# $\mathcal{I}_2$-INVARIANT CONVERGENCE OF DOUBLE SEQUENCES

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ABSTRACT. In this paper, we study the concepts of invariant convergence, $p$-strongly invariant convergence $\big([V_\sigma^2]_p\big)$, $\mathcal{I}_2$-invariant convergence $\big(\mathcal{I}_2^\sigma\big)$, $\mathcal{I}_2^*$-invariant convergence $\big(\mathcal{I}_2^{\sigma *}\big)$ of double sequences and investigate the relationships among invariant convergence, $[V_\sigma^2]_p$, $\mathcal{I}_2^\sigma$ and $\mathcal{I}_2^{\sigma *}$. Also, we introduce the concepts of $\mathcal{I}_2^\sigma$-Cauchy double sequence and $\mathcal{I}_2^{\sigma *}$-Cauchy double sequence.

## 1. Introduction and Background

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [5] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. $\mathcal{I}$-convergence of double sequences in a metric space and some properties of this convergence and similar concepts which are noted following can be seen in [2, 4, 6].

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
$(i)$ $\emptyset \in \mathcal{I}$, $(ii)$ For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, $(iii)$ For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout the paper we take $\mathcal{I}$ as an admissible ideal in $\mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if
$(i)$ $\emptyset \notin \mathcal{F}$, $(ii)$ For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, $(iii)$ For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

For any ideal there is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$, given by

$$\mathcal{F}(\mathcal{I}) = \big\{M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N} \backslash A)\big\}.$$

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

$\mathcal{I}_2^0 = \big\{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\big\}$. Then $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

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A double sequence $x = (x_{kj})_{k,j \in \mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case, we write $P - \lim\limits_{k,j \to \infty} x_{kj} = L$ *or* $\lim\limits_{k,j \to \infty} x_{kj} = L$.

A double sequence $x = (x_{kj})$ is said to be bounded if $\sup\limits_{k,j} x_{kj} < \infty$. The set of all bounded double sequences of sets will be denoted by $\ell^2_\infty$.

Let $(X, \rho)$ be a metric space. A sequence $x = (x_{mn})$ in $X$ is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$,

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2 - \lim\limits_{m,n \to \infty} x_{mn} = L$.

A double sequence $x = (x_{kj})$ is $\mathcal{I}_2^*$-convergent to $L$ if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2$) such that

$$\lim_{\substack{k,j \to \infty \\ (k,j) \in M_2}} x_{kj} = L.$$

In this case, we write $\mathcal{I}_2^* - \lim\limits_{k,j \to \infty} x_{kj} = L$.

A double sequence $x = (x_{kj})$ is $\mathcal{I}_2$-Cauchy sequence if for every $\varepsilon > 0$, there exists $(r, s)$ in $\mathbb{N} \times \mathbb{N}$ such that

$$\{(k, j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - x_{rs}| \geq \varepsilon\} \in \mathcal{I}_2.$$

A double sequence $x = (x_{kj})$ is $\mathcal{I}_2^*$-Cauchy if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ $(\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2)$ such that,

$$\lim_{k,j,r,s \to \infty} |x_{kj} - x_{rs}| = 0,$$

for $(k, j), (r, s) \in M_2$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that $E_j \Delta F_j \in \mathcal{I}_2^0$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2$ (hence $F_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Several authors have studied invariant convergent sequences (see, [1, 7–9, 11–16]).

Let $\sigma$ be a mapping of the positive integers into themselves. A continuous linear functional $\phi$ on $\ell_\infty$, the space of real bounded sequences, is said to be an invariant mean or a $\sigma$-mean if it satisfies following conditions:

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all $n$,
(2) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings $\sigma$ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers $n$ and $m$, where $\sigma^m(n)$ denotes the $m$ th iterate of the mapping $\sigma$ at $n$. Thus, $\phi$ extends the limit functional on $c$, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.

In the case $\sigma$ is translation mappings $\sigma(n) = n + 1$, the $\sigma$-mean is often called a Banach limit and the space $V_\sigma$, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences $\hat{c}$.

It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \quad \text{uniformly in } n \right\}.$$

The concept of strongly $\sigma$-convergence was defined by Mursaleen in [7]:

A bounded sequence $x = (x_k)$ is said to be strongly $\sigma$-convergent to $L$ if

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L| = 0,$$

uniformly in $n$. It is denoted by $x_k \to L[V_\sigma]$.

By $[V_\sigma]$, we denote the set of all strongly $\sigma$-convergent sequences. In the case $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

The concept of strongly $\sigma$-convergence was generalized by Savaş [13] as below:

$$[V_\sigma]_p = \left\{ x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0, \text{ uniformly in } n \right\},$$

where $0 < p < \infty$. If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.

Recently, the concepts of $\sigma$-uniform density of the set $A \subseteq \mathbb{N}$, $\mathcal{I}_\sigma$-convergence and $\mathcal{I}_\sigma^*$-convergence of sequences of real numbers were defined by Nuray et al. [11]. Also, the concept of $\sigma$-convergence of double sequences was studied by Çakan et al. [1] and the concept of $\sigma$-uniform density of $A \subseteq \mathbb{N} \times \mathbb{N}$ was defined by Tortop and Dündar [16].

Let $A \subseteq \mathbb{N}$ and

$$s_m = \min_n \left| A \cap \{\sigma(n), \sigma^2(n), \cdots, \sigma^m(n)\} \right|$$

and

$$S_m = \max_n \left| A \cap \{\sigma(n), \sigma^2(n), \cdots, \sigma^m(n)\} \right|.$$

If the following limits exist

$$\underline{V}(A) = \lim_{m\to\infty} \frac{s_m}{m}, \quad \overline{V}(A) = \lim_{m\to\infty} \frac{S_m}{m}$$

then they are called a lower and upper $\sigma$-uniform density of the set $A$, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called $\sigma$-uniform density of $A$.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathcal{I}_\sigma$-convergent to the number $L$ if for every $\varepsilon > 0$ $A_\varepsilon = \{k : |x_k - L| \geq \varepsilon\} \in \mathcal{I}_\sigma$; i.e., $V(A_\varepsilon) = 0$. In this case, we write $\mathcal{I}_\sigma - \lim_k = L$.

The set of all $\mathcal{I}_\sigma$-convergent sequences will be deneted by $\mathfrak{I}_\sigma$.

Let $\mathcal{I}_\sigma \subset 2^{\mathbb{N}}$ be an admissible ideal. A sequence $x = (x_k)$ is said to be $\mathcal{I}^*_\sigma$-convergent to the number $L$ if there exists a set $M = \{m_1 < m_2 < \cdots\} \in \mathcal{F}(\mathcal{I}_\sigma)$ such that $\lim_{k\to\infty} x_{m_k} = L$. In this case, we write $\mathcal{I}^*_\sigma - \lim_k = L$.

A bounded double sequences $x = (x_{kj})$ of real numbers is said to be $\sigma$-convergent to a limit $L$ if

$$\lim_{mn} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} x_{\sigma^k(s),\sigma^j(t)} = L$$

uniformly in $s, t$. In this case, we write $\sigma_2 - \lim x = L$.

Let $A \subseteq \mathbb{N}\times\mathbb{N}$ and

$$s_{mn} := \min_{k,j} \left| A \cap \left\{ \big(\sigma(k),\sigma(j)\big), \big(\sigma^2(k),\sigma^2(j)\big), ..., \big(\sigma^m(k),\sigma^n(j)\big) \right\} \right|$$

and

$$S_{mn} := \max_{k,j} \left| A \cap \left\{ \big(\sigma(k),\sigma(j)\big), \big(\sigma^2(k),\sigma^2(j)\big), ..., \big(\sigma^m(k),\sigma^n(j)\big) \right\} \right|.$$

If the following limits exists

$$\underline{V_2}(A) := \lim_{m,n\to\infty} \frac{s_{mn}}{mn}, \qquad \overline{V_2}(A) := \lim_{m,n\to\infty} \frac{S_{mn}}{mn}$$

then they are called a lower and an upper $\sigma$-uniform density of the set $A$, respectively. If $\underline{V_2}(A) = \overline{V_2}(A)$, then $V_2(A) = \underline{V_2}(A) = \overline{V_2}(A)$ is called the $\sigma$-uniform density of $A$.

Denote by $\mathcal{I}^\sigma_2$ the class of all $A \subseteq \mathbb{N}\times\mathbb{N}$ with $V_2(A) = 0$.

Throughout the paper we let $\mathcal{I}^\sigma_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal

## 2. $\mathcal{I}_2$-Invariant Convergence

In this section, we introduce the concepts of strongly invariant convergence $\big([V^2_\sigma]\big)$, $p$-strongly invariant convergence $\big([V^2_\sigma]_p\big)$, $\mathcal{I}_2$-invariant convergence $\big(\mathcal{I}^\sigma_2\big)$ of double sequences and investigate the relationships among invariant convergence, $[V^2_\sigma]_p$ and $\mathcal{I}^\sigma_2$.

**Definition 2.1.** A double sequence $x = (x_{kj})$ is said to be $\mathcal{I}_2$-invariant convergent or $\mathcal{I}^\sigma_2$-convergent to $L$, if for every $\varepsilon > 0$

$$A(\varepsilon) = \big\{(k,j) : |x_{kj} - L| \geq \varepsilon\big\} \in \mathcal{I}^\sigma_2$$

that is, $V_2\big(A(\varepsilon)\big) = 0$. In this case, we write $\mathcal{I}^\sigma_2 - \lim x = L$ or $x_{kj} \to L(\mathcal{I}^\sigma_2)$.

The set of all $\mathcal{I}_2$-invariant convergent double sequences will be denoted by $\mathfrak{I}^\sigma_2$.

**Theorem 2.2.** *If $\mathcal{I}^\sigma_2 - \lim x_{kj} = L_1$ and $\mathcal{I}^\sigma_2 - \lim y_{kj} = L_2$, then*

(i) $\mathcal{I}^\sigma_2 - \lim(x_{kj} + y_{kj}) = L_1 + L_2$
(ii) $\mathcal{I}^\sigma_2 - \lim \alpha x_{kj} = \alpha L_1$ *($\alpha$ is a constant).*

*Proof.* The proof is clear so we omit it. □

**Theorem 2.3.** *Suppose that $x = (x_{kj})$ is a bounded double sequence. If $x = (x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent to $L$, then $x = (x_{kj})$ is invariant convergent to $L$.*

*Proof.* Let $m, n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. We estimate

$$u(m,n,s,t) = \left| \frac{1}{mn} \sum_{k,j=1,1}^{m,n} x_{\sigma^k(s),\sigma^j(t)} - L \right|.$$

Then, we have

$$u(m,n,s,t) \le u^1(m,n,s,t) + u^2(m,n,s,t)$$

where

$$u^1(m,n,s,t) = \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L| \ge \varepsilon}}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|$$

and

$$u^2(m,n,s,t) = \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L| < \varepsilon}}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|.$$

Therefore, we have

$$u^2(m,n,s,t) < \varepsilon,$$

for every $s, t = 1, 2, \dots$ . The boundedness of $(x_{kj})$ implies that there exists $K > 0$ such that

$$|x_{\sigma^k(s),\sigma^j(t)} - L| \le K, \quad (k, j, s, t = 1, 2, \dots),$$

then this implies that

$$\begin{aligned} u^1(m,n,s,t) &\le \frac{K}{mn} \left| \left\{ 1 \le k \le m, 1 \le j \le n : |x_{\sigma^k(s),\sigma^j(t)} - L| \ge \varepsilon \right\} \right| \\ &\le K \frac{\max_{s,t} \left| \left\{ 1 \le k \le m, 1 \le j \le n : |x_{\sigma^k(s),\sigma^j(t)} - L| \ge \varepsilon \right\} \right|}{mn} \\ &= K \frac{S_{mn}}{mn}. \end{aligned}$$

Hence, $(x_{kj})$ is invariant convergent to $L$. □

The converse of Theorem 2.3 does not hold. For example, $x = (x_{kj})$ is the double sequence defined by following;

$$x_{kj} := \begin{cases} 1 \ , & \text{if } k+j \text{ is even integer,} \\ 0 \ , & \text{if } k+j \text{ is odd integer.} \end{cases}$$

When $\sigma(s) = s + 1$ and $\sigma(t) = t + 1$, this sequence is invariant convergent to $\frac{1}{2}$ but it is not $\mathcal{I}_2^\sigma$-convergent.

In [11], Nuray et al. gave some inclusion relations between $[V_\sigma]_p$-convergence and $\mathcal{I}$-invariant convergence, and showed that these are equivalent for bounded sequences. Now, we shall give analogous theorems which states inclusion relations between $[V_2^\sigma]_p$-convergence and $\mathcal{I}_2$-invariant convergence, and show that these are equivalent for bounded double sequences.

**Definition 2.4.** A double sequence $x = (x_{kj})$ is said to be strongly invariant convergent to $L$, if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right| = 0,$$

uniformly in $s,t$. In this case, we write $x_{kj} \to L\big([V_\sigma^2]\big)$.

**Definition 2.5.** A double sequence $x = (x_{kj})$ is said to be $p$-strongly invariant convergent to $L$, if

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right|^p = 0,$$

uniformly in $s,t$, where $0 < p < \infty$. In this case, we write $x_{kj} \to L\big([V_\sigma^2]_p\big)$.

The set of all $p$-strongly invariant convergent double sequences will be denoted by $[V_\sigma^2]_p$.

**Theorem 2.6.** *Let* $0 < p < \infty$.

(i) *If* $x_{kj} \to L\big([V_\sigma^2]_p\big)$, *then* $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$.

(ii) *If* $(x_{kj}) \in \ell_\infty^2$ *and* $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$, *then* $x_{kj} \to L\big([V_\sigma^2]_p\big)$.

(iii) *If* $(x_{kj}) \in \ell_\infty^2$, *then* $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$ *if and only if* $x_{kj} \to L\big([V_\sigma^2]_p\big)$.

*Proof.* $(i)$ : Assume that $x_{kj} \to L\big([V_\sigma^2]_p\big)$. Then, for every $\varepsilon > 0$, we can write

$$\begin{aligned} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right|^p &\geq \sum_{\substack{k,j=1,1 \\ |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon}}^{m,n} |x_{\sigma^k(s),\sigma^j(t)} - L|^p \\ &\geq \varepsilon^p \left|\left\{k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon\right\}\right| \\ &\geq \varepsilon^p \max_{s,t} \left|\left\{k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon\right\}\right| \end{aligned}$$

and

$$\begin{aligned} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \left|x_{\sigma^k(s),\sigma^j(t)} - L\right|^p &\geq \varepsilon^p \frac{\max_{s,t} \left|\left\{k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon\right\}\right|}{mn} \\ &= \varepsilon^p \frac{S_{mn}}{mn} \end{aligned}$$

for every $s,t = 1,2,\ldots$ . This implies

$$\lim_{m,n\to\infty} \frac{S_{mn}}{mn} = 0$$

and so $(x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent to $L$.

$(ii)$ : Suppose that $(x_{kj}) \in \ell^2_\infty$ and $x_{kj} \to L\big(\mathcal{I}_2^\sigma\big)$. Let $0 < p < \infty$ and $\varepsilon > 0$. By assumption we have $V_2\big(A(\varepsilon)\big) = 0$. Since $(x_{kj})$ is bounded, $(x_{kj})$ implies that there exists $K > 0$ such that

$$|x_{\sigma^k(s),\sigma^j(t)} - L| \leq K,$$

for all $k, j, s, t$. Then, we have

$$\begin{aligned}
\frac{1}{mn}\sum_{k,j=1,1}^{m,n}\big|x_{\sigma^k(s),\sigma^j(t)} - L\big|^p &= \frac{1}{mn}\sum_{\substack{k,j=1,1\\ |x_{\sigma^k(s),\sigma^j(t)}-L|\geq\varepsilon}}^{m,n}|x_{\sigma^k(s),\sigma^j(t)} - L|^p \\
&\quad + \frac{1}{mn}\sum_{\substack{k,j=1,1\\ |x_{\sigma^k(s),\sigma^j(t)}-L|<\varepsilon}}^{m,n}|x_{\sigma^k(s),\sigma^j(t)} - L|^p \\
&\leq K\frac{\max_{s,t}\big|\big\{k \leq m, j \leq n : |x_{\sigma^k(s),\sigma^j(t)} - L| \geq \varepsilon\big\}\big|}{mn} + \varepsilon^p \\
&\leq K\frac{S_{mn}}{mn} + \varepsilon^p.
\end{aligned}$$

Hence, we obtain

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,j=1,1}^{m,n}\big|x_{\sigma^k(s),\sigma^j(t)} - L\big|^p = 0,$$

uniformly in $s, t$.

$(iii)$ : This is immediate consequence of $(i)$ and $(ii)$. □

Now, we introduce $\mathcal{I}_2^*$-invariant convergence concept, $\mathcal{I}_2$-invariant Cauchy double sequence and $\mathcal{I}_2^*$-invariant Cauchy double sequence concepts and give the relationships among these concepts and relationships with $\mathcal{I}_2$-invariant convergence concept.

**Definition 2.7.** A double sequence $(x_{kj})$ is $\mathcal{I}_2^*$-invariant convergent or $\mathcal{I}_2^{\sigma*}$-convergent to $L$ if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ $(\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^\sigma)$ such that

$$\lim_{\substack{k,j\to\infty\\ (k,j)\in M_2}} x_{kj} = L.$$

In this case, we write $\mathcal{I}_2^{\sigma*} - \lim_{k,j\to\infty} x_{kj} = L$ or $x_{kj} \to L(\mathcal{I}_2^{\sigma*})$.

**Theorem 2.8.** *If a double sequence $(x_{kj})$ is $\mathcal{I}_2^{\sigma*}$-convergent to $L$, then this sequence is $\mathcal{I}_2^\sigma$-convergent to $L$.*

*Proof.* Since $\mathcal{I}_2^{\sigma*} - \lim_{k,j\to\infty} x_{kj} = L$, there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ $(\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^\sigma)$ such that

$$\lim_{\substack{k,j\to\infty\\ (k,j)\in M_2}} x_{kj} = L.$$

Let $\varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that

$$|x_{kj} - L| < \varepsilon,$$

for all $(k,j) \in M_2$ and $k, j \geq k_0$. Hence, for every $\varepsilon > 0$, we have

$$\begin{aligned} T(\varepsilon) &= \big\{(k,j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - L| \geq \varepsilon\big\} \\ &\subset H \cup \Big(M_2 \cap \big((\{1,2,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,...,(k_0-1)\})\big)\Big). \end{aligned}$$

Since $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal,

$$H \cup \Big(M_2 \cap \big((\{1,2,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,...,(k_0-1)\})\big)\Big) \in \mathcal{I}_2^\sigma,$$

so we have $T(\varepsilon) \in \mathcal{I}_2^\sigma$ that is $V_2\big(T(\varepsilon)\big) = 0$. Hence,

$$\mathcal{I}_2^\sigma - \lim_{k,j\to\infty} x_{kj} = L.$$

□

**Theorem 2.9.** *Let $\mathcal{I}_2^\sigma$ has property $(AP2)$. If $(x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent to $L$, then $(x_{kj})$ is $\mathcal{I}_2^{\sigma *}$-convergent to $L$.*

*Proof.* Suppose that $\mathcal{I}_2^\sigma$ satisfies property $(AP2)$. Let $(x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent to $L$. Then,

$$T(\varepsilon) = \big\{(k,j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - L| \geq \varepsilon\big\} \in \mathcal{I}_2^\sigma \tag{2.1}$$

for every $\varepsilon > 0$. Put

$$T_1 = \big\{(k,j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - L| \geq 1\big\}$$

and

$$T_v = \left\{(k,j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{v} \leq |x_{kj} - L| < \frac{1}{v-1}\right\}$$

for $v \geq 2$ and $v \in \mathbb{N}$. Obviously $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}_2^\sigma$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exits a sequence of sets $\{E_v\}_{v\in\mathbb{N}}$ such that $T_i \Delta E_i$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $i$ and $E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{I}_2^\sigma$.

We shall prove that for $M_2 = \mathbb{N} \times \mathbb{N} \backslash E$ we have

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} x_{kj} = L.$$

Let $\eta > 0$ be given. Choose $v \in \mathbb{N}$ such that $\frac{1}{v} < \eta$ . Then,

$$\big\{(k,j) \in \mathbb{N} \times \mathbb{N} : |x_{kj} - L| \geq \eta\big\} \subset \bigcup_{i=1}^{v} T_i.$$

Since $T_i \Delta E_i$, $i = 1, 2, ...$ are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{v} T_i\right) \cap \big\{(k,j) : k \geq n_0 \wedge j \geq n_0\big\} = \left(\bigcup_{i=1}^{v} E_i\right) \cap \big\{(k,j) : k \geq n_0 \wedge j \geq n_0\big\}. \tag{2.2}$$

If $k,j > n_0$ and $(k,j) \notin E$, then

$$(k,j) \notin \bigcup_{i=1}^{v} E_i \quad \text{and} \quad (k,j) \notin \bigcup_{i=1}^{v} T_i.$$

This implies that

$$|x_{kj} - L| < \frac{1}{v} < \eta.$$

Hence, we have

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} x_{kj} = L.$$

□

Finally, we define the concepts of $\mathcal{I}_2^\sigma$-Cauchy and $\mathcal{I}_2^{\sigma *}$-Cauchy double sequences.

**Definition 2.10.** A double sequence $(x_{kj})$ is said to be $\mathcal{I}_2$-invariant Cauchy or $\mathcal{I}_2^\sigma$-Cauchy sequence, if for every $\varepsilon > 0$, there exist numbers $r = r(\varepsilon), s = s(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \big\{(k,j) : |x_{kj} - x_{rs}| \geq \varepsilon\big\} \in \mathcal{I}_2^\sigma,$$

that is, $V_2\big(A(\varepsilon)\big) = 0$.

**Definition 2.11.** A double sequence $(x_{kj})$ is $\mathcal{I}_2^*$-invariant Cauchy or $\mathcal{I}_2^{\sigma *}$-Cauchy sequence if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2^\sigma$) such that for every $(k,j),(r,s) \in M_2$

$$\lim_{k,j,r,s\to\infty} |x_{kj} - x_{rs}| = 0.$$

We give following theorems which show relationships between $\mathcal{I}_2^\sigma$-convergence, $\mathcal{I}_2^\sigma$-Cauchy double sequence and $\mathcal{I}_2^{\sigma *}$-Cauchy double sequence. The proof of them are similar to the proof of Theorems in [3, 4, 10], so we omit them.

**Theorem 2.12.** *If a double sequence $(x_{kj})$ is $\mathcal{I}_2^\sigma$-convergent, then $(x_{kj})$ is an $\mathcal{I}_2^\sigma$-Cauchy double sequence.*

**Theorem 2.13.** *If a double sequence $(x_{kj})$ is $\mathcal{I}_2^{\sigma *}$-Cauchy double sequence, then $(x_{kj})$ is $\mathcal{I}_2^\sigma$-Cauchy double sequence.*

**Theorem 2.14.** *Let $\mathcal{I}_2^\sigma$ has property $(AP2)$. Then, the concepts $\mathcal{I}_2^\sigma$-Cauchy double sequence and $\mathcal{I}_2^{\sigma *}$-Cauchy double sequence coincides.*

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