ON IDEAL CONVERGENCE AND IDEAL CAUCHY SEQUENCES OF FUNCTIONS IN 2-NORMED SPACES AND SOME PROPERTIES

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Abstract. In this paper, we study concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions and investigate relationships between them and some properties in 2-normed spaces.

1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [7] and Schoenberg [26].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [19] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$ [7, 8]. Nabiev et al. [22] studied $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequence, and then study their certain properties. Gökhan et al. [12] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued functions. Gezer and Karakuş [11] investigated $\mathcal{I}$-pointwise and uniform convergence and $\mathcal{I}^*$-pointwise and uniform convergence of function sequences and they examined the relation between them. Baláz et al. [1] investigated $\mathcal{I}$-convergence and $\mathcal{I}$-continuity of real functions. Balcerzak et al. [2] studied statistical convergence and ideal convergence for sequences of functions Dündar and Altay [4, 5] studied the concepts of pointwise and uniformly $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [6] investigated some results of $\mathcal{I}_2$-convergence of double sequences of functions.

The concept of 2-normed spaces was initially introduced by Gähler [9, 10] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [16] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Şahiner et al. [28] and Gürdal [18] studied $\mathcal{I}$-convergence in 2-normed spaces. Gürdal and Açık [17] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [24] presented various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathcal{I}$-equistatistically convergence and study $\mathcal{I}$-equistatistically convergence of sequences of functions. Recently, Savaş and Gürdal [25] concerned with $\mathcal{I}$-convergence of sequences of functions in random 2-normed spaces and introduce the concepts of ideal uniform convergence and ideal pointwise convergence in the topology induced by random 2-normed spaces, and gave some basic properties of these concepts. Yegül and Dündar [30] studied statistical convergence of sequence of functions in 2-normed spaces. A lot of development have been made in this area after the works of [3, 20, 21, 23, 27, 29].

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2. Definitions and Notations

Now, we recall the concept of 2-normed space, ideal convergence and some fundamental definitions and notations (See [1, 2, 7, 8, 13, 14, 15, 16, 17, 18, 19, 24, 28]).

If $K \subseteq \mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $\delta(K) = \lim_n \frac{1}{n}|K_n|$, if it exists. The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set

$$K = K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has natural density zero; in this case, we write $st - \lim x = L$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1 ([19])**.** *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$, for each $x \in X$.

Example 2.1. *Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then, $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ convergence is the usual convergence.*

Throughout the paper, we let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.

A sequence (f_n) of functions is said to be $\mathcal{I}$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if for every $\varepsilon > 0$ and each $x \in D$,

$$\{n : |f_n(x) - f(x) \geq \varepsilon|\} \in \mathcal{I}.$$

In this case, we will write $f_n \xrightarrow{\mathcal{I}} f$ on D.

A sequence (f_n) of functions is said to be $\mathcal{I}^*$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if $\forall \varepsilon > 0$ and $\forall x \in D$, $\exists K_x \notin \mathcal{I}$ and $\exists n_0 = n_0(\varepsilon, x) \in K_x : \forall n \geq n_0$ and $n \in K_x$, $|f_n(x) - f(x)| < \varepsilon$.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot, \cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x, y\| = \|y, x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot, \cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if

$$\lim_{n\to\infty} \|x_n - L, y\| = 0$$

for every $y \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ and each nonzero $z \in X$,

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim_{n\to\infty} \|x_n - L, z\| = 0$ or $\mathcal{I} - \lim_{n\to\infty} \|x_n, z\| = \|L, z\|$.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathcal{F}$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $\lim_{n\to\infty} \|x_{m_k} - L, z\| = 0$, for each nonzero $z \in X$.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}$-Cauchy sequence in X, if for each $\varepsilon > 0$ and nonzero $z \in X$ there exists a number $N = N(\varepsilon, z)$ such that

$$\{k \in \mathbb{N} : \|x_k - x_N, z\| \geq \varepsilon\} \in \mathcal{I}.$$

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}^*$-Cauchy sequence in X, if there exists a set $M \in \mathcal{F}$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that the subsequence $x_M = (x_{m_k})$ is a Cauchy sequence on X, i.e.,

$$\lim_{k,p\to\infty} \|x_{m_k} - x_{m_p}, z\| = 0, \text{ for each nonzero } z \in X.$$

Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. $\{f_n\}$ is said to be convergent to f if $f_n(x) \overset{\|.,.\|_Y}{\longrightarrow} f(x)$ for each $x \in X$. We write $f_n \overset{\|.,.\|_Y}{\longrightarrow} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x), z\| < \varepsilon.$$

Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. $\{f_n\}$ is said to be $\mathcal{I}$-pointwise convergent to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I},$$

or $\mathcal{I} - \lim_{n\to\infty} \|f_n(x) - f(x), z\|_Y = 0$ (in $(Y, \|.,.\|_Y)$), for each $x \in X$. In this case, we write $f_n \overset{\|.,.\|_Y}{\longrightarrow}_{\mathcal{I}} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall \varepsilon > 0)(\exists M \in \mathcal{I})(\forall n_0 \in \mathbb{N}\backslash M)(\forall x \in X)(\forall n \geq n_0)\|f_n(x) - f(x), z\| \leq \varepsilon.$$

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_i \Delta B_i$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}$.

Now we begin with quoting the lemma due to Nabiev et al. [22] which are needed throughout the paper.

Lemma 2.2 ([22]). *Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}$ such that $P_i \in \mathcal{F}(\mathcal{I})$ for each i, where $\mathcal{F}(\mathcal{I})$ is a filter associated by an admissible ideal $\mathcal{I}$ with property (AP). Then, there is a set $P \subset \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I})$ and the set $P\backslash P_i$ is finite for all i.*

3. Main Results

In this paper, we study concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions and investigate relationships between them and some properties in 2-normed spaces.

Throughout the paper, we let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal, X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f, g be two functions from X to Y.

Definition 3.1. *The sequence of functions $\{f_n\}$ is said to be (pointwise) $\mathcal{I}^*$-convergent to f, if there exists a set $M \in \mathcal{F}(\mathcal{I})$, (i.e., $\mathbb{N}\backslash M \in \mathcal{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, such that for each $x \in X$ and each nonzero $z \in Y$*

$$\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|$$

and we write

$$\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ or } f_n \xrightarrow{\mathcal{I}^*} f.$$

Theorem 3.1. *For each $x \in X$ and each nonzero $z \in Y$,*

$$\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ implies } \mathcal{I} - lim_{n\to\infty}\|f_n(x), z\| = \|f(x), z\|.$$

Proof. Since for each $x \in X$ and each nonzero $z \in Y$,

$$\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|,$$

so there exists a set $H \in \mathcal{I}$ such that for $M = \mathbb{N}\backslash H = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ we have

$$\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|.$$

Let $\varepsilon > 0$. Then, for each $x \in X$ there exists a $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that for each nonzero $z \in Y$, $\|f_n(x) - f(x), z\| < \varepsilon$, for all $n \in M$ such that $n \geq k_0$. Then, obviously we have

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < \cdots < m_{k_0}\},$$

for each $x \in X$ and each nonzero $z \in Y$. Since $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal, then

$$H \cup \{m_1 < m_2 < \cdots < m_{k_0}\} \in \mathcal{I}$$

and therefore, $A(\varepsilon, z) \in \mathcal{I}$. This implies that $\mathcal{I} - \lim_{n\to\infty}\|f_n(x), z\| = \|f(x), z\|$. □

Theorem 3.2. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal having the property (AP). Then,*

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ implies } \mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|,$$

Proof. Let $\mathcal{I} \subset 2^{\mathbb{N}}$ satisfy the property (AP) and $\mathcal{I} - \lim_{n\to\infty}\|f_n(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. Then, for any $\varepsilon > 0$

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I},$$

for each $x \in X$ and each nonzero $z \in Y$. Now put

$$A_1(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq 1\}$$

and

$$A_k(\varepsilon, z) = \{n \in \mathbb{N} : \frac{1}{k} \leq \|f_n(x) - f(x), z\| < \frac{1}{k-1}\}$$

for $k \geq 2$. It is clear that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathcal{I}$ for each $i \in \mathbb{N}$. By property (AP) there exists a sequence $\{B_k\}_{k\in\mathbb{N}}$ of sets such that $A_j \Delta B_j$ is finite and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

We shall prove that, for each $x \in X$ and each nonzero $z \in Y$,

$$\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|, \quad k \in M,$$

for $M = \mathbb{N} \backslash B \in \mathcal{F}(\mathcal{I})$. Let $\delta > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k}$. Then we have

$$\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \delta\} \subset \bigcup_{j=1}^{k} A_j.$$

Since $A_j \Delta B_j$, $j = 1, 2, ..., k$, is finite set there exists $n_0 \in \mathbb{N}$ such that

$$\Big(\bigcup_{j=1}^{k} B_j\Big) \cap \{n \in \mathbb{N} : n \geq n_0\} = \Big(\bigcup_{j=1}^{k} A_j\Big) \cap \{n \in \mathbb{N} : n \geq n_0\}.$$

If $n \geq n_0$ and $n \notin B$ then

$$n \notin \bigcup_{j=1}^{k} B_j \text{ and so } n \notin \bigcup_{j=1}^{k} A_j.$$

Hence, we have $\|f_n(x) - f(x), z\| < \frac{1}{k} < \delta$, for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|, \quad k \in M,$$

and so, we have

$$\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$$

for each $x \in X$ and each nonzero $z \in Y$. □

Now we give the concepts of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence and investigate some properties about them.

Definition 3.2. *$\{f_n\}$ is said to be $\mathcal{I}$-Cauchy sequence, if for every $\varepsilon > 0$ and each $x \in X$ there exists $s = s(\varepsilon, x) \in \mathbb{N}$ such that*

$$\{n \in \mathbb{N} : \|f_n(x) - f_s(x), z\| \geq \varepsilon\} \in \mathcal{I},$$

for each nonzero $z \in Y$.

Theorem 3.3. *If $\{f_n\}$ is $\mathcal{I}$-convergent, then it is $\mathcal{I}$-Cauchy sequence.*

Proof. Suppose that $\{f_n\}$ is $\mathcal{I}$-convergent to f. Then, for $\varepsilon > 0$

$$A\left(\frac{\varepsilon}{2}, z\right) = \left\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I},$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$A^c\left(\frac{\varepsilon}{2}, z\right) = \left\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}),$$

for each $x \in X$ and each nonzero $z \in Y$ and therefore $A^c\left(\frac{\varepsilon}{2}, z\right)$ is non-empty. So, we can choose a positive integer k such that $k \notin A\left(\frac{\varepsilon}{2}, z\right)$ and $\|f_k(x) - f(x), z\| < \frac{\varepsilon}{2}$. Now, we define the set

$$B(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f_k(x), z\| \geq \varepsilon\},$$

for each $x \in X$ and each nonzero $z \in Y$, such that we show that $B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z)$. Let $n \in B(\varepsilon, z)$, then we have

$$\begin{aligned}\varepsilon \leq \|f_n(x) - f_k(x), z\| &\leq \|f_n(x) - f(x), z\| + \|f_k(x) - f(x), z\| \\ &< \|f_n(x) - f(x), z\| + \frac{\varepsilon}{2},\end{aligned}$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$\frac{\varepsilon}{2} < \|f_n(x) - f(x), z\|$$

and so, $n \in A(\frac{\varepsilon}{2}, z)$. Hence, we have $B(\varepsilon, z) \subset A(\frac{\varepsilon}{2}, z)$ and $\{f_n\}$ is $\mathcal{I}$-Cauchy sequence. □

Definition 3.3. *The sequence $\{f_n\}$ is said to be $\mathcal{I}^*$-Cauchy sequence, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, such that the subsequence $\{f_M\} = \{f_{m_k}\}$ is a Cauchy sequence, i.e.,*

$$\lim_{k,p\to\infty} \|f_{m_k}(x) - f_{m_p}(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$.

Theorem 3.4. *If $\{f_n\}$ is a $\mathcal{I}^*$-Cauchy sequence, then it is $\mathcal{I}$-Cauchy sequence in 2-normed spaces.*

Proof. Let (f_n) is a $\mathcal{I}^*$-Cauchy sequence in 2-normed spaces. Then, by definition there exists the set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that for every $\varepsilon > 0$,

$$\|f_{n_k}(x) - f_{n_p}(x), z\| < \varepsilon,$$

for every $\varepsilon > 0$, for each $x \in X$, each nonzero $z \in Y$ and $k, p > k_0 = k_0(\varepsilon, x)$. Let $N = N(\varepsilon, x) = m_{k_0} + 1$. Then, for every $\varepsilon > 0$ we have

$$\|f_{n_k}(x) - f_N(x), z\| < \varepsilon,$$

for each $x \in X$, each nonzero $z \in Y$ and $k > k_0$. Now put $H = \mathbb{N}\backslash M$. It is clear that $H \in \mathcal{I}$ and

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f_N(x), z\| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < ... < m_{k_0}\}.$$

Since $\mathcal{I}$ is an admissible ideal then, $H \cup \{m_1 < m_2 < ... < m_{k_0}\} \in \mathcal{I}$. Hence, for every $\varepsilon > 0$ we find $N = N(\varepsilon, x)$ such that $A(\varepsilon, z) \in \mathcal{I}$, i.e., (f_n) is a $\mathcal{I}$-Cauchy sequence. □

Theorem 3.5. *If $\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x) - f(x), z\| = 0$ then, $\{f_n\}$ is a $\mathcal{I}$-Cauchy sequence.*

Proof. By assumption there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that $\lim_{n\to\infty} \|f_n(x) - f(x), z\| = 0$ for each $x \in X$ and each nonzero $z \in Y$. It shows that there exists $k_0 = k_0(\varepsilon, x)$ such that

$$\|f_n(x) - f(x), z\| < \frac{\varepsilon}{2},$$

for every $\varepsilon > 0$, each $x \in X$, each nonzero $z \in Y$ and $k > k_0$. Since

$$\begin{aligned} \|f_{n_k}(x) - f_{n_p}(x), z\| &< \|f_{n_k}(x) - f(x), z\| + \|f_{n_p}(x) - f(x), z\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for each $x \in X$, each nonzero $z \in Y$ and $k > k_0$, $p > k_0$ so we have

$$\lim_{k,p\to\infty} \|f_{n_k}(x) - f_{n_p}(x), z\| = 0,$$

i.e., (f_n) is a $\mathcal{I}^*$-Cauchy sequence. Then by Theorem 3.4 (f_n) is a $\mathcal{I}$-Cauchy sequence. □

Theorem 3.6. *Let $\mathcal{I}$ be an admissible ideal with property (AP). Then the concepts $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence coincide.*

Proof. By Theorem 3.4, $\mathcal{I}^*$-Cauchy sequence implies $\mathcal{I}$-Cauchy sequence (in this case $\mathcal{I}$ need not to have (AP) condition). Then, under assumption that (f_n) is a $\mathcal{I}$-Cauchy sequence, it suffices to prove (f_n) is a $\mathcal{I}^*$-Cauchy sequence. Let (f_n) is a $\mathcal{I}$-Cauchy sequence. Then, for every $\varepsilon > 0$ and each $x \in X$ there exists $s = s(\varepsilon, x) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : \|f_n(x) - f_s(x), z\| \geq \varepsilon\} \in \mathcal{I},$$

for each nonzero $z \in Y$. Let

$$P_i = \Big\{n \in \mathbb{N} : \|f_n(x) - f_{m_i}(x), z\| < \frac{1}{i}\Big\},$$

$i = 1, 2, ...$ where $m_i = s\Big(\frac{1}{i}\Big)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I})$, $i = 1, 2, ...$ Since $\mathcal{I}$ has (AP) property then by Lemma 2.2 there exists a set $P \subset \mathbb{N}$ such that $P \subset \mathcal{F}(\mathcal{I})$ and $P \backslash P_i$ is finite for all i.

Now, we show that

$$\lim_{m,n\to\infty} \|f_n(x) - f(x), z\| = 0$$

for each $x \in X$, each nonzero $z \in Y$. Let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > \frac{2}{\varepsilon}$. If $m, n \in P$ then $P \backslash P_j$ is a finite set, so there exists $k = k(j)$ such that $m \in P_j$ and $n \in P_j$, for all $m, n > k(j)$. Therefore,

$$\|f_n(x) - f_{m_j}(x), z\| < \frac{1}{j} \text{ and } \|f_m(x) - f_{m_j}(x), z\| < \frac{1}{j}$$

for all $m, n > k(j)$, each $x \in X$ and each nonzero $z \in Y$ and so, we get

$$\begin{aligned} \|f_n(x) - f_m(x), z\| &< \|f_n(x) - f_{m_j}(x), z\| + \|f_m(x) - f_{m_j}(x), z\| \\ &< \frac{1}{j} + \frac{1}{j} = \frac{2}{j} < \varepsilon \end{aligned}$$

for $m, n > k(j)$, each $x \in X$ and each nonzero $z \in Y$. Thus, for any $\varepsilon > 0$ and each $x \in X$ there exists $k = k(\varepsilon, x)$ such that for any $m, n > k$ and $m, n \in P \in \mathcal{F}(\mathcal{I})$,

$$\|f_n(x) - f_m(x), z\| < \varepsilon$$

for every nonzero $z \in Y$ and so, the sequence (f_n) is a $\mathcal{I}^*$-Cauchy sequence in 2-normed spaces. □

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