Asymptotically $\mathcal{I}_2^{\theta}$-Equivalence of Double Sequences Defined by Modulus Functions

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Abstract

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast (1951) and Schoenberg (1959). Fridy and Orhan (1993) studied lacunary statistical convergence. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. (2000) as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$. The idea of $\mathcal{I}_2$-convergence and some properties of this convergence were studied by Das et al. (2008).

Marouf (1993) peresented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson (2003) presented asymptotically statistical equivalent sequences for nonnegative summability matrices. Dündar et al. (in review) defined asymptotically $\mathcal{I}_2^{\sigma}$-equivalent, asymptotically invariant equivalent, strongly asymptotically invariant equivalent and p-strongly asymptotically invariant equivalent for double sequences. Hazarika and Kumar (2013) studied on asymptotically double lacunary statistical equivalent sequences in ideal context. Ulusu and Dündar (in press) introduced the concepts of asymptotically lacunary $\mathcal{I}_2$-invariant equivalence, asymptotically lacunary σ_2-equivalence and asymptotically lacunary invariant S_2-equivalence for double sequences. Modulus function was introduced by Nakano (1953). Maddox (1986), Pehlivan (1995) and many authors used a modulus function f to new some new concepts and inclusion theorems. Kumar and Sharma (2012) studied lacunary equivalent sequences by ideals and modulus function.

In this study, we present the notions of f-asymptotically $\mathcal{I}_2$-equivalence, strongly f-asymptotically $\mathcal{I}_2$-equivalence, f-asymptotically lacunary $\mathcal{I}_2$-equivalence and strongly f-asymptotically lacunary $\mathcal{I}_2$-equivalence for real double sequences and investigate some important relationships between them.

Key Words: Asymptotic equivalence, Lacunary equivalence, Double sequence, $\mathcal{I}_2$-equivalence, Modulus Function.

Özet

Bu çalışmada $\mathbb{N}$ pozitif tam sayıları $\mathbb{R}$ reel sayıları gösterir. Reel sayı dizilerinin yakınsaklığı genişletilerek oluşturulan istatistiksel yakınsaklık kavramı ile ilgili Fast (1951) ve Schoenberg (1959) çalışma yapmıştır. Fridy ve Orhan (1993) lacunary istatistiksel yakınsaklık kavramı ile ilgili bir çalışma yaptı. Lacunary istatistiksel yakınsaklık kavramı çift reel sayı dizilerine Mursaleen ve Edely (2003) tarafından genişletilmiştir. İstatistiksel yakınsaklığın bir genelleştirmesi olan $\mathcal{I}$-yakınsaklık Kostyrko vd. (2000) tarafından tanımlanmış olup, bu kavram $\mathbb{N}$ doğal sayılar kümesinin alt kümelerinin sınıfı olan $\mathcal{I}$ idealinin yapısına bağlıdır.

Marouf (1993) asimptotik denklik ve asimptotik regüler matris kavramlarını tanımladı. Asimptotik denklik ile ilgili birçok yazar çalışmalar yaptı. Patterson (2003) negatif olmayan toplanabilir matrisler için asimptotik istatistiksel denk diziler ile ilgili çalışma yaptı. Dündar vd. (incelemede) çift reel sayı dizileri için asimptotik $\mathcal{I}_2^{\sigma}$-denklik, asimptotik invariant denklik, kuvvetli asimptotik invariant denklik ve p-kuvvetli asimptotik invariant denklik kavramlarını tanımladı. Hazarika ve Kumar (2013) asimptotik ideal lacunary istatistiksel denk çift diziler ile ilgili çalışmalar yaptı. Ulusu ve Dündar (basımda) çift diziler için asimptotik lacunary $\mathcal{I}_2$-invariant denklik, asimptotik lacunary σ_2-denklik ve asimptotik lacunary invariant S_2-denklik kavramlarını tanımladı. Modülüs fonksiyonu ilk defa Nakano (1953) tarafından tanımlandı. Maddox (1986), Pehlivan (1995) ve birçok yazar tarafından f modülüs fonksiyonu kullanılarak bazı yeni kavramları ve sonuç teoremlerini içeren çalışmalar yapıldı. Modülüs fonsiyonunu kullanılarak lacunary ideal denk diziler ile ilgili Kumar ve Sharma (2012) tarafından bir çalışma yapıldı.

Bu çalışmada, reel çift sayı dizileri için f-asimptotik $\mathcal{I}_2$-denklik, kuvvetli f-asimptotik $\mathcal{I}_2$-denklik, f-asimptotik lacunary $\mathcal{I}_2$-denklik ve kuvvetli f-asimptotik lacunary $\mathcal{I}_2$-denklik kavramları tanımlandı ve bu kavramlar arasındaki bazı önemli ilişkiler incelendi.

Anahtar Kelimeler: Asimptotik denklik, Lacunary denklik, Çift dizi, $\mathcal{I}_2$-denklik, Modulus fonksiyonu.

Introduction and Definitions

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers Das et al. [2] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of development have been made in this area after the works of [3-6].
Dündar et al. [7] defined asymptotically $\mathcal{I}_2^{\sigma}$-equivalent, asymptotically invariant equivalent, strongly asymptotically invariant equivalent and p-strongly asymptotically invariant equivalent for double sequences. Ulusu and Dündar [29] introduced the concepts of asymptotically lacunary $\mathcal{I}_2$-invariant equivalence, asymptotically lacunary σ_2-equivalence and asymptotically lacunary invariant S_2-equivalence for double sequences. Hazarika and Kumar [10] studied on asymptotically double lacunary statistical equivalent sequences in ideal context. Several authors define some new concepts and give inclusion theorems using a modulus function f (see, [11, 12]).

Now, we recall the basic concepts and some definitions (See [1, 9, 13, 14, 16-18, 21-24, 28]).

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty.$$

Throughout the paper, we let θ a lacunary sequence.

The double sequence $\theta_2 = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \quad and \quad j_0 = 0,\ \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad r, u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u,\ \ h_{ru} = h_r \bar{h}_u,\ \ I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \quad and \quad j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \quad and \quad q_u = \frac{j_u}{j_{u-1}}.$$

Throughout the paper, we let $\theta_2 = \{(k_r, j_u)\}$ a double lacunary sequence.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$. Throughout the paper we let $\mathcal{I}$ be an admissible ideal.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L, if for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $x \overset{S_L}{\sim} y$) and simply asymptotically statistical equivalent if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically equivalent of multiple

L with respect to the ideal $\mathcal{I}$ if for every $\varepsilon > 0$,

$$\left\{n \in \mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} |\frac{x_k}{y_k} - L| \geq \varepsilon\right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega)}{\sim} y_k$) and simply strongly asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly asymptotically lacunary equivalent of multiple L respect to the ideal $\mathcal{I}$ provided that for every $\varepsilon > 0$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k \in I_r} |\frac{x_k}{y_k} - L| \geq \varepsilon\right\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(N_\theta)}{\sim} y_k$) and simply strongly asymptotically lacunary $\mathcal{I}$-equivalent with respect to the ideal $\mathcal{I}$, if $L = 1$.

The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for every $\varepsilon > 0$ and $\gamma > 0$,

$$\{r \in \mathbb{N}: \frac{1}{h_r}|\{k \in I_r: |\frac{x_k}{y_k} - L| \geq \varepsilon\}| \geq \gamma\} \in \mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{S}_\theta)}{\sim} y_k$) and simply asymptotically lacunary $\mathcal{I}$-statistical equivalent if $L = 1$.

A function $f: [0, \infty) \to [0, \infty)$ is called a modulus if

1. $f(x) = 0$ if and if only if $x = 0$,
2. $f(x + y) \leq f(x) + f(y)$,
3. f is increasing,
4. f is continuous from the right at 0.

A modulus may be unbounded (for example $f(x) = x^p,\ 0 < p < 1$) or bounded (for example $f(x) = \frac{x}{x+1}$).

Let f be modulus function. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that, for every $\varepsilon > 0$,

$$\left\{k \in \mathbb{N}: f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}$$

denoted by $x_k \overset{\mathcal{I}(f)}{\sim} y_k$ and simply f-asymptotically $\mathcal{I}$-equivalent if $L = 1$.

Let f be modulus function. The two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that, for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} f(|\frac{x_k}{y_k} - L|) \geq \varepsilon\right\} \in \mathcal{I}$$

denoted by $x_k \overset{\mathcal{I}(\omega_f)}{\sim} y_k$ and simply strongly f-asymptotically $\mathcal{I}$-equivalent if $L = 1$.

Let f be a modulus function. The two nonnegative $x = (x_k)$ and $y = (y_k)$ are said to be strongly f-asymptotically lacunary equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that for every $\varepsilon > 0$,

$$\left\{r \in \mathbb{N}: \frac{1}{h_r}\sum_{k \in I_r} f\left(\left|\frac{x_k}{y_k} - L\right|\right) \geq \varepsilon\right\} \in \mathcal{I}$$

denoted by $x_k \overset{\mathcal{I}(N_\theta^f)}{\sim} y_k$ and simply strongly f-asymptotically lacunary $\mathcal{I}$-equivalent if $L = 1$.

The two non-negative sequences x_{kj} and y_{kj} are said to be asymptotically strongly $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} \sum_{k,j=1}^{m,n} \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2$$

denoted by $x_{kj} \overset{\mathcal{I}_2(S)}{\sim} y_{kj}$ and simply asymptotically $\mathcal{I}_2$ statistical equivalent if $L = 1$.

The two non-negative sequences x_{kj} and y_{kj} are said to be asymptotically $\mathcal{I}_2$-statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,

$$\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn} |\{k,j \leq m,n: \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon\}| \geq \gamma\} \in \mathcal{I}_2$$

denoted by $x_{kj} \overset{\mathcal{I}_2(S)}{\sim} y_{kj}$ and simply asymptotically $\mathcal{I}_2$ statistical equivalent if $L = 1$.

The two non-negative sequences x_{kj} and y_{kj} are said to be asymptotically lacunary $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{ru}} \sum_{(k,j) \in I_{ru}} \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon \right\} \in \mathcal{I}_2$$

denoted by $x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj}$ and simply strongly asymptotically lacunary $\mathcal{I}_2$-equivalent if $L = 1$.

The two non-negative sequences x_{kj} and y_{kj} are said to be asymptotically lacunary $\mathcal{I}_2$-statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,

$$\{(r,u) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{ru}} |\{(k,j) \in I_{ru}: \left| \frac{x_{kj}}{y_{kj}} - L \right| \geq \varepsilon\}| \geq \gamma\} \in \mathcal{I}_2$$

denoted by $x_{kj} \overset{\mathcal{I}_2(S_\theta)}{\sim} y_{kj}$ and simply asymptotically $\mathcal{I}_2$ statistical equivalent if $L = 1$.

Lemma 1 *[23] Let f be a modulus and $0 < \delta < 1$. Then, for each $x \geq \delta$ we have $f(x) \leq 2f(1)\delta^{-1}x$.*

Method

In the proofs of the theorems obtained in this study, used frequently in mathematics,

i. Direct proof method,

ii. Reverse proof method

iii. Contrapositive method,

iv. Induction method

methods were used as needed.

Main Results

Definition 2.1 *Let f be a modulus function. The two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be f-asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,*

$$\left\{(k,j) \in \mathbb{N} \times \mathbb{N}: f(|\frac{x_{kj}}{y_{kj}} - L|) \geq \varepsilon\right\} \in \mathcal{I}_2$$

denoted by

$$x_{kj} \overset{\mathcal{I}_2^L(f)}{\sim} y_{kj}$$

and simply f-asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

Definition 2.2 *Let f be a modulus function. The two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strongly f-asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,*

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mn}\sum_{k,j=1}^{m,n} f(|\frac{x_{kj}}{y_{kj}} - L|) \geq \varepsilon\right\} \in \mathcal{I}_2$$

denoted by

$$x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}$$

and simply strongly f-asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

Theorem 2.1 *Let f be a modulus function. Then,*

$$x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj}.$$

Theorem 2.2 *If $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then*

$$x_{kj} \overset{[\mathcal{I}_2^L(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_2^L]}{\sim} y_{kj}.$$

Definition 2.3 *Let f be a modulus function. The two non-negative sequences $x = (x_{kj})$ and $y = (y_{kj})$ are said to be strongly f-asymptotically lacunary $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$,*

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{ru}}\sum_{(k,j)\in I_{ru}} f(|\frac{x_{kj}}{y_{kj}} - L|) \geq \varepsilon\right\} \in \mathcal{I}_2$$

denoted by

$$x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}$$

and simply strongly f-asymptotically lacunary $\mathcal{I}_2$-equivalent if $L = 1$.

Theorem 2.3 *Let f be a modulus function. Then,*

$$x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj} \Rightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj}.$$

Theorem 2.4 *If $\lim_{t\to\infty} \frac{f(t)}{t} = \alpha > 0$, then*

$$x_{kj} \overset{[\mathcal{I}_{\theta_2}^L(f)]}{\sim} y_{kj} \Leftrightarrow x_{kj} \overset{[\mathcal{I}_{\theta_2}^L]}{\sim} y_{kj}.$$

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