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Abstract

In this paper, we study the concepts of Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-convergence and Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-Cauchy double sequences of sets and investigate the relationships among them.

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1 Introduction

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [10] and Schoenberg [23]. This concept was extended to the double sequences by Mursaleen and Edely [16]. Fridy and Orhan [12] have introduced the concepts of statistical limit superior and statistical limit inferior. Çakan and Altay [7] presented multidimensional analogues of the results presented by Fridy and Orhan [12].

Nuray and Ruckle [19] indepedently introduced the same with another name generalized statistical convergence. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko, Šalát and Wilczyński [14] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Das, Kostyrko, Wilczyński and Malik [8] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of development have been made in this area after the works of [9, 15, 17].

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, [3, 4, 5, 18, 25, 26]). Nuray and Rhoades [18] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [24] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wiijsman statistical convergence, which was defined by Nuray and Rhoades. Nuray et al. [20] studied Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigate the relationships between them. Kişi and Nuray [13] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence.

In this paper, we study the concepts of Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-convergence and Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-Cauchy double sequences of sets and investigate the relationships among them.

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2 Definitions and Notations

Now, we recall the basic definitions and concepts (See [1, 2, 3, 4, 5, 8, 9, 14, 18, 21, 25, 26]). For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

We let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X that use following. We say that the sequence $\{A_k\}$ is Wijsman convergent to A if $\lim_{k\to\infty} d(x, A_k) = d(x, A)$, for each $x \in X$. In this case we write $W - \lim A_k = A$. We say that the sequence $\{A_k\}$ is Wijsman Cauchy sequence, if for $\varepsilon > 0$ and for each $x \in X$, there is a positive integer k_0 such that for all $m, n > k_0$, $|d(x, A_m) - d(x, A_n)| < \varepsilon$. A double sequence $x = (x_{kj})_{k,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case we write $P - \lim_{k,j\to\infty} x_{kj} = L \quad or \quad \lim_{k,j\to\infty} x_{kj} = L$.

Throughout the paper, we let A, A_{kj} be any non-empty closed subsets of X. The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A) \quad or \quad \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$$

for each $x \in X$. In this case we write $W_2 - \lim A_{kj} = A$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided: (i) $\emptyset \in \mathcal{I}$, (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$. $\mathcal{I}$ is called nontrivial ideal if $X \notin \mathcal{I}$. Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided: (i) $\emptyset \notin \mathcal{F}$, (ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, (iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1 [14] If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$ is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$. Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$. A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$. It is evident that a strongly admissible ideal is admissible also.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (X, ρ) be a metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2$. In this case we say that x is $\mathcal{I}_2$-convergent and we write $\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$.

Let (X, ρ) be a metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2^*$- convergent to $L \in X$ if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M_2 \in \mathcal{I}_2$) such that $\lim_{m,n\to\infty} x_{mn} = L$, for $(m, n) \in M_2$ and we write $\mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L$.

Let (X, ρ) be a metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2$-Cauchy if for every $\varepsilon > 0$ there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \in \mathcal{I}_2$.

Let (X, ρ) be a metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2^*$-Cauchy sequence if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N} \times \mathbb{N} \backslash M_2 \in \mathcal{I}_2$) such that for every $\varepsilon > 0$ and for $(m, n), (s, t) \in M_2$, $m, n, s, t > k_0 = k_0(\varepsilon)$ $\rho(x_{mn}, x_{st}) < \varepsilon$. In this case we write $\lim_{m,n,s,t\to\infty} \rho(x_{mn}, x_{st}) = 0$.

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Throughout the paper, we let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal, (X, ρ) be a separable metric space and A, A_k be any non-empty closed subsets of X. We say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$ the set $A(x, \varepsilon) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \geq \varepsilon\}$ belongs to $\mathcal{I}$. In this case we write $\mathcal{I}_W - \lim A_k = A$ or $A_k \to A(\mathcal{I}_W)$. We say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-convergent to A, if and only if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that for each $x \in X$ $\lim_{k\to\infty} d(x, A_{m_k}) = d(x, A)$. In this case, we write $\mathcal{I}_W^* - \lim A_k = A$. We say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-Cauchy sequence if for each ε and for each $x \in X$, there exists a number $N = N(\varepsilon)$ such that $\{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\} \in \mathcal{I}$. We say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}^*$-Cauchy sequence if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m = (m_i) : m_i < m_{i+1}, i \in \mathbb{N}\} \subset \mathbb{N}$ such that the subsequence $A_M = \{A_{m_k}\}$ is Wijsman Cauchy in X that is, $\lim_{k,p\to\infty} \left| d(x, A_{m_k}) - d\left(x, A_{m_p}\right) \right| = 0$. The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A) \quad or \quad \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$$

for each $x \in X$. In this case we write $W_2 - \lim A_{kj} = A$.

3 Main Results

Throughout the paper, we let (X,ρ) be a separable metric space, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and A, A_{kj} be any non-empty closed subsets of X.

Definition 3.1 We say that a double sequence of sets $\{A_{kj}\}$ *is* $\mathcal{I}_{W_2}$-convergent to A, if for every $x \in X$ and for every $\varepsilon > 0$,

$$\{(k,j) \in \mathbb{N}\times\mathbb{N}: |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we write $\mathcal{I}_{W_2} - \lim_{k,j\to\infty} d(x,A_{kj}) = d(x,A)$.

Definition 3.2 We say that the double sequence of sets $\{A_{kj}\}$ *is* $\mathcal{I}^*_{W_2}$-convergent to A, if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ *(i.e.,* $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2$*)* such that for every $x \in X$

$$\lim_{\substack{k,j\to\infty\\(k,j)\in M_2}} d(x,A_{kj}) = d(x,A).$$

In this case we write $\mathcal{I}^*_{W_2} - \lim_{k,j\to\infty} d(x,A_{kj}) = d(x,A)$.

Theorem 3.1 $\mathcal{I}^*_{W_2}$-convergence implies $\mathcal{I}_{W_2}$-convergence for double sequence of sets.

Proof. Since $\mathcal{I}^*_{W_2} - \lim_{k,j\to\infty} d(x,A_{kj}) = d(x,A)$, so there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2$) such that for each $x \in X$

$$\lim_{\substack{k,j\to\infty\\(k,j)\in M_2}} d(x,A_{kj}) = d(x,A).$$

Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that for each $x \in X$, $|d(x,A_{kj}) - d(x,A)| < \varepsilon$ for all $(k,j) \in M_2$ and $k,j \geq k_0$. Then for each $\varepsilon > 0$ and $x \in X$, we have

$$\begin{aligned} T(\varepsilon,x) &= \{(k,j) \in \mathbb{N}\times\mathbb{N}: |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\} \\ &\subset H \cup (M_2 \cap ((\{1,2,\ldots,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,\ldots,(k_0-1)\}))). \end{aligned}$$

Since

$$H \cup (M_2 \cap ((\{1,2,\ldots,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,\ldots,(k_0-1)\}))) \in \mathcal{I}_2,$$

so we have $T(\varepsilon,x) \in \mathcal{I}_2$. Hence, $\mathcal{I}_{W_2} - \lim_{k,j\to\infty} d(x,A_{kj}) = d(x,A)$.

Theorem 3.2 If the idea*l* $\mathcal{I}_2$ has the property $(AP2)$*,* then $\mathcal{I}_{W_2}$-convergence implies $\mathcal{I}^*_{W_2}$-convergence for double sequence of sets.

Proof. Suppose that $\mathcal{I}_2$ satisfies property $(AP2)$. Let $\mathcal{I}_{W_2} - \lim_{k,j\to\infty} d(x,A_{kj}) = d(x,A)$. Then

$$T(\varepsilon,x) = T_\varepsilon = \{(k,j) \in \mathbb{N}\times\mathbb{N}: |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\} \in \mathcal{I}_2 \tag{1}$$

for each $\varepsilon > 0$ and for each $x \in X$. Put

$$T_1 = T(1,x) = \{(k,j) \in \mathbb{N}\times\mathbb{N}: |d(x,A_{kj}) - d(x,A)| \geq 1\}$$

and

$$T_k = T(k,x) = \left\{(k,j) \in \mathbb{N}\times\mathbb{N}: \frac{1}{k} \leq |d(x,A_{kj}) - d(x,A)| < \frac{1}{k-1}\right\}$$

for $k \geq 2$ and $k \in \mathbb{N}$. Obviously, $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exits a sequence of sets $\{V_k\}_{k\in\mathbb{N}}$ such that $T_j \Delta V_j$ is included in finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each j and $V = \bigcup_{j=1}^{\infty} V_j \in \mathcal{I}_2$. We shall prove that for $M_2 = \mathbb{N}\times\mathbb{N}\backslash V$ we have

$$\lim_{\substack{k,j\to\infty\\(k,j)\in M_2}} d(x,A_{kj}) = d(x,A).$$

Let $\eta > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \eta$. Then

$$\{(k,j) \in \mathbb{N}\times\mathbb{N}: |d(x,A_{kj}) - d(x,A)| \geq \eta\} \subset \bigcup_{j=1}^{k} T_j.$$

Since, $T_j \Delta V_j$, $j = 1,2,\ldots$ are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{k} T_j\right) \cap \{(k,j): k \geq n_0 \wedge j \geq n_0\} = \left(\bigcup_{j=1}^{k} V_j\right) \cap \{(k,j): k \geq n_0 \wedge j \geq n_0\}. \tag{2}$$

If $k,j > n_0$ and $(k,j) \notin V$, then $(k,j) \notin \bigcup_{j=1}^{k} V_j$ and $(k,j) \notin \bigcup_{j=1}^{k} T_j$. This implies that $\left|d\left(x,A_{kj}\right) - d(x,A)\right| < \frac{1}{k} < \eta$. Hence, we have

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} d(x,A_{kj}) = d(x,A).$$

Definition 3.3 We say that the double sequence $\{A_{kj}\}$ is Wijsman Cauchy, if for each $\varepsilon > 0$ and for each $x \in X$, there is positive integers (p,q) such that for all $(m,n) > (p,q)$ we have

$$|d(x,A_{kj}) - d(x,A_{mn})| < \varepsilon.$$

Definition 3.4 We say that the double set sequence $\{A_{kj}\}$ is $\mathcal{I}_2$-Cauchy sequence in Pringsheim's sense if for every $x \in X$ and for every $\varepsilon > 0$, there exists (p,q) in $\mathbb{N}\times\mathbb{N}$ such that

$$\left\{(k,j) \in \mathbb{N}\times\mathbb{N} : \left|d(x,A_{kj}) - d(x,A_{pq})\right| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Theorem 3.3 A double sequence of sets $\{A_{kj}\}$ if $\mathcal{I}_{W_2}$-convergent then it is $\mathcal{I}_{W_2}$-Cauchy.

Proof. Let $\mathcal{I}_{W_2} - \lim A_{kj} = A$. Then for each $\varepsilon > 0$ and for each $x \in X$, we have

$$A(x,\varepsilon) = \left\{(k,j) \in \mathbb{N}\times\mathbb{N} : \left|d\left(x,A_{kj}\right) - d(x,A)\right| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Since $\mathcal{I}_2$ is a strongly admissible ideal, there exists an $p,q \in \mathbb{N}$ such that $(p,q) \notin A(x,\varepsilon)$. Let

$$B(x,\varepsilon) = \left\{(k,j) \in \mathbb{N}\times\mathbb{N} : \left|d\left(x,A_{kj}\right) - d\left(x,A_{pq}\right)\right| \geq 2\varepsilon\right\}.$$

Taking into account the inequality

$$\left|d\left(x,A_{kj}\right) - d\left(x,A_{pq}\right)\right| \leq \left|d\left(x,A_{kj}\right) - d(x,A)\right| + \left|d\left(x,A_{pq}\right) - d(x,A)\right|;$$

we observe that if $(k,j) \in B(x,\varepsilon)$ then $\left|d\left(x,A_{kj}\right) - d(x,A)\right| + \left|d\left(x,A_{pq}\right) - d(x,A)\right| \geq 2\varepsilon$.

On the other hand, since $(k,j) \notin A(x,\varepsilon)$ we have $\left|d\left(x,A_{pq}\right) - d(x,A)\right| < \varepsilon$.

Here we conclude that

$$\left|d\left(x,A_{kj}\right) - d(x,A)\right| \geq \varepsilon,$$

hence $(k,j) \in A(x,\varepsilon)$. Observe that

$$B(x,\varepsilon) \subset A(x,\varepsilon) \in \mathcal{I}_2$$

for each $\varepsilon > 0$ and for each $x \in X$. This gives that $B(x,\varepsilon) \in \mathcal{I}_2$ that is $\left\{A_{kj}\right\}$ is Wijsman $\mathcal{I}_2$-Cauchy double sequence.

Definition 3.5 We say that the double sequence of sets $\{A_{kj}\}$ *is* $\mathcal{I}^*_{W_2}$-Cauchy, if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ *(i.e.,* $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2$*)* such that for every $x \in X$ and $(k,j),(p,q) \in M_2$

$$\lim_{k,j,p,q\to\infty} |d(x,A_{kj}) - d(x,A_{pq})| = 0.$$

Theorem 3.4 A double sequence of sets $\{A_{kj}\}$ if $\mathcal{I}^*_{W_2}$-Cauchy then it is $\mathcal{I}_{W_2}$-Cauchy.

Proof. Let $\left\{A_{kj}\right\}$ is Wijsman $\mathcal{I}^*_{W_2}$-Cauchy sequence then by the definition, there exits a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2$) such that for each $\varepsilon > 0$ and for each $x \in X$,

$$\left|d\left(x,A_{kj}\right) - d\left(x,A_{pq}\right)\right| < \varepsilon,$$

for all $(k,j),(p,q) \in M_2$, $k,j,p,q > N = N(x,\varepsilon)$ and $N \in \mathbb{N}$. Then, for each $\varepsilon > 0$ and $x \in X$, we have

$$A(\varepsilon,x) = \left\{(k,j) \in \mathbb{N}\times\mathbb{N} : \left|d\left(x,A_{kj}\right) - d\left(x,A_{pq}\right)\right| \geq \varepsilon\right\}$$
$$\subset H \cup (M_2 \cap ((\{1,2,\ldots,(N-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,\ldots,(N-1)\}))).$$

Since

$$H \cup (M_2 \cap ((\{1,2,\ldots,(N-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,\ldots,(N-1)\}))) \in \mathcal{I}_2,$$

so we have $A(\varepsilon,x) \in \mathcal{I}_2$. Hence, $\{A_{kj}\}$ is $\mathcal{I}_{W_2}$-Cauchy double sequence.

Theorem 3.5 A double sequence of sets $\{A_{kj}\}$ if $\mathcal{I}^*_{W_2}$-convergent, then it is $\mathcal{I}_{W_2}$-Cauchy.

Proof. Let $\mathcal{I}^*_{W_2} - \lim_{k,j\to\infty} d(x,A_{kj}) = d(x,A)$, so there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2$) such that for each $x \in X$

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} d(x,A_{kj}) = d(x,A).$$

Let $\varepsilon > 0$. Then there exists $k_0 \in \mathbb{N}$ such that for each $x \in X$,

$$|d(x,A_{kj}) - d(x,A)| < \frac{\varepsilon}{2},$$

for all $(k,j) \in M_2$ and $k,j \geq k_0$. Then for each $\varepsilon > 0$ and $x \in X$, we have

$$\left|d\left(x, A_{kj}\right) - d\left(x, A_{pq}\right)\right| < \left|d\left(x, A_{kj}\right) - d(x, A)\right| + \left|d\left(x, A_{pq}\right) - d(x, A)\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, for each $x \in X$ and $(k,j), (p,q) \in M_2$ we have

$$\lim_{k,j,p,q\to\infty} |d(x, A_{kj}) - d(x, A_{pq})| = 0.$$

Hence, $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{*}$-Cauchy double sequence and so by Theorem 3.4 $\{A_{kj}\}$ is $\mathcal{I}_{W_2}$-Cauchy double sequence.

References

[1] *B. Altay, F. Başar, Some new spaces of double sequences, J. Math. Anal. Appl., 309 (1) (2005), pp. 70–90.*

[2] *J.-P. Aubin, and H. Frankowska, Set-valued analysis, Birkhauser, Boston, (1990).*

[3] *M. Baronti, and P. Papini, Convergence of sequences of sets, In: Methods of functional analysis in approximation theory, ISNM 76, Birkhauser-Verlag, Basel, (1986), pp. 133-155.*

[4] *G. Beer, On convergence of closed sets in a metric space and distance functions, Bull. Aust. Math. Soc., 31 (1985), pp. 421–432.*

[5] *G. Beer, Wijsman convergence: A survey, Set-Valued Var. Anal. 2 (1994), pp. 77–94.*

[6] *J.S. Connor, The statistical and strong p-Cesàro convergence of sequences, Analysis, 8 (1988), pp. 46–63.*

[7] *C. Çakan, B. Altay, Statistically boundedness and statistical core of double sequences, J. Math. Anal. Appl. 317 (2006), pp. 690–697.*

[8] *P. Das, P. Kostyrko, W. Wilczyński, P. Malik, I and I^*-convergence of double sequences, Math. Slovaca, 58 (5) (2008), pp. 605–620.*

[9] *E. Dündar, B. Altay, $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences, Acta Mathematica Scientia 34B(2) (2014), pp. 343–353.*

[10] *H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951), pp. 241–244.*

[11] *A.R. Freedman, J.J. Sember, M. Raphael, Some Cesàro type summability spaces, Proc. London Math. Soc., 37 (1978), pp. 508–520.*

[12] *J. A. Fridy, C. Orhan, Statistical limit superior and inferior, Proc. Amer. Math. Soc., 125 (1997), pp. 3625–3631.*

[13] *Kişi, Ö. and Nuray, F. A new convergence for sequences of sets, Abstr. Appl. Anal., (2013), Article ID 852796, 6 pages.*

[14] *P. Kostyrko, T. Šalát and W. Wilczyński, I-convergence, Real Anal. Exchange, 26 (2) (2000), pp. 669-686.*

[15] *V. Kumar, On I and I^*-convergence of double sequences, Math. Commun., 12 (2007), pp. 171–181.*

[16] *Mursaleen, O. H. H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288 (2003), pp. 223–231.*

[17] *A. Nabiev, S. Pehlivan, M. Gürdal, On I-Cauchy sequence, Taiwanese J. Math., 11 (2) (2007), pp. 569–576.*

[18] *F. Nuray, B. E. Rhoades, Statistical convergence of sequences of sets, Fasc. Math., 49 (2012), pp. 87–99.*

[19] *F. Nuray, W.H. Ruckle, Generalized statistical convergence and convergence free spaces, J. Math. Anal. Appl., 245 (2000), pp. 513–527.*

[20] *F. Nuray, E. Dündar, U. Ulusu, Wijsman statistical convergence of double sequences of sets, (communicated).*

[21] *A. Pringsheim, Zur theorie der zweifach unendlichen Zahlenfolgen, Math. Ann., 53 (1900), pp. 289–321.*

[22] *R.T. Rockafellar and R.J-B Wets, Variational Analysis, Grundlehren der Mathematischen Wissenschaften 317, Springer-Verlag 2009.*

[23] *I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66 (1959), pp. 361–375.*

[24] *U. Ulusu, F. Nuray, Lacunary statistical convergence of sequence of sets, Progress in Applied Mathematics, 4(2) (2012), pp. 99–109.*

[25] *R. A. Wijsman, Convergence of sequences of convex sets, cones and functions, Bull. Amer. Math. Soc., 70 (1964), pp. 186–188.*

[26] *R. A. Wijsman, Convergence of Sequences of Convex sets, Cones and Functions II, Trans. Amer. Math. Soc., 123 (1) (1966), pp. 32–45.*