ON $\mathcal{I}$-UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS IN 2-NORMED SPACES

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Abstract. In this work, we discuss various kinds of $\mathcal{I}$-uniform convergence for sequences of functions and introduce the concepts of $\mathcal{I}^*$-uniform convergence, $\mathcal{I}$ and $\mathcal{I}^*$-uniformly Cauchy sequences for sequences of functions in 2-normed spaces. Then, we show the relation between them.

1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [9] and Schoenberg [27].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [21] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$ [9, 10]. Gökhan et al. [14] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued functions. Gezer and Karakuş [13] investigated $\mathcal{I}$-pointwise and uniform convergence and $\mathcal{I}^*$-pointwise and uniform convergence of function sequences and they examined the relation between them. Baláz et al. [3] investigated $\mathcal{I}$-convergence and $\mathcal{I}$-continuity of real functions. Balcerzak et al. [4] studied statistical convergence and ideal convergence for sequences of functions Dündar and Altay [6, 7] studied the concepts of pointwise and uniformly $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [8] investigated some results of $\mathcal{I}_2$-convergence of double sequences of functions.

The concept of 2-normed spaces was initially introduced by Gähler [11, 12] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [18] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Şahiner et al. [29] and Gürdal [20] studied $\mathcal{I}$-convergence in 2-normed spaces. Gürdal and Açık [19] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [25] presented various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathcal{I}$-equistatistically convergence and study $\mathcal{I}$-equistatistically convergence of sequences of functions. Recently, Savaş and Gürdal [26] concerned with $\mathcal{I}$-convergence of sequences of functions in random 2-normed spaces and introduce the concepts of ideal uniform convergence and ideal pointwise convergence in the topology induced by random 2-normed spaces, and gave some basic properties of these concepts. Arslan and Dündar [1, 2] investigated the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions in 2-normed spaces and

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showed relationships between them. Also, Yegül and Dündar [31] studied statistical convergence of sequence of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [5, 22, 23, 24, 28, 30]).

2. Definitions and Notations

Now, we recall the concept of 2-normed space, ideal convergence and some fundamental definitions and notations (See [1, 2, 3, 4, 9, 10, 15, 16, 17, 18, 19, 20, 21, 25, 29]).

If $K \subseteq \mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $\delta(K) = \lim_n \frac{1}{n}|K_n|$, if it exists.

The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$ the set

$$K = K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has natural density zero; in this case, we write $st - \lim x = L$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$, for each $x \in X$.

Example 2.1. *Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then, $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ convergence is the usual convergence.*

Throughout the paper, we let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1 ([21]). *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A sequence (f_n) of functions is said to be $\mathcal{I}$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if for every $\varepsilon > 0$ and each $x \in D$,

$$\{n : |f_n(x) - f(x) \geq \varepsilon|\} \in \mathcal{I}.$$

In this case, we will write $f_n \xrightarrow{\mathcal{I}} f$ on D.

A sequence (f_n) of functions is said to be $\mathcal{I}^*$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if $\forall \varepsilon > 0$ and $\forall x \in D$, $\exists K_x \notin \mathcal{I}$ and $\exists n_0 = n_0(\varepsilon, x) \in K_x : \forall n \geq n_0$ and $n \in K_x$, $|f_n(x) - f(x)| < \varepsilon$.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot, \cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x, y\| = \|y, x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1y_2 - x_2y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if

$$\lim_{n\to\infty} \|x_n - L, y\| = 0,$$

for every $y \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ and each nonzero $z \in X$,

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $\mathcal{I} - \lim_{n\to\infty} \|x_n - L, z\| = 0$ or $\mathcal{I} - \lim_{n\to\infty} \|x_n, z\| = \|L, z\|$.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathcal{F}$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $\lim_{n\to\infty} \|x_{m_k} - L, z\| = 0$, for each nonzero $z \in X$.

Throughout the paper, we let X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f, g be two functions from X to Y.

The sequence $\{f_n\}$ is said to be convergent to f if $f_n(x) \stackrel{\|.,.\|_Y}{\longrightarrow} f(x)$ for each $x \in X$. We write $f_n \stackrel{\|.,.\|_Y}{\longrightarrow} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x), z\| < \varepsilon.$$

We introduce uniform convergent of $(f_n)_{n\in\mathbb{N}}$ to f by the formula

$$(\forall y \in Y)\,(\forall \varepsilon > 0)\,(\exists n_0 \in \mathbb{N})\,(\forall n > n_0)\,(\forall x \in X)\,\|f_n(x) - f(x), y\|_Y < \varepsilon$$

and we write it as $f_n \stackrel{\|.,.\|_Y}{\Longrightarrow} f$.

The sequence $(f_n)_{n\in\mathbb{N}}$ is equi-continuous (on X) if

$$(\forall z \in X)\,(\forall \varepsilon > 0)\,(\exists \delta > 0)\,(\forall x, x_0 \in X)\,\|x - x_0, z\|_X < \delta \Rightarrow \|f_n(x) - f_n(x_0)\|_\infty < \varepsilon.$$

The sequence $\{f_n\}$ is said to be $\mathcal{I}$-pointwise convergent to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, $A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I}$ or $\mathcal{I} - \lim_{n\to\infty} \|f_n(x) - f(x), z\|_Y = 0$ (in $(Y, \|.,.\|_Y)$), for each $x \in X$. In this case, we write $f_n \stackrel{\|.,.\|_Y}{\longrightarrow}_{\mathcal{I}} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall \varepsilon > 0)(\exists M \in \mathcal{I})(\forall n_0 \in \mathbb{N}\backslash M)(\forall x \in X)(\forall n \geq n_0)\|f_n(x) - f(x), z\| \leq \varepsilon.$$

The sequence $\{f_n\}$ is said to be pointwise $\mathcal{I}^*$-convergent to f, if there exists a set $M \in \mathcal{F}(\mathcal{I})$, (i.e., $\mathbb{N}\backslash M \in \mathcal{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, such that for each $x \in X$ and each nonzero $z \in Y$ $\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|$ and we write

$$\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ or } f_n \stackrel{\mathcal{I}^*}{\to} f.$$

The sequence $\{f_n\}$ is said to be $\mathcal{I}$-Cauchy sequence, if for every $\varepsilon > 0$ and each $x \in X$ there exists $s = s(\varepsilon, x) \in \mathbb{N}$ such that

$$\{n \in \mathbb{N} : \|f_n(x) - f_s(x), z\| \geq \varepsilon\} \in \mathcal{I},$$

for each nonzero $z \in Y$.

The sequence $\{f_n\}$ is said to be $\mathcal{I}^*$-Cauchy sequence, if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$, such that the subsequence $\{f_M\} = \{f_{m_k}\}$ is a Cauchy sequence, i.e.,

$$\lim_{k,p\to\infty} \|f_{m_k}(x) - f_{m_p}(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$.

The sequence $(f_n)_{n\in\mathbb{N}}$ is said to be $\mathcal{I}$-uniformly convergent to f (on X) if and only if

$$(\forall z \in Y)\,(\forall \varepsilon > 0)\,(\exists M \in \mathcal{I})\,(\forall n \in \mathbb{N}\backslash M)\,(\forall x \in X)\,\|f_n(x) - f(x), z\|_Y \leq \varepsilon.$$

We write $f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}} f$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_i \Delta B_i$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}$.

Now we begin with quoting the lemmas due to Arslan and Dündar [1], Gezer and Karakuş [13] and Sarabadan and Talebi [25] which are needed throughout the paper.

Lemma 2.2 ([1]). *Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. For each $x \in X$ and each nonzero $z \in Y$,*

$$\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ implies } \mathcal{I} - lim_{n\to\infty}\|f_n(x), z\| = \|f(x), z\|.$$

Lemma 2.3 ([1]). *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal having the property (AP), X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. If the sequence of functions $\{f_n\}$ is $\mathcal{I}$-convergent, then it is $\mathcal{I}^*$-convergent.*

Lemma 2.4 ([13]). *Let $\mathcal{I} \subset 2^{\mathbb{N}}$, X and Y be two 2-normed spaces and (f_n) be a sequence of functions on X. (f_n) is $\mathcal{I}$-convergent if and only if (f_n) is $\mathcal{I}$-Cauchy.*

Lemma 2.5 ([25]). *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal, X and Y be two 2-normed spaces with $dimY < \infty$. Assume that $f_n \overset{\|.,.\|}{\longrightarrow}_{\mathcal{I}} f$ on X, where $f_n : X \to Y, (n \in \mathbb{N})$ are equi-continuous and $f : X \to Y$. Then, f is sequentially continuous on X.*

3. Main Results

In this paper, we study concepts of convergence, $\mathcal{I}$-uniform convergence, $\mathcal{I}^*$-uniform convergence of functions and investigate relationships between them and some properties such as continuity in 2-normed spaces.

Definition 3.1. *The sequence of functions $\{f_n\}$ is said to be $\mathcal{I}^*$-uniformly convergent to f, if there exists a set $M \in \mathcal{F}(\mathcal{I}), (i.e., N\backslash M \in \mathcal{I}), M = \{m_1 < m_2 < ... < m_k < ...\}$, such that for each nonzero $z \in Y$,*

$$\lim_{n\to\infty} \|f_n(x) - f(x), z\|,$$

for each $x \in X$ and we write

$$f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}^*} f.$$

Theorem 3.1. *Let f_n be a sequence of continuous functions and f be function from X to Y. If $f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}^*} f$, then f is continuous on X.*

Proof 3.1. *Assume $f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}^*} f$ on X.Then, for every $\varepsilon > 0$, there exists a set $M \in \mathcal{F}(\mathcal{I})$, (i.e., $H = N\backslash M \in \mathcal{I}$) and $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$*

$$\|f_n(x) - f(x), z\| < \frac{\varepsilon}{3}, n \in M$$

for each $x \in X$ *and for all* $n > k_0$. *Now we let* $x_0 \in X$ *is arbitrary. Since* $\{f_{k_0}\}$ *is continuous at* $x_0 \in X$, *there is a* $\delta > 0$ *such that for each nonzero* $z \in Y$,

$$\|x - x_0, z\| < \delta$$

implies

$$\|f_{k_0}(x) - f_{k_0}(x_0), z\| < \frac{\varepsilon}{3}.$$

Then for all $x \in X$*for which* $\|x - x_0, z\| < \delta$, *we have*

$$\begin{aligned} \|f(x) - f(x_0), z\| &\leq \|f(x) - f_{k_0}(x_0), z\| + \|f_{k_0}(x) - f_{k_0}(x_0), z\| + \|f_{k_0}(x) - f(x_0), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

for each nonzero $z \in Y$. *Since* $x_0 \in X$ *is arbitrary, f is continuous on X.*

Theorem 3.2. *Let* $\mathcal{I} \subset 2^{\mathbb{N}}$ *be an admissible ideal with the property (AP), S be a compact subset of X and* $\{f_n\}$ *be a sequence of continuous function on S. Assume that* $\{f_n\}$ *be monotonic decreasing on S, i.e.,*

$$f_{n+1} \leq f_n(x), (n = 1, 2, ...)$$

for every $x \in S$, *f is continuous and for each nonzero* $z \in Y$,

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$$

on S. Then,

$$f_n \overset{\|.,.\|}{\Longrightarrow}_{\mathcal{I}} f$$

on S.

Proof 3.2. *Let*

$$g_n = f_n - f \tag{3.1}$$

be a sequence of functions on S. Since $\{f_n\}$ *is continuous on monotonic decreasing and f is continuous on S. Then* $\{g_n\}$ *is continuous on monotonic decreasing and f is continuous on S. Since for each nonzero* $z \in Y$,

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|,$$

for each nonzero $z \in Y$. *Then by 3.1, for each nonzero* $z \in Y$,

$$\mathcal{I} - \lim_{n\to\infty} \|g_n(x), z\| = 0$$

on S and since $\mathcal{I}$ *satisfy condition (AP), then we have for each nonzero* $z \in Y$,

$$\mathcal{I}^* - \lim_{n\to\infty} \|g_n(x), z\| = 0,$$

for each $x \in S$. *Hence, for every* $\varepsilon > 0$ *and each* $x \in S$ *there exists* $K_x \in \mathcal{F}(\mathcal{I})$ *such that* $0 \leq g_n(x) < \frac{\varepsilon}{2}$, *for all* $n \geq n(x)(n(x) = n(x, \varepsilon) \in K_x)$. *Since* $\{g_n\}$ *is continuous at* $x \in S$ *for every* $\varepsilon > 0$, *there exists an open set* $A(x)$ *which contains X such that for each nonzero* $z \in Y$,

$$\|g_n(t) - g_n(x), z\| \leq \frac{\varepsilon}{2}$$

for all $t \in A(x)$. *Then for* $\varepsilon > 0$,*by monotonicity we have*

$$0 \leq g_n(x) \leq g_{n(x)}(t) \leq g_{n(x)}(t) - g_{n(x)}(x) + g_{n(x)}(x) \leq \|g_{n(x)}(t) - g_{n(x)}(x), z\| + \|g_{n(x)}(x)\|$$

$(n \in K_X)$ *for every* $t \in A(x)$ *and for all* $n \geq n(x)$ *and for each* $x \in S$. *Since* $S \subset \cup_{x\in S} A(x)$ *and S is a compact set, by the Heine- Borel theorem S has a finite open covering such that*

$$S \subset A(x_1) \cup A(x_2) \cup A(x_3) ... \cup A(x_{x_i}).$$

Now, let

$$K = K_{x_1} \cap K_{x_2} \cap K_{x_3} \cap ... \cap K_{x_i}$$

and define

$$N = max\{n(x_1), n(x_2), n(x_3), ..., n(x_i)\}.$$

Since for every K_{x_1}*, belong to* $\mathcal{F}(\mathcal{I})$*, we have* $K \in \mathcal{F}(\mathcal{I})$*. Then when all* $n \geq N$ *and* $n \in K, 0 \leq g_n(t) < \varepsilon$ *for every* $t \in A(x)$*. So*

$$g_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}^*} 0.$$

Since $\mathcal{I}$ *is an admissible ideal* $g_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}} 0$ *on* S *and by (2.1) we have* $f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}} f$ *on* S*.*

Sarabadan and Talebi proved the lemma 2.5. In addition to this lemma, if X is compact, we have following theorem:

Theorem 3.3. *If* X *is compact then, we have* $f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}} f$ *on* X*.*

Proof 3.3. *First we will prove that f is continuous on X. Let* $x_0 \in X$ *and* $\varepsilon > 0$*. By the equi-continuity of* f_n*'s, there exists* $\delta > 0$ *such that for each nonzero* $\|z \in Y$*.*

$$f_n(x) - f_n(x_0), z\| \leq \frac{\varepsilon}{3}$$

for every $n \in \mathbb{N}$ *and* $x \in B_\delta(x_0)$*. (*$x \in B_\delta(x_0)$ *stands for an open ball in X with center to* x_0 *and radius* δ*). Let* $x \in B_\delta(x_0)$ *be fixed. Since* $f_n \overset{\|.,.\|_Y}{\longrightarrow}_{\mathcal{I}} f$*, the set for each nonzero* $z \in Y$*,*

$$\{n \in \mathbb{N} : \|f_n(x_0) - f(x_0), z\| \geq \frac{\varepsilon}{0}\} \cup \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \frac{\varepsilon}{3}\}$$

is in $\mathcal{I}$ *and is different from* $\mathbb{N}$*. Hence, there exists a* $n \in \mathbb{N}$ *such that for each nonzero* $z \in Y$*,*

$$\|f_n(x_0) - f(x_0), z\| \geq \frac{\varepsilon}{3}\} \;\; and \;\; \|f_n(x) - f(x), z\| \geq \frac{\varepsilon}{3}\}.$$

Thus, we have

$$\begin{aligned} \|f(x_0) - f(x), z\| &\leq \|f(x_0) - f_n(x_0), z\| + \|f_n(x_0) - f_n(x), z\| + \|f_n(x) - f(x), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

So f is continuous on X*. Now assume that* X *is compact. Let* $\varepsilon > 0$*. Since* X *is compact, it follows that f is uniformly continuous and* f_n*'s are equi-uniformly continuous on X. So, pick* $\delta > 0$ *such that for any* $x, x^{'} \in X$ *and for each nonzero* $z \in Y$ *with* $\|x - x^{'}, z\| < \delta$*. Then by equi-uniformly continuouty we have for each nonzero* $z \in Y$ $\|f_n(x) - f_n(x'), z\| < \frac{\varepsilon}{3}$ *and* $\|f_(x) - f(x'), z\| < \frac{\varepsilon}{3}$*. By the compactness of X, we can choose a finite subcover*

$$B_{x_1}(\delta), B_{x_2}(\delta), B_{x_3}(\delta), ..., B_{x_k}(\delta)$$

from the cover $\{B_x(\delta)\}_{x \in X}$ *of* X*. Using* $f_n \Longrightarrow_{\mathcal{I}}^{\|.,.\|_Y} f$ *and a set* $M \in \mathcal{I}$ *such that for each nonzero* $z \in Y$*,* $\|f_n(x_i) - f_(x_i), z\| < \frac{\varepsilon}{3}, i \in \{1, 2, ..., k\}$ *for all* $n \notin M$*.*

Let $n \notin M$ $x \in X$. Thus, $x \in B_{x_i}(\delta)$ for since $i \in \{1, 2, ..., k\}$. Hence,for each nonzero $z \in Y$ we have

$$\begin{aligned} \|f_n(x) - f(x), z\| &\leq \|f_n(x) - f_n(x_i), z\| + \|f_n(x_i) - f(x_i), z\| + \|f(x_i) - f(x), z\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}, \end{aligned}$$

and so $f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}} f$ on X.

Definition 3.2. *$\{f_n\}$ is said to be $\mathcal{I}$-uniformly Cauchy if for every $\varepsilon > 0$ here exists $s = s(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,*

$$\{n \in \mathbb{N} : \|f_n(x) - f_s(x), z\} \geq \varepsilon\} \in \mathcal{I} \quad , for\ each\ x \in X.$$

Now, we give $\mathcal{I}$-Cauchy criteria for $\mathcal{I}$-uniformly Convergence.

Theorem 3.4. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be admissible ideal with the property (AP) and let f_n be a sequence of bounded function on X. Then f_n is $\mathcal{I}$-uniformly convergent if and only if for every $\varepsilon > 0$. There is a $n(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$*

$$\|f_n(x) - f_s(x), z\} < \varepsilon\} \notin \mathcal{I} \tag{3.2}$$

Note 3.5. *The sequence $\{f_n\}$ satisfying property 3.2 is said to be $\mathcal{I}$-uniformly Cauchy on X.*

Proof 3.4. *Assume that $\{f_n\}$ converges $\mathcal{I}$-Uniformly to a function f defined on X.Let $\varepsilon > 0$ then for each nonzero $z \in Y$,we have*

$$\{n : \|f_n(x) - f(x), z\} < \varepsilon\} \notin \mathcal{I}$$

for each $x \in X$.We can select on $n(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$

$$\{n : \|f_{n(\varepsilon)}(x) - f(x), z\} < \varepsilon\} \notin \mathcal{I}$$

for each $x \in X$.The triangle inequality yields that

$$\{n : \|f_n(x) - f_{n(\varepsilon)}(x), z\} < \varepsilon\} \notin \mathcal{I}.$$

Since ε is arbitrary , $\{f_n\}$ is $\mathcal{I}$-uniformly Cauchy on it.

Conversely,assume that $\{f_n\}$ is $\mathcal{I}$-uniformly Cauchy on X.Let $x \in X$ be fixed by 3.2 for every $\varepsilon > 0$ there is on $n(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,

$$\{n : \|f_n(x) - f_{n(\varepsilon)}(x), z\} < \varepsilon\} \notin \mathcal{I}.$$

Hence $\{f_n(x)\}$ is $\mathcal{I}$-Cauchy ,so by Lemma 2.4 we have that $\{f_n(x)\}$ is $\mathcal{I}$-convergent to $f(x)$.Then $f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}} f$ on X.

Now we shall show that this convergence must be uniform.Note that since $\mathcal{I}$ satisfy the condition (AP),by 3.2 there is a $K \notin \mathcal{I}$ such that for each nonzero $z \in Y$,

$$\|f_n(x) - f_{n(\varepsilon)}(x), z\} < \frac{\varepsilon}{2}$$

for all $n(\varepsilon) \in \mathbb{N}$ and $n \in K$. So for every $\varepsilon > 0$. Here is $K \notin \mathcal{I}$ and $n(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$,

$$\|f_n(x) - f(x), z\| < \varepsilon \tag{3.3}$$

for all $n \geq n(\varepsilon)$ and $n \in K$ and for each $x \in X$.Fixing n on applying the limit operator in 3.3 , we conclude that for every $\varepsilon > 0$ there is a $K \notin \mathcal{I}$ and $n(\varepsilon) \in \mathbb{N}$ such that for each nonzero $z \in Y$. $\|f_n(x) - f(x), z\| < \varepsilon$ for all $n \geq n_0$ and for each $x \in X$.Hence $f_n \Longrightarrow^{\|.,.\|_y}{}_{\mathcal{I}^*} f$ on X, consequently $f_n \overset{\|.,.\|_Y}{\Longrightarrow}_{\mathcal{I}} f$ on X.

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Definition 3.3. *The sequence of functions $\{f(x)\}$ is said to be $\mathcal{I}^*$-uniformly Cauchy sequence, if there exist a set*

$$M \in \mathcal{F}(\mathcal{I}), M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}$$

such that for each subsequence $\{f_M\} = \{f_{mk}\}$ *is a Cauchy Sequence ,i,e., for each nonzero* $z \in Y$,

$$\lim_{k,p\to\infty} \|f_{mk}(x) - f_{mp}(x), z\| = 0$$

for each $x \in X$.

Theorem 3.6. *If* $\{f_n\}$ *is a* $\mathcal{I}^*$*-uniformly Cauchy sequence than is* $\mathcal{I}$*-uniformly Cauchy sequence in 2-normed spaces.*

Proof 3.5. *Let* $\{f_n\}$ *is a* $\mathcal{I}^*$ *-uniformly Cauchy sequence is 2-normed spaces then, by definition there exist the set*

$$M = \{m_1 < m_2 < ... < m_k < ...\} \subset \mathbb{N}, M \in \mathcal{F}(\mathcal{I})$$

such that for every $\varepsilon > 0$ *and for each nonzero* $z \in Y$,

$$\|f_{nk}(x) - f_{np}(x), z\| < \varepsilon$$

for each $x \in X$ *and* $k_p > k_0 = k_0(\varepsilon, x)$.*Let* $N = N(\varepsilon, X) = m_{k_0} + 1$.*Then for* $\varepsilon > 0$ *and for each nonzero* $z \in Y$ *we have*

$$\|f_{nk}(x) - f_N(x), z\| < \varepsilon$$

for each $x \in X$ *and* $k > k_0$. *Now put* $H = \frac{\mathbb{N}}{M}$.*It is clear that* $H \in \mathcal{I}$ *and*

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f_N(x)\| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < ... < m_{k_0}\}.$$

Since $\mathcal{I}$*s an admissible ideal then,*$H \cup \{m_1 < m_2 < ... < m_{k_0} \in \mathcal{I}$. *Hence for every* $\varepsilon > 0$ *we find* $N = N(\varepsilon, X)$ *such that* $A(\varepsilon, z) \in \mathcal{I}$*,i.e.,*$\{f_n\}$ *is* $\mathcal{I}$ *uniformly Cauchy sequence.*

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