cases}
$$
$$
\text{st- lim sup } x = \begin{cases} 
\sup B_x & \text{if } B_x \neq \emptyset, \\
-\infty & \text{if } B_x = \emptyset.
\end{cases}
$$

For every $\varepsilon > 0$, a sequence of functions $(f_n)$ is uniformly statistically convergent to $f$ on a set $S$ if,
$$
\lim_{k \to \infty} \left\lfloor n \leq k : |f_n(x) - f(x)| \geq \varepsilon \text{ for all } x \in S \right\rfloor = 0.
$$

For a sequence of functions $f_n : X \to \mathbb{R}$, if it is statistically alpha convergent to a function $f$, then it is uniformly statistically convergent to $f$ (Caserta and Kočinac 2012).

Let $\sigma \in X$ and $(x_n)$ is a sequence. If there exists a set $K = \{ n_1 < n_2 < n_3 < \ldots \}$ with $\delta(K) \neq 0$ satisfying $x_{n_k} \to \sigma$ while $k \to \infty$, then $\sigma$ is a statistical limit point of $(x_n)$. Let $A_x$ denote the set of all statistical limit points of $(x_n)$.

Let $\mu \in X$ and $(x_n)$ is a sequence of real numbers. If for any $\varepsilon > 0$, $\mu$ is a statistical cluster point of $(x_n)$, then the following statement holds
$$
\delta(\{ n \in N : d(x_n, \mu) < \varepsilon \}) \neq 0.
$$

$\Gamma_x$ will denote the set of all statistical cluster points of $(x_n)$.

Let $\gamma \in X$ and $(x_n)$ is a sequence of real numbers. If there exists a set $K = \{ n_1 < n_2 < n_3 < \ldots \}$ satisfying $x_{n_k} \to \gamma$ while $k \to \infty$, then $\gamma$ is a limit point of $(x_n)$. The set of all limit points of $(x_n)$ will be denoted by $L_x$.

Obviously we have $\Lambda_x \subseteq \Gamma_x \subseteq L_x$.

Following definitions are statistical inner and outer limits on the concept of set convergence which is fundamental to define statistical epi-limit using sets. In this paper, we deal with Painlevé-Kuratowski (1958) convergence and actually its statistical version will be studied here which is defined by Talo et al. (2016). Now we start with the following collections of subsets of $N$.

$$
S^\#: = \{ N \subset N : \delta(N) \neq 0 \},
$$
$$
S = \{ N \subset N : \delta(N) = 1 \}.
$$

Let $(X, d)$ be a metric space. Statistical outer and inner limit of $(A_n)$ are defined in the following equalities:

$$
\text{st- lim inf } A_n = \{ x | \forall V \in N(x), \exists N \in S, \forall n \in N : A_n \cap V \neq \emptyset \}
$$
$$
= \{ x | \exists N \in S, \forall n \in N, \exists y_n \in A_n : \text{lim}_{n} y_n = x \}.
$$
$$
\text{st- lim sup } A_n = \{ x | \forall V \in N(x), \exists N \in S^\#, \forall n \in N : A_n \cap V \neq \emptyset \}
$$
$$
= \{ x | \exists N \in S^\#, \forall n \in N, \exists y_n \in A_n : x \in \Gamma_y \}.
$$
Let $f$ be a function defined on $X$, the epigraph of $f$ is the set $\text{epi} f := \{(x, \alpha) \in X \times \mathbb{R} | \alpha \geq f(x)\}$ and its level set is defined by

$$\text{lev}_{\alpha} f := \{x \in X | f(x) \leq \alpha\}.$$ 

Let $f_n: X \to \mathbb{R}$ be a sequence consisting of lower semicontinuous functions and $(X, d)$ a metric space. The lower statistical epi-limit, $e_{st} - \liminf_n f_n$ is defined by the help of the sequence of sets:

$$\text{epi}(e_{st} - \liminf_n f_n) = st - \limsup_n (\text{epi} f_n).$$

Similarly, the upper statistical epi-limit $e_{st} - \limsup_n f_n$ is defined by:

$$\text{epi}(e_{st} - \limsup_n f_n) = st - \liminf_n (\text{epi} f_n).$$

If we have the following equality, it is called statistical epi-convergence:

$$f = st - \lim_n f_n = e_{st} - \limsup_n f_n = e_{st} - \liminf_n f_n.$$ 

Following definition is a sequential characterization of epi-convergence.

For each $x \in X$ the sequence $f_n: X \to \mathbb{R}$ is epi-convergent to $f$, if and only if the following conditions hold.

(i) for all $x_n \in X$ whenever $(x_n)$ is convergent to $x$, we have $f(x) \leq \liminf_n f_n(x_n)$,

(ii) there exists a sequence $(x_n)$ convergent to $x$ such that $f(x) = \lim_n f_n(x_n)$

Let $\mathcal{G}(f)$ be the set of all lower semicontinuous functions denoted by $h$ on $X$ satisfying $h(y) \leq f(y)$ for every $y \in X$. For every function $f: X \to \mathbb{R}$, the lower semicontinuous envelope $sc^{-} f$ of $f$ is defined by

$$(sc^{-} f)(x) = \sup_{g \in \mathcal{G}(f)} g(x)$$

for every $x \in X$.

Let $f: X \to \mathbb{R}$ be a function. Then

$$(sc^{-} f)(x) = \sup_{y \in N(x), y \in U} \inf_{y \in U} f(y)$$

for every $x \in X$ where $N(x)$ is the neighbourhood of $x$.

More information about epi-convergence and statistical convergence we advise to look at papers in the reference part (Di Maio and Kočinac 2008, Rockafellar and Wets 2009, Šala’t 1980).

3. Main Result

**Theorem 3.1** Let $f_n: X \to \mathbb{R}$ be a sequence of functions. If $(f_n)$ is uniformly statistically convergent to $f$, then $(f_n)$ is statistically epi-convergent to $sc^{-} f$.

**Proof:** Assume that $(f_n)$ is uniformly statistically convergent to $f$. Then, for every $\varepsilon > 0$, there exists $K \in \mathcal{S}$ such that for all $n \in K$ and for all $y \in X$ we have $|f_n(y) - f(y)| < \varepsilon$. Hence,

$$f(y) - \varepsilon < f_n(y) < f(y) + \varepsilon.$$ 

Since uniform statistical convergence is independent of $y$, the following equality is valid for an open set $U \subseteq X$ and all $n \in K$.

$$\inf_{y \in U} f(y) - \varepsilon < \inf_{y \in U} f_n(y) < \inf_{y \in U} f(y) + \varepsilon.$$ 

Then we have

$$st - \liminf_n f_n(y) = \inf_{y \in U} f(y),$$

hence for every $x \in X$

$$\sup_{y \in U} st - \liminf_n f_n(y) = \sup_{y \in U} \inf_{y \in U} f(y),$$

which implies that $(f_n)$ is statistically epi-convergent to $sc^{-} f$.

In statistical pointwise convergence, $\varepsilon$ is dependent on every point $x \in X$ hence it gives us an idea about why statistical pointwise convergence and statistical epi-convergence do not coincide in general.

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Remark 3.2 Let each function \( f_n : X \to \mathbb{R} \) be lower semicontinuous. If \( (f_n) \) statistically uniformly converges to \( f \), then \( f \) is lower semicontinuous and \( (f_n) \) statistically epi-converges to \( f \).

Theorem 3.3 Let \( f_n : X \to \mathbb{R} \) be a sequence of functions and \( g : \mathbb{R} \to \mathbb{R} \) be a continuous and increasing function. Then

\[
\begin{align*}
\varepsilon_n - \liminf (g \circ f_n) &= g \circ (\varepsilon_n - \liminf f_n), \\
\varepsilon_n - \limsup (g \circ f_n) &= g \circ (\varepsilon_n - \limsup f_n).
\end{align*}
\]

(1)

(2)

Proof: As we know, \( g \) is a continuous and increasing function, then we have

\[ g(\inf S) = \inf g(S) \quad \text{and} \quad g(\sup S) = \sup g(S) \]

for each subset \( S \) of \( \mathbb{R} \). Since

\[
\varepsilon_n - \liminf f_n(x) = \sup_{y \in U} \sup_{n \in \mathbb{N}} \inf_{y \in U} g(f_n(y)).
\]

Hence the equation can be rewritten as

\[
\sup_{y \in U} \sup_{n \in \mathbb{N}} \inf_{y \in U} g(f_n(y)) = g(\sup_{y \in U} \sup_{n \in \mathbb{N}} \inf_{y \in U} g(f_n(y))).
\]

It gives the proof of (1). The proof of (2) is analogous to the previous one.

Theorem 3.4 If \( f_n : X \to \mathbb{R} \) and \( g_n : X \to \mathbb{R} \) be sequences of functions and their sum is well defined, then the following inequalities are valid.

\[
\begin{align*}
\varepsilon_n - \liminf (f_n + g_n) &\geq \varepsilon_n - \liminf f_n + \varepsilon_n - \liminf g_n, \\
\varepsilon_n - \limsup (f_n + g_n) &\geq \varepsilon_n - \limsup f_n + \varepsilon_n - \liminf g_n.
\end{align*}
\]

(3)

(4)

Proof: First, we apply some additional restrictions for \( (f_n) \) and \( (g_n) \). \( \exists \alpha \in \mathbb{R} \) such that \( f_n \leq \alpha \) and \( g_n \leq \alpha \) on \( X \) for every \( n \in \mathbb{N} \). In our operations, all sums have become well defined. Let \( U \in X \) be an open set. For every \( U \),

\[
\inf_{y \in U} (f_n + g_n)(y) \geq \inf_{y \in U} f_n(y) + \inf_{y \in U} g_n(y).
\]

Hence, by using properties of statistical upper and lower limits, we get

\[
\begin{align*}
st - \limsup_{n \to \infty} &\inf_{y \in U} (f_n + g_n)(y) \\
&\geq \inf_{y \in U} f_n(y) + \inf_{y \in U} g_n(y),
\end{align*}
\]

(5)

Now fix \( x \in X \). If

\[
\varepsilon_n - \limsup f_n(x) + \varepsilon_n - \liminf g_n(x) = -\infty
\]

then we are done. Otherwise, for each \( \varepsilon > 0 \) there exists \( V, W \in \mathcal{N}(x) \) such that

\[
\begin{align*}
\varepsilon_n - \limsup f_n(x) - \varepsilon &< \inf_{y \in U} f_n(y), \\
\varepsilon_n - \liminf g_n(x) - \varepsilon &< \inf_{y \in U} g_n(y).
\end{align*}
\]

(6)

(7)

Let \( U = V \cap W \). Since \( U \in \mathcal{N}(x) \) and

\[
\inf_{y \in W} f_n(y) \leq \inf_{y \in U} f_n(y), \quad \inf_{y \in W} g_n(y) \leq \inf_{y \in U} g_n(y).
\]

By using definition of statistical upper epi-limit, (5), (6) and (7) we obtain

\[
\begin{align*}
\left( \varepsilon_n - \limsup f_n(x) + \varepsilon_n - \liminf g_n \right) (x) &\geq st - \limsup_{n \to \infty} \inf_{y \in U} (f_n + g_n)(y) \\
&\geq (\varepsilon_n - \limsup f_n)(x) + \\
&\left( \varepsilon_n - \liminf g_n \right) (x) - 2\varepsilon.
\end{align*}
\]

\( \varepsilon \) was arbitrary, hence the proof is completed.

Now we deal with the general case. Assume that the sequences \( (f_n) \) and \( (g_n) \) are not restricted from above. Let us define a function \( h_\alpha : \mathbb{R} \to \mathbb{R} \) as
\[ h_a(t) = \min\{t, a\} \text{ for every } a \in \mathbb{R}. \] For every \( n \in \mathbb{N} \) we know
\[ h_a \circ f_n \leq a \text{ and } h_a \circ g_n \leq a \]
on \( X \) from previous part of the proof we get
\[ e_{st} = \limsup \left( (h_a \circ f_n) + (h_a \circ g_n) \right) \]
\[ \geq e_{st} - \limsup (h_a \circ f_n) + e_{st} - \liminf (h_a \circ g_n). \]
Theorem 3.3 implies that
\[ e_{st} - \limsup (f_n + g_n) \]
\[ \geq e_{st} - \limsup ((h_a \circ f_n) + (h_a \circ g_n)) \]
\[ \geq h_a \circ (e_{st} - \limsup f_n) + h_a \circ (e_{st} - \limsup g_n). \]

When taking \( a \to \infty \), the proof for the case of unboundedness is completed.

Even if \( (f_n) \) and \( (g_n) \) are statistically epi-convergent, the inequalities (3) and (4) can be strict. This situation can be seen in the following example.

**Example 3.5** Let \( (f_n) \) and \( (g_n) \) be real valued functions defined on \( \mathbb{R} \) as,
\[ f_n(x) = \begin{cases} 
-2 & \text{if } n \text{ is even square}, \\
\sin(nx) & \text{if otherwise}.
\end{cases} \]
\[ g_n(x) = \begin{cases} 
-2 & \text{if } n \text{ is odd square}, \\
-\sin(nx) & \text{if otherwise}.
\end{cases} \]
Then \( (f_n) \) and \( (g_n) \) are statistically epi-convergent to \( h(x) = -1 \) while \( (f_n + g_n) \) is statistically epi-convergent to \( h(x) = 0 \).

A sequence \( f_n : X \to \mathbb{R} \) is statistically \( \alpha \)-convergent to \( f \) if for every \( x \in X \) and every sequence \( (x_n) \) in \( X \) converging to \( x \), the sequence \( f_n(x_n) \) statistically converges to \( f(x) \). By Theorem 3.5 by (Caserta and Kočinac 2012), we know statistical \( \alpha \)-convergence implies statistical uniform convergence. Also, we proved in Theorem 3.1 that statistical epi-convergence is implied by statistical uniform convergence. Hence we will use it in the following Corollary.

**Corollary 3.6** Assume that \( f_n : X \to \mathbb{R} \) and \( g_n : X \to \mathbb{R} \) are sequences of functions. If \( (g_n) \) is statistically \( \alpha \)-convergent to a function \( g \) provided that \( (g_n) \) and \( g \) are finite, then the following equalities hold.
\[ e_{st} - \liminf \left( f_n + g_n \right) = e_{st} - \liminf f_n + g, \] (8)
\[ e_{st} - \limsup \left( f_n + g_n \right) = e_{st} - \limsup f_n + g. \] (9)

**Proof:** We shall prove only (9), the other one being analogous. First of all, we know that if \( g_n \xrightarrow{st-\alpha} g \) then \( g \) is continuous and \( g_n \xrightarrow{st-\alpha} g \). Hence by Theorem 3.1 we have
\[ g_n \xrightarrow{e_{st}} g. \] (10)

From now on, we continue by using Theorem 3.4 and we get
\[ e_{st} - \liminf \left( f_n + g_n \right) \geq e_{st} - \liminf f_n + g. \] (11)
On the other hand, \((-g_n)\) is statistically epi-convergent to \(-g\) in \( X \) and by Theorem 3.4 we have
\[ e_{st} - \limsup f_n = e_{st} - \limsup (f_n + g_n - g_n) \]
\[ \geq e_{st} - \limsup (f_n + g_n) - g. \]

Hence,
\[ e_{st} - \liminf f_n + g \geq e_{st} - \liminf (f_n + g_n). \] (12)

Equality (9) follows from (11) and (12).

**Corollary 3.7** Let \( f_n : X \to \mathbb{R} \) be a sequence of functions. Suppose that \( g : X \to \mathbb{R} \) is a continuous function. Then the following equalities hold.
\[ e_{st} - \liminf \left( f_n + g \right) = e_{st} - \liminf f_n + g, \]
\[ e_{st} - \limsup_n (f_n + g) = e_{st} - \limsup_n f_n + g. \]

(14)

**Proof:** The sequence \((g_n)\) is statistically alpha convergent to \(g\), since \(g\) is a continuous function. Then the result follows by using Corollary 3.6.

Continuity of \(g\) is essential in Corollary 3.7, as the following example shows.

**Example 3.8** Let \((f_n)\) and \(g\) be real valued functions defined on \(\mathbb{R}\) as,

\[ f_n(x) = \begin{cases} 2nx e^{-2n^2 x^2} & \text{if } n \text{ is square}, \\ nx e^{-2n^2 x^2} & \text{if otherwise} \end{cases} \]

and \(g(x) = \begin{cases} 1 & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \)

The function \(g\) is lower semicontinuous and each of \(f_n\) is continuous. \((f_n)\) statistically epi-converges to \(-\frac{1}{2}e^{-12}\) while \((f_n + g)\) statistically epi-converges to \(1 - \frac{1}{2}e^{-12}\) at the point 0 where \(g\) is not continuous.

**Corollary 3.9** Let \((f_n)\) and \((g_n)\) be functions from \(X\) to \(\overline{\mathbb{R}}\). Suppose that \((f_n)\) is statistically epi-convergent and statistically pointwise convergent to \(f\) and \((g_n)\) is statistically epi-convergent and statistically pointwise convergent to \(g\). Then \((f_n + g_n)\) is statistically epi-convergent and statistically pointwise convergent to \((f + g)\), provided that the functions \((f_n + g_n)\) and \((f + g)\) are well defined on \(X\).

**Proof:** By Theorem 3.4 we have

\[ f + g = e_{st} - \liminf_n f_n + e_{st} - \liminf_n g_n \leq e_{st} - \liminf_n (f_n + g_n) \leq e_{st} - \limsup_n (f_n + g_n) \leq st - \limsup_n (f_n + g_n) = f + g. \]

**Theorem 3.10** For any sequence \((f_n)\) of convex functions on \(X\), the function \(e_{st} - \limsup_n f_n\) is convex.

**Proof:** Since each \(f_n\) is convex function on \(X\), each of \(epi f_n\) is convex set. Let \(x, y \in st - \liminf_n (epi f_n)\), then \(\exists x_n \in epi f_n\) such that \(x_n \xrightarrow{st} x, \forall n \in N\) with \(N \in S\). Similarly there exists a sequence \(y_n \in epi f_n\) such that for all \(n \in K\) with \(K \in S\), \(y_n \xrightarrow{st} y\). Let \(W = N \cap K\) that is \(\delta(W) = 1\). For arbitrary \(\lambda \in [0,1]\), define \(z^n_\lambda := (1 - \lambda)x_n + \lambda y_n\) and \(z^\lambda := (1 - \lambda)x + \lambda y\), then we have \(z^n_\lambda \in epi f_n\) and \(z_n \xrightarrow{st} z\) for all \(n \in W\), hence \(z^\lambda \in st - \liminf_n (epi f_n)\) and proves the convexity of this set. Consequently, \(e_{st} - \limsup_n f_n\) is convex.

Following example shows that \(e_{st} - \liminf_n f_n\) function need not be convex.

**Example 3.11** Let \(f_n : \mathbb{R} \to \mathbb{R}\) be defined as \(f_n = (x + (-1)^n)^2\). Indeed, \(f = e_{st} - \liminf_n f_n\) function is

\[ f(x) = \begin{cases} (x + 1)^2 & \text{if } x \leq 0, \\ (x - 1)^2 & \text{if } x > 0 \end{cases} \]

which is not convex.

**References**


Basic Properties of Statistical Epi-convergence, Tortop


