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$\mathcal{I}$-Lacunary Statistical Convergence in Intuitionistic Fuzzy Normed Spaces

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Abstract

In this study, first, we investigate the notions of $\mathcal{I}$-lacunary statistical convergence and strongly $\mathcal{I}$-lacunary convergence with regards to the intuitionistic fuzzy norm (IFN for short) (μ,ν). Then, we investigate relationships among this new concepts and make important observations about them. Futhermore, we examine the relations among $\mathcal{I}$-lacunary statistical convergence and $\mathcal{I}$-statistical convergence in terms of IFN (μ,ν) in the corresponding intuitionistic fuzzy normed space.

Sezgisel Fuzzy Normlu Uzaylarda $\mathcal{I}$-Lacunary İstatiksel Yakınsaklık

Anahtar kelimeler

İdeal yakınsaklık; $\mathcal{I}$-lacunary istatistiksel yakınsaklık; Sezgisel fuzzy normlu uzay; Banach uzayı; Cauchy dizisi

Öz

Bu çalışmada, ilk olarak (μ,ν) sezgisel normuna göre $\mathcal{I}$-lacunary istatistiksel yakınsaklık ve kuvvetli $\mathcal{I}$-lacunary yakınsaklık kavramları tanımlandı. Daha sonra bu kavramlar arasındaki ilişkiler incelendi ve bu kavramlar üzerine önemli gözlemler yapıldı. Bununla birlikte, ilgili sezgisel fuzzy normlu uzayda (μ,ν) sezgisel normuna göre $\mathcal{I}$-lacunary istatistiksel yakınsaklık ile $\mathcal{I}$-istatistiksel yakınsaklık arasındaki ilişkiler incelendi.

1. Introduction and Definitions

Statistical convergence of a real number sequence was firstly originated by Fast (1951). It became a notable topic in summability theory after the work of Fridy (1985) and Šalát (1980).

Theory of $\mathcal{I}$-convergence of sequences was given by Kostyrko et al. (2000). It is known that $\mathcal{I}$-convergence is a significant generality of statistical convergence. The other studies of ideals can be done by Das and Ghosal (2010), Das et al. (2011) and Savaş and Das (2011).

Using lacunary sequence, Fridy and Orhan (1993a) examined lacunary statistical convergence. Subsequently, it was studied by Fridy and Orhan (1993b), Li (2000), Mursaleen and Mohiuddine (2009), Bakery (2014). Lacunary ideal convergence for real sequences was investigated by Tripathy et al. (2012). The other studies of this concept can be examined by Yamancı and Gürdal (2014a, 2014b), Ulusu and Dündar (2014), Kişi and Dündar (2018), Savaş et al. (2019).

Several authors have studied invariant convergent sequences (see, Ulusu et al. (2018), Pancaroğlu Akın et al. (2018)).

Fuzzy set theory has become an important working area after the study of Zadeh (1965). Atanassov (1986) investigated intuitionistic fuzzy set; this concept was utilized by Atanassov et al. (2002) in the study of decision-making problems. The idea of an intuitionistic fuzzy metric space was put forward by Park (2004). Saadati and Park (2006) initially introduced the concept of an intuitionistic fuzzy normed space (IFNS). Several studies of the convergence of sequences in some normed linear spaces with a fuzzy setting might be revealed by the research of the Hosseini et al. (2008), Debnath (2012), Sen and Debnath (2011), Debnath and Sen (2014), Debnath (2015). Karakuş et al. (2008) defined statistical convergence in IFNS. Savaş and Gürdal (2015) studied the concept of $\mathcal{I}-[V,\lambda]$-summability and $\mathcal{I}-\lambda$-statistical convergence with regards to the IFN (μ,ν).

2. Main Results

Definition 2.1 Let $(X, \Phi, \omega, *, \Theta)$ be an IFNLS. A sequence $x = (x_k)$ is called to be $\mathcal{J}$-lacunary statistically convergent to $\xi \in X$ in terms of IFN (μ, ν), and is demonstrated by $S_\theta(\mathcal{J})^{(\mu,\nu)} - lim x = \xi$ or $x_k \xrightarrow{(\mu,\nu)} \xi\big(S_\theta(\mathcal{J})\big)$, if for every $\varepsilon > 0$, every $\delta > 0$, and $t > 0$,

$$\left\{r \in N:\ \frac{1}{h_r}|\{k \in I_r\ :\ \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \delta\right\} \in I.$$

We consider $\mathcal{J}_f$ as the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{J}_f$ is an admissible ideal and $\mathcal{J}$-lacunary statistically convergence coincides with lacunary statistically convergence introduced in the study of Mursaleen and Mohiuddine (2009).

Definition 2.2 Let $(X, \Phi, \omega, *, \Theta)$ be an IFNLS. $x = (x_k)$ in X is called strongly $\mathcal{J}$-lacunary convergent to $\xi \in X$ or $N_\theta(\mathcal{J})$-convergent to $\xi \in X$ in terms of IFN (μ, ν), and is demonstrated by $x_k \xrightarrow{(\mu,\nu)} \xi\big(N_\theta(\mathcal{J})\big)$, if for every $\delta > 0$, and $t > 0$,

$$\left\{\boldsymbol{r \in N}:\ \frac{\boldsymbol{1}}{\boldsymbol{h_r}}\sum_{\boldsymbol{k\in I_r}} \boldsymbol{\mu(x_k - \xi, t) \leq 1 - \delta} \text{ or } \frac{\boldsymbol{1}}{\boldsymbol{h_r}}\sum_{\boldsymbol{k\in I_r}} \boldsymbol{\nu(x_k - \xi, t) \geq \delta}\right\} \in \boldsymbol{I}.$$

Theorem 2.1 Let $(X, \Phi, \omega, *, \Theta)$ be an IFNLS, and $x = \{x_k\} \in X$, then

1.
   a. If $x_k \xrightarrow{(\mu,\nu)} \xi\big(N_\theta(\mathcal{J})\big)$ then $x_k \xrightarrow{(\mu,\nu)} \xi\big(S_\theta(\mathcal{J})\big)$.
   b. If $x \in m(X)$, the space of all bounded sequences of X and $x_k \xrightarrow{(\mu,\nu)} \xi\big(S_\theta(\mathcal{J})\big)$ then $x_k \xrightarrow{(\mu,\nu)} \xi\big(N_\theta(\mathcal{J})\big)$.
2. $S_\theta(\mathcal{J})^{(\mu,\nu)} \cap m(X) = N_\theta(\mathcal{J})^{(\mu,\nu)} \cap m(X)$.

Proof. 1 (a) By assumption, for each $\varepsilon > 0$, $\delta > 0$, and t>0, let $x_k \xrightarrow{(\mu,\nu)} \xi\big(N_\theta(\mathcal{J})\big)$. We have

$$\sum_{k\in I_r}\big(\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)\big) \geq \sum_{\substack{k\in I_r \\ \mu(x_k-\xi,t)\leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t)\geq\varepsilon}}\big(\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)\big)$$
$$\geq \varepsilon.\,|\{k \in I_r\ :\ \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}|.$$

Then, observe that

$$\frac{1}{h_r}|\{k \in I_r\ :\ \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \delta$$
$$\Rightarrow \frac{1}{h_r}\sum_{k\in I_r}\mu(x_k - \xi, t) \leq (1 - \varepsilon)\delta \text{ or } \frac{1}{h_r}\sum_{k\in I_r}\nu(x_k - \xi, t) \geq \varepsilon\delta,$$

which implies

$$\left\{r \in N:\ \frac{1}{h_r}|\{k \in I_r\ :\ \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \delta\right\}$$
$$\subset \left\{r \in N:\ \frac{1}{h_r}\left\{\sum_{k\in I_r}\mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \sum_{k\in I_r}\nu(x_k - \xi, t) \geq \varepsilon\right\} \geq \varepsilon\delta\right\}.$$

Since $x_k \xrightarrow{(\mu,\nu)} \xi\big(N_\theta(\mathcal{J})\big)$, we see that $x_k \xrightarrow{(\mu,\nu)} \xi\big(S_\theta(\mathcal{J})\big)$, hence we get the result.

1 (b) Assume that $x_k \xrightarrow{(\mu,\nu)} \xi\big(S_\theta(\mathcal{J})\big)$ and $x \in l_\infty^{(\mu,\nu)}$. The inequalities $\mu(x_k - \xi, t) \geq 1 - M$ or $\nu(x_k - \xi, t) \leq M$ hold for all k. For $\varepsilon > 0$, we get

$$\frac{1}{h_r}\sum_{k\in I_r}\big(\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)\big)$$
$$= \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \mu(x_k-\xi,t)\leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t)\geq\varepsilon}}\big(\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)\big)$$
$$+ \frac{1}{h_r}\sum_{\substack{k\in I_r \\ \mu(x_k-\xi,t)> 1-\varepsilon \text{ or } \nu(x_k-\xi,t)<\varepsilon}}\big(\mu(x_k - \xi, t) \text{ or } \nu(x_k - \xi, t)\big)$$
$$\leq \frac{M}{h_r}|\{k \in I_r\ :\ \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| + \varepsilon.$$

Note that

$$A_{\mu,\nu}(\varepsilon, t) = \left\{r \in N:\ \frac{1}{h_r}|\{k \in I_r\ :\ \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}| \geq \frac{\varepsilon}{M}\right\}$$

belongs to $\mathcal{J}$. If $r \in \big(A_{\mu,\nu}(\varepsilon, t)\big)^c$ then we have

$$\frac{1}{h_r}\sum_{k\in I_r}\mu(x_k - \xi, t) > 1 - 2\varepsilon \text{ or } \frac{1}{h_r}\sum_{k\in I_r}\nu(x_k - \xi, t) < 2\varepsilon.$$

Now

$$T_{\mu,\nu}(\varepsilon, t) = \left\{r \in N:\ \frac{1}{h_r}\sum_{k\in I_r}\mu(x_k - \xi, t) \leq 1 - 2\varepsilon \text{ or } \frac{1}{h_r}\sum_{k\in I_r}\nu(x_k - \xi, t) \geq 2\varepsilon\right\}.$$

Hence, $T_{\mu,\nu}(\varepsilon,t) \subseteq B_{\mu,\nu}(\varepsilon,t)$ and so, by the definition of an ideal,

$$T_{\mu,\nu}(\varepsilon,t) \in \mathcal{J}.$$

Therefore, we conclude that $x_k \xrightarrow{(\mu,\nu)} \xi(N_\theta(\mathcal{J}))$.

2. This readily follows from 1 (a) and 1 (b).

Theorem 2.2 $(X,\Phi,\omega,*,\Theta)$ be an IFNLS. If $\theta = \{k_r\}$ be a lacunary sequence with $liminf\, q_r > 1$, then

$$x_k \xrightarrow{(\mu,\nu)} \xi(S(\mathcal{J})) \Rightarrow x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J})).$$

Proof. Suppose first that $liminf\, q_r > 1$, then there exists a $\alpha > 0$ such that $q_r \geq 1+\alpha$ for quite large r, which emphasize that

$$\frac{h_r}{k_r} \geq \frac{\alpha}{1+\alpha}.$$

If $x_k \xrightarrow{(\mu,\nu)} \xi(S(\mathcal{J}))$, then for every $\varepsilon > 0$, for each $x \in X$ and for quite large r, we have

$$\frac{1}{k_r}|\{k \leq k_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$\geq \frac{1}{k_r}|\{k \in I_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$\geq \frac{\alpha}{1+\alpha}\frac{1}{h_r}|\{k \in I_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$

For any $\delta > 0$, we get

$$\left\{r \in N: \ \frac{1}{h_r}|\{k \in I_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}| \geq \delta\right\}$$
$$\subseteq \left\{r \in N: \ \frac{1}{k_r}|\{k \leq k_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}| \geq \frac{\delta\alpha}{(\alpha+1)}\right\}.$$

This shows that $x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J}))$.

To prove of the following theorem, we accept that the lacunary sequence θ satisfies the situation that for any set $C \in F(\mathcal{J})$, $\bigcup\{n: k_{r-1} < n \leq k_r, r \in C\} \in F(\mathcal{J})$.

Theorem 2.3 $(X,\Phi,\omega,*,\Theta)$ be an IFNLS. If $\theta = \{k_r\}$ be a lacunary sequence with $limsup\, q_r <\propto$, then

$$x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J})) \Rightarrow x_k \xrightarrow{(\mu,\nu)} \xi(S(\mathcal{J})).$$

Proof. If $limsup\, q_r < \infty$ then without any loss of generalization we can accept that there exists a $0 < M < \infty$ such that $q_r < M$ for all $r \geq 1$. Suppose that $x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J}))$, and for $\varepsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{r \in N: \ \frac{1}{h_r}|\{k \in I_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}| < \delta\right\}$$

and

$$T = \left\{n \in N: \ \frac{1}{n}|\{k \leq n \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}| < \delta_1\right\}.$$

It is clear from our assumption that $C \in F(\mathcal{J})$, the filter associated with the ideal $\mathcal{J}$. Observe that

$$K_j = \frac{1}{h_j}|\{k \in I_j \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}| < \delta$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n \leq k_r$ for some $r \in C$. Now

$$\frac{1}{n}|\{k \leq n \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$\leq \frac{1}{k_{r-1}}|\{k \leq k_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$= \frac{1}{k_{r-1}}|\{k \in I_1 \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$+\frac{1}{k_{r-1}}|\{k \in I_2 \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$+\ldots+\frac{1}{k_{r-1}}|\{k \in I_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$= \frac{k_1}{k_{r-1}}\frac{1}{h_1}|\{k \in I_1 \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$+\frac{k_2-k_1}{k_{r-1}}\frac{1}{h_2}|\{k \in I_2 \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$+\ldots+\frac{k_r-k_{r-1}}{k_{r-1}}\frac{1}{h_r}|\{k \in I_r \ : \ \mu(x_k-\xi,t) \leq 1-\varepsilon \text{ or } \nu(x_k-\xi,t) \geq \varepsilon\}|$$
$$= \frac{k_1}{k_{r-1}}K_1 + \frac{k_2-k_1}{k_{r-1}}K_2 + \ldots + \frac{k_r-k_{r-1}}{k_{r-1}}K_r$$
$$\leq \{sup_{j\in C} K_j\}\frac{k_r}{k_{r-1}}$$
$$< M\delta.$$

Choosing $\delta_1 = \frac{\delta}{M}$ and given the fact that

$$\bigcup\{n: k_{r-1} < n \leq k_r, r \in C\} \subset T$$

where $C \in F(\mathcal{J})$ it follows from our assumption on θ that the set $T \in F(\mathcal{J})$.

Combination of Theorem 2.2 and Theorem 2.3 we have

Theorem 2.4 $(X,\Phi,\omega,*,\Theta)$ be an IFNLS. If $\theta = \{k_r\}$ be a lacunary sequence with $1 < liminf\, q_r \leq limsup\, q_r < \infty$, then

$$x_k \xrightarrow{(\mu,\nu)} \xi(S_\theta(\mathcal{J})) \Leftrightarrow x_k \xrightarrow{(\mu,\nu)} \xi(S(\mathcal{J}))$$

Proof. This is an immediate consequence of Theorem 2.2 and Theorem 2.3.

Theorem 2.5 $(X,\Phi,\omega,*,\Theta)$ be an IFNLS such that

$$\frac{1}{4}\varepsilon_n\Theta\frac{1}{4}\varepsilon_n<\frac{1}{2}\varepsilon_n$$

and

$$\left(1-\tfrac{1}{4}\varepsilon_n\right)*\left(1-\tfrac{1}{4}\varepsilon_n\right)>1-\tfrac{1}{2}\varepsilon_n.$$

If X is a Banach space, then $S_\theta(\mathcal{I})^{(\mu,\nu)}\cap m(x)$ is a closed subset of $m(x)$.

Proof. We first assume that $(x^n)\subset S_\theta(\mathcal{I})^{(\mu,\nu)}\cap m(x)$ is a convergent sequence that it converges to $x\in m(x)$. We have to show that $x\in S_\theta(\mathcal{I})^{(\mu,\nu)}\cap m(x)$. Suppose that $x^n\xrightarrow{(\mu,\nu)}L_n(S_\theta(\mathcal{I}))$ for all $n\in\mathbb{N}$. Consider a sequence $\{\varepsilon_n\}$ of reducing positive numbers converging to zero. We can determinate an $n\in\mathbb{N}$ such that $sup_j\nu(x-x^j,t)<\frac{1}{4}\varepsilon_n$ for all $j\geq n$. Choose $0<\beta<\frac{1}{5}$. Let

$$A_{\mu,\nu}(\varepsilon_n,t)=\left\{\begin{array}{c} r\in N:\ \frac{1}{h_r}|\{k\in I_r\ : \\ \mu(x_k^n-L_n,t)\leq 1-\frac{\varepsilon_n}{4}\text{ or} \\ \nu(x_k^n-L_n,t)\geq\frac{\varepsilon_n}{4}\}|<\delta\end{array}\right\}$$

belongs to $F(\mathcal{I})$ and

$$B_{\mu,\nu}(\varepsilon_n,t)=\left\{\begin{array}{c} r\in N:\ \frac{1}{h_r}|\{k\in I_r\ : \\ \mu(x_k^{n+1}-L_{n+1},t)\leq 1-\frac{\varepsilon_n}{4}\text{ or} \\ \nu(x_k^{n+1}-L_{n+1},t)\geq\frac{\varepsilon_n}{4}\}|<\delta\end{array}\right\}$$

belongs to $F(\mathcal{I})$. Since $A_{\mu,\nu}(\varepsilon_n,t)\cap B_{\mu,\nu}(\varepsilon_n,t)\in F(\mathcal{I})$ and $\emptyset\notin F(\mathcal{I})$, we can select $r\in A_{\mu,\nu}(\varepsilon_n,t)\cap B_{\mu,\nu}(\varepsilon_n,t)$. Then

$$\begin{aligned}\frac{1}{h_r}\Big|\Big\{k\in I_r\ :\ \mu(x_k^n-L_n,t)\leq 1-\frac{\varepsilon_n}{4}\\ \text{or }\nu(x_k^n-L_n,t)\geq\frac{\varepsilon_n}{4}\ \vee\\ \mu(x_k^{n+1}-L_{n+1},t)\leq 1-\frac{\varepsilon_n}{4}\text{ or}\\ \nu(x_k^{n+1}-L_{n+1},t)\geq\frac{\varepsilon_n}{4}\Big\}\Big|\leq 2\delta<1.\end{aligned}$$

Since $h_r\to\infty$ and $A_{\mu,\nu}(\varepsilon_n,t)\cap B_{\mu,\nu}(\varepsilon_n,t)\in F(\mathcal{I})$ is finite, we can select the above r so that $h_r>5$. So, there must exist a $k\in I_r$ for which we get at the same time, $\mu(x_k^n-L_n,t)>1-\frac{\varepsilon_n}{4}$ or $\nu(x_k^n-L_n,t)<\frac{\varepsilon_n}{4}$ and $\mu(x_k^{n+1}-L_{n+1},t)>1-\frac{\varepsilon_n}{4}$ or $\nu(x_k^{n+1}-L_{n+1},t)<\frac{\varepsilon_n}{4}$. For a given $\varepsilon_n>0$ select $\frac{\varepsilon_n}{2}$ such that $\left(1-\frac{\varepsilon_n}{2}\right)*\left(1-\frac{\varepsilon_n}{2}\right)>1-\varepsilon_n$ and $\frac{1}{2}\varepsilon_n\Theta\frac{1}{2}\varepsilon_n<\varepsilon_n$. Then, it follows that

$$\begin{aligned}\nu\left(L_n-x_k^n,\frac{t}{2}\right)\Theta\nu\left(L_{n+1}-x_k^{n+1},\frac{t}{2}\right)\\ \leq\frac{\varepsilon_n}{4}\Theta\frac{\varepsilon_n}{4}<\frac{\varepsilon_n}{2}\end{aligned}$$

and

$$\begin{aligned}\nu(x_k^n-x_k^{n+1},t)&\leq sup_n\,\nu\left(x-x^n,\frac{t}{2}\right)\\ &\quad\Theta\, sup_n\,\nu\left(x-x^{n+1},\frac{t}{2}\right)\\ &\leq\frac{1}{4}\varepsilon_n\Theta\frac{1}{4}\varepsilon_n<\frac{1}{2}\varepsilon_n.\end{aligned}$$

Hence, we have

$$\begin{aligned}\nu(L_n-L_{n+1},t)&\leq\Big[\nu\left(L_n-x_k^n,\frac{t}{3}\right)\\ &\quad\Theta\nu\left(x_k^{n+1}-L_{n+1},\frac{t}{3}\right)\\ &\quad\Theta\nu\left(x_k^n-x_k^{n+1},\frac{t}{3}\right)\Big]\\ &\leq\frac{1}{2}\varepsilon_n\Theta\frac{1}{2}\varepsilon_n<\varepsilon_n\end{aligned}$$

and likewise $\mu(L_n-L_{n+1},t)>1-\varepsilon_n$. This implies that $\{L_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and let $L_n\to L\in X$ as $n\to\infty$. We have to show that $x\xrightarrow{(\mu,\nu)}L(S_\theta(\mathcal{I}))$. For any $\varepsilon>0$ and $t>0$, choose $n\in\mathbb{N}$ such that $\varepsilon_n<\frac{\varepsilon}{4}$, $sup_n\,\nu(x-x^n,t)<\frac{1}{4}\varepsilon$, $\mu(L_n-L,t)>1-\frac{1}{4}\varepsilon$ or $\nu(L_n-L,t)<\frac{1}{4}\varepsilon$. Now, since

$$\begin{aligned}&\frac{1}{h_r}|\{k\in I_r\ :\ \nu(x_k-L,t)\geq\varepsilon\}|\\ &\leq\frac{1}{h_r}\Big|\Big\{k\in I_r\ :\ \nu\left(x_k-x_k^n,\frac{t}{3}\right)\Theta\\ &\left[\nu\left(x_k^n-L_n,\frac{t}{3}\right)\Theta\nu\left(L_n-L,\frac{t}{3}\right)\right]\geq\varepsilon\Big\}\Big|\\ &\leq\frac{1}{h_r}\Big|\Big\{k\in I_r\ :\ \nu\left(x_k^n-L_n,\frac{t}{3}\right)\geq\frac{\varepsilon}{2}\Big\}\Big|\end{aligned}$$

and similarly

$$\begin{aligned}&\frac{1}{h_r}|\{k\in I_r\ :\ \mu(x_k-L,t)\leq 1-\varepsilon\}|\\ &>\frac{1}{h_r}\Big|\Big\{k\in I_r\ :\ \mu\left(x_k^n-L,\frac{t}{3}\right)\leq 1-\frac{\varepsilon}{2}\Big\}\Big|.\end{aligned}$$

It follows that

$$\begin{aligned}&\Big\{r\in N:\ \tfrac{1}{h_r}|\{k\in I_r\ :\ \mu(x_k-L,t)\leq 1-\varepsilon\\ &\text{or }\nu(x_k-L,t)\geq\varepsilon\}|\geq\delta\Big\}\\ &\subset\Big\{r\in N:\ \tfrac{1}{h_r}\Big|\Big\{k\in I_r\ :\ \mu\left(x_k^n-L,\tfrac{t}{3}\right)\leq\\ &1-\tfrac{\varepsilon}{2}\text{ or }\nu\left(x_k^n-L,\tfrac{t}{3}\right)\geq\tfrac{\varepsilon}{2}\Big\}\Big|\geq\delta\Big\}\end{aligned}$$

for given $\delta>0$. Hence, we have $x\xrightarrow{(\mu,\nu)}L(S_\theta(\mathcal{I}))$.

3. References

Atanassov, K.T., 1986. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, **20**, 87-96.

Atanassov, K., Pasi, G. and Yager, R., 2002. Intuitionistic fuzzy interpretations of multi-person multicriteria decision making. *Proceedings of First International IEEE Symposium Intelligent Systems*, **1**, 115-119.

Bakery, A.A., 2014. Operator ideal of Cesàro type sequence spaces involving lacunary sequence. *Abstract and Applied Analysis*, **2014**, 6 pp. Article ID 419560.

Das, P. and Ghosal, S., 2010. Some further results on $\mathcal{I}$-Cauchy sequences and condition (AP). *Computers & Mathematics with Applications,* **59**, 2597-2600.

Das, P. and Debnath, P., 2011. On generalizations of certain summability methods using ideals. *Applied Mathematics Letters*, **24**, 1509-1514.

Debnath, P., 2012. Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces. *Computers & Mathematics with Applications*, **63**, 708-715.

Debnath, P. and Sen, M., 2014. Some completeness results in terms of infinite series and quotient spaces in intuitionistic fuzzy *n*-normed linear spaces. *Journal of Intelligent & Fuzzy Systems*, **26**, 975-782.

Debnath, P., 2015. Results on lacunary difference ideal convergence in intuitionistic fuzzy normed linear spaces. *Journal of Intelligent & Fuzzy Systems,* **28**, 1299-1306.

Fast, H., 1951. Sur la convergence statistique. *Colloquium Mathematicum*, **2**, 241-244.

Fridy, JA., 1985. On statistical convergence. *Analysis*, **5**, 301-313.

Fridy, J.A. and Orhan, C., 1993a. Lacunary statistical convergence. *Pacific Journal of Mathematics*, **160**(1), 43-51.

Fridy, J.A. and Orhan, C., 1993b. Lacunary statistical summability. *Journal of Mathematical Analysis and Applications*, **173**, 497-504.

Hosseini, S.B., Regan, D.O. and Saadati, R., 2007. Some results of intuitionistic fuzzy spaces. *Iranian Journal of Fuzzy Systems,* **4**(1), 53-64.

Karakuş, S., Demirci, K. and Duman, O*., 2008.* Statistical convergence on intuitionistic fuzzy normed spaces. *Chaos, Solitons & Fractals,* **35***,* 763-769.

Kişi, Ö. and Dündar, E., 2018. Rough $\mathcal{I}_2$-lacunary statistical convergence of double sequences. *Journal of Inequalities and Applications*, 2018:**230**, 1-16.

Kostyrko, P., Šalát, T. and Wilczynki, W., 2000. $\mathcal{I}$-convergence. *Real Analysis Exchange*, **26**(2), 669-686.

Li, J., 2000. Lacunary statistical convergence and inclusion properties between lacunary methods. *International Journal of Mathematics and Mathematical Sciences*, **23**(3), 175-180.

Mursaleen, M. and Mohiuddine, S.A., 2009. On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space. *Journal of Computational and Applied Mathematics*, **233**(2), 142-149.

Pancaroğlu, Akın, N., Dündar, E. and Ulusu, U., 2018. Asymptotically $\mathcal{I}_{\sigma\theta}$-statistical equivalence of sequences of sets defined by a modulus functions. *Sakarya University Journal of Science***, 22**(6), 1857-1862.

Park, J.H., 2004. Intuitionistic fuzzy metric spaces. *Chaos, Solitons & Fractals*, **22**, 1039-1046.

Saadati, R. and Park, J.H., 2006. On the intuitionistic fuzzy topological spaces. *Chaos, Solitons & Fractals,* **27**, 331-344.

Savaş, E. and Das, P., 2011. A generalized statistical convergence via ideals. *Applied Mathematics Letters,* **24**, 826–30.

Savaş, E. and Gürdal, M., 2015. A generalized statistical convergence in intuitionistic fuzzy normed spaces. *Science Asia,* **41**, 289-294.

Savaş, E., Yamancı, U. and Gürdal, M., 2019. $\mathcal{I}$-lacunary statistical convergence of weighted g via modulus functions in 2-normed spaces. *Communications*

Faculty of Sciences University of Ankara Series A1-Mathematics and Statistics, **68**(2), 2324-2332.

Šalát, T., 1980. On statistically convergent sequences of real numbers. *Mathematica Slovaca*, **30**, 139-150.

Sen, M. and Debnath, P., 2011. Lacunary statistical convergence in intuitionistic fuzzy *n* normed spaces. *Mathematical and Computer Modelling*, **54**(11-12), 2978-2985.

Sen, M., Savaş, E. and Ghosal, SKr., 2011. On generalizations of certain summability methods using ideals. *Applied Mathematics Letters*, **24**, 1509-1514.

Tripathy, B.C., Hazarika, B. and Choudhary, B., 2012. Lacunary $\mathcal{I}$-convergent sequences. *Kyungpook Mathematical Journal*, **52**(4), 473-482.

Ulusu, U., Dündar, E. and Nuray, F., 2018. Lacunary $\mathcal{I}_2$-invariant convergence and some properties. *International Journal of Analysis and Applications,* **16**(3), 317-327.

Ulusu, U. and Dündar, E., 2018. $\mathcal{I}$-lacunary statistical convergence of sequences of sets. *Filomat*, **28**(8), 1567-1574.

Yamancı, U. and Gürdal, M., 2014a. $\mathcal{I}$-statistical convergence in 2-normed space. *Arab Journal of Mathematical Sciences*, **20**(1), 41-47.

Yamancı, U. and Gürdal, M., 2014b. $\mathcal{I}$-statistically pre-Cauchy double sequences. *Global Journal of Mathematical Analysis*, **2**(4), 297-303.

Zadeh, L.A., 1965. Fuzzy sets. *Information and Control*, **8**, 338-353.