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Author(s): Özer Talo and Yurdal Sever

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On Kuratowski $\mathcal{I}-$Convergence of Sequences of Closed Sets

Özer Talo[a], Yurdal Sever[b]

[a]Manisa Celal Bayar University, Faculty of Art and Sciences, Department of Mathematics 45040 Manisa, TURKEY
[b]Afyon Kocatepe University, Faculty of Arts and Sciences, Department of Mathematics 03200 Afyonkarahisar, TURKEY

Abstract. In this paper we extend the concepts of statistical inner and outer limits (as introduced by Talo, Sever and Başar) to $\mathcal{I}-$inner and $\mathcal{I}-$outer limits and give some $\mathcal{I}-$analogue of properties of statistical inner and outer limits for sequences of closed sets in metric spaces, where $\mathcal{I}$ is an ideal of subsets of the set $\mathbb{N}$ of positive integers. We extend the concept of Kuratowski statistical convergence to Kuratowski $\mathcal{I}-$convergence for a sequence of closed sets and get some properties for Kuratowski $\mathcal{I}-$convergent sequences. Also, we examine the relationship between Kuratowski $\mathcal{I}-$convergence and Hausdorff $\mathcal{I}-$convergence.

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [9] and Schoenberg [23]. The idea of $\mathcal{I}-$convergence was introduced by Kostyrko et al. [11] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of positive integers. Nuray and Ruckle [18] independently introduced the same with another name generalized statistical convergence. Kostyrko et al. [12] gave some of basic properties of $\mathcal{I}-$convergence and dealt with extremal $\mathcal{I}-$limit points.

For the last few years, study of $\mathcal{I}-$convergence of sequences has become one of the most active areas of research in classical analysis. Balcerzak et al. [2] studied on statistical convergence and ideal convergence for sequences of functions. Komisarski [10] discussed the pointwise $\mathcal{I}-$convergence and $\mathcal{I}-$convergence in measure of sequences of functions. Mursaleen et al. [16] defined and studied the concept of $\mathcal{I}-$convergence in probabilistic normed space. Nabiev et al. [17] gave Cauchy condition for $\mathcal{I}-$convergence. Şahiner et al. [26] introduced and investigated $\mathcal{I}-$convergence in 2-normed spaces and examined some new sequence spaces. Kumar and Kumar [13] studied the concepts of $\mathcal{I}-$convergence and $\mathcal{I}^*-$convergence for sequences of fuzzy numbers.

In set valued and variational analysis, limits of sequences of sets have the leading role. See [1, 8, 20]. The concepts of inner and outer limits for a sequence of sets are due to Painlevé, who introduced them in 1902 in his lectures on analysis at the University of Paris; set convergence was defined as the equality of these two limits. This convergence has been popularized by Kuratowski in his famous book Topologie [14] and thus, often called Kuratowski convergence of sequences of sets. For some properties of inner and outer limits we refer to [4, 5, 15, 20, 22, 24, 25, 28, 29]. Other convergence notions for sets are not equivalent to Kuratowski

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Email addresses: ozertalo@hotmail.com (Özer Talo), yurdalsever@hotmail.com (Yurdal Sever)

convergence but have significance for certain applications. One of them is Hausdorff convergence. We mention some references related to Hausdorff convergence: [3, 4, 14, 22, 25]. Nuray and Rhoades [19] first defined the statistical convergence for sequences of sets and studied Hausdorff and Wijsman statistical convergence.

In this paper our aim is to discuss two kinds of $\mathcal{I}-$convergence for sequences of closed sets which are called Kuratowski $\mathcal{I}-$convergence and Hausdorff $\mathcal{I}-$convergence. For our purpose we give the definitions of $\mathcal{I}-$outer and $\mathcal{I}-$inner limits for a sequence of closed sets and investigate some properties of them.

2. Definitions and Notation

Let K be a subset of positive integers $\mathbb{N}$ and $K(n) = |\{k \leq n : k \in K\}|$, where $|A|$ denotes the number of elements in A. The natural density of K is given by $\delta(K) = \lim_{n\to\infty} \frac{1}{n}K(n)$ if this limit exists.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero for every $\varepsilon > 0$. In this case we write $st-\lim_{k\in\mathbb{N}} x_k = L$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1. *[11] If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : X\backslash M \in \mathcal{I}\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

Lemma 2.2. *[21, Lemma 2.5] $K \in F(\mathcal{I})$ and $M \subseteq \mathbb{N}$. If $M \notin \mathcal{I}$ then $M \cap K \notin \mathcal{I}$.*

In what follows (X, d) is a fixed metric space and $\mathcal{I}$ denotes a non-trivial ideal of subsets of $\mathbb{N}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}-$convergent to $\xi \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : d(x_n, \xi) \geq \varepsilon\}$ belongs to $\mathcal{I}$. The element ξ is called the $\mathcal{I}-$limit of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$. In this case we write $\mathcal{I} - \lim_{n\to\infty} x_n = \xi$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}^*-$convergent to $\xi \in X$ if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} d(x_{m_k}, \xi) = 0$. In this case we write $\mathcal{I}^* - \lim_{n\to\infty} x_n = \xi$.

We say that an admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ satisfies the property (AP), if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ of sets such that each symmetric difference $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. (Hence $B_j \in \mathcal{I}$ for each $j \in \mathbb{N}$).

Lemma 2.3. *[11, Proposition 3.2] Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^* - \lim_{n\to\infty} x_n = \xi$, then $\mathcal{I} - \lim_{n\to\infty} x_n = \xi$.*

Lemma 2.4. *[11, Theorem 3.2] Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. If the ideal $\mathcal{I}$ has property (AP) and (X, d) is an arbitrary metric space, then for arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X we have $\mathcal{I} - \lim_{n\to\infty} x_n = \xi$ implies $\mathcal{I}^* - \lim_{n\to\infty} x_n = \xi$.*

An element $\xi \in X$ is said to be an $\mathcal{I}-$limit point of a sequence $x = (x_k)$ if there is a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = \xi$. The set of all $\mathcal{I}-$limit points of a sequence x will be denoted by $\mathcal{I}(\Lambda_x)$.

An element $\xi \in X$ is said to be an $\mathcal{I}-$cluster point of a sequence $x = (x_k)$ if for each $\varepsilon > 0$, we have $\{k \in \mathbb{N} : d(x_k, \xi) < \varepsilon\} \notin \mathcal{I}$. The set of all $\mathcal{I}-$cluster points of x will be denoted by $\mathcal{I}(\Gamma_x)$.

Let L_x denote the set of all limit points ξ (accumulation points) of the sequence x; i.e., $\xi \in L_x$ if there exists an infinite set $K = \{k_1 < k_2 < k_3 < \cdots\}$ such that $x_{k_n} \to \xi$ as $n \to \infty$.

Obviously, for an admissible ideal $\mathcal{I}$ we have $\mathcal{I}(\Lambda_x) \subseteq \mathcal{I}(\Gamma_x) \subseteq L_x$.

Lemma 2.5. *[6, Lemma 3.1] K be a compact subset of X. Then we have $K \cap \mathcal{I}(\Gamma_x) \neq \emptyset$ for every $x = (x_n)$ with $\{n \in \mathbb{N} : x_n \in K\} \notin \mathcal{I}$.*

The concepts of $\mathcal{I}-$limit superior and inferior were introduced by Demirci [7] as follows: Let $\mathcal{I}$ be an admissible ideal and $x = (x_k)$ be a real number sequence.

$$\mathcal{I} - \limsup_{k\to\infty} x_k := \begin{cases} \sup B_x, & B_x \neq \emptyset, \\ -\infty, & B_x = \emptyset, \end{cases}$$

$$\mathcal{I} - \liminf_{k\to\infty} x_k := \begin{cases} \inf A_x, & A_x \neq \emptyset, \\ \infty, & A_x = \emptyset, \end{cases}$$

where $A_x := \{a \in \mathbb{R} : \{k \in \mathbb{N} : x_k < a\} \notin \mathcal{I}\}$ and $B_x := \{b \in \mathbb{R} : \{k \in \mathbb{N} : x_k > b\} \notin \mathcal{I}\}$.

Lemma 2.6. *[7, Theorem 1] If $\beta = \mathcal{I} - \limsup_{k\to\infty} x_k$ is finite, then for every $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : x_k > \beta - \varepsilon\} \notin \mathcal{I} \quad \textit{and} \quad \{k \in \mathbb{N} : x_k > \beta + \varepsilon\} \in \mathcal{I}. \tag{1}$$

Conversely, if (1) holds for every $\varepsilon > 0$ then $\beta = \mathcal{I} - \limsup_{k\to\infty} x_k$.

The dual statement for $\mathcal{I} - \liminf$ is as follows:

Lemma 2.7. *[7, Theorem 2] If $\alpha = \mathcal{I} - \liminf_{k\to\infty} x_k$ is finite, then for every $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\} \notin \mathcal{I} \quad \textit{and} \quad \{k \in \mathbb{N} : x_k < \alpha - \varepsilon\} \in \mathcal{I}. \tag{2}$$

Conversely, if (2) holds for every $\varepsilon > 0$ then $\alpha = \mathcal{I} - \liminf_{k\to\infty} x_k$.

Let (X, d) be a metric space. The distance between a subset A of X and $x \in X$ is given by $d(x, A) = \inf\{d(x, y) : y \in A\}$, where it is understood that the infimum of $d(x, .)$ is ∞ if $A = \emptyset$. For each closed subset A of X, the function $x \to d(., A)$ is Lipschitz continuous, i.e. for each $x, y \in X$

$$\left| d(x, A) - d(y, A) \right| \leq d(x, y).$$

The open ball with center x and radius $\varepsilon > 0$ in X is denoted by $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$. Also, for any set A and $\varepsilon > 0$, we write $B(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}$.

Now we recall some basic properties of Kuratowski convergence. We use the following notation:

$$\begin{aligned} \mathcal{N} &:= \{N \subseteq \mathbb{N} : \mathbb{N} \backslash N \ \text{ finite}\} \\ &:= \{\text{subsequences of } \mathbb{N} \text{ containing all n beyond some } n_0\} \\ \mathcal{N}^{\#} &:= \{N \subseteq \mathbb{N} : N \text{ infinite}\} = \{\text{all subsequences of } \mathbb{N}\}. \end{aligned}$$

We write $\lim_{n\to\infty}$ when $n \to \infty$ as usual in $\mathbb{N}$, but $\lim_{n\in N}$ in the case of convergence of a subsequence designated by an index set N in $\mathcal{N}^{\#}$.

Definition 2.8. *For a sequence (A_n) of closed subsets of X; the outer limit is the set*

$$\limsup_{n\to\infty} A_n := \left\{x \mid \forall\varepsilon>0,\ \exists N\in\mathcal{N}^{\#},\ \forall n\in N : A_n\cap B(x,\varepsilon)\neq\emptyset\right\}$$
$$:= \left\{x \mid \exists N\in\mathcal{N}^{\#},\ \forall n\in N,\ \exists x_n\in A_n : \lim_{n\in N} x_n = x\right\},$$

while the inner limit is the set

$$\liminf_{n\to\infty} A_n := \left\{x \mid \forall\varepsilon>0,\ \exists N\in\mathcal{N},\ \forall n\in N : A_n\cap B(x,\varepsilon)\neq\emptyset\right\}$$
$$:= \left\{x \mid \exists N\in\mathcal{N},\ \forall n\in N,\ \exists x_n\in A_n : \lim_{n\in N} x_n = x\right\}.$$

The limit of a sequence (A_n) of closed subsets of X exists if the outer and inner limit sets are equal, that is, $\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$.

Talo et al. [27] introduced Kuratowski statistical convergence of sequences of closed sets. The statistical outer limit and statistical inner limit of a sequence (A_n) of closed subsets of X are defined by

$$st-\limsup_{n\to\infty} A_n := \left\{x \mid \forall\varepsilon>0,\ \exists N\in\mathcal{S}^{\#},\ \forall n\in N : A_n\cap B(x,\varepsilon)\neq\emptyset\right\},$$
$$st-\liminf_{n\to\infty} A_n := \left\{x \mid \forall\varepsilon>0,\ \exists N\in\mathcal{S},\ \forall n\in N : A_n\cap B(x,\varepsilon)\neq\emptyset\right\},$$

where

$$\mathcal{S} := \{N\subseteq\mathbb{N} : \delta(N)=1\} \quad \text{and} \quad \mathcal{S}^{\#} := \{N\subseteq\mathbb{N} : \delta(N)\neq 0\}.$$

The statistical limit of a sequence (A_n) exists if its statistical outer and statistical inner limits coincide; i.e., $st-\lim_{n\to\infty} A_n = st-\limsup_{n\to\infty} A_n = st-\liminf_{n\to\infty} A_n$.

3. Kuratowski $\mathcal{I}-$Convergence

In this section, we introduce Kuratowski $\mathcal{I}-$convergence of sequences of closed sets. We use the analogous idea employed by Kuratowski [14] and Talo et al. [27] for convergence and statistical convergence of sequences closed sets. Let us consider

$$\mathcal{N}_{\mathcal{I}} := \{N\subseteq\mathbb{N} : \mathbb{N}\backslash N\in\mathcal{I}\} = \mathcal{F}(\mathcal{I}) \quad \text{and} \quad \mathcal{N}_{\mathcal{I}}^{\#} := \{N\subseteq\mathbb{N} : N\notin\mathcal{I}\}.$$

Firstly, we define the $\mathcal{I}$ analogues for outer and inner limits of a sequence of closed sets.

Definition 3.1. *The $\mathcal{I}-$outer limit and $\mathcal{I}-$inner limit of a sequence (A_n) of closed subsets of X are defined as follows:*

$$\mathcal{I}-\limsup_{n\to\infty} A_n := \left\{x \mid \forall\varepsilon>0,\ \exists N\in\mathcal{N}_{\mathcal{I}}^{\#},\ \forall n\in N : A_n\cap B(x,\varepsilon)\neq\emptyset\right\},$$

and

$$\mathcal{I}-\liminf_{n\to\infty} A_n := \left\{x \mid \forall\varepsilon>0,\ \exists N\in\mathcal{N}_{\mathcal{I}},\ \forall n\in N : A_n\cap B(x,\varepsilon)\neq\emptyset\right\}.$$

The $\mathcal{I}-$limit of a sequence (A_n) exists if its $\mathcal{I}-$outer and $\mathcal{I}-$inner limits coincide. In this situation we say that the sequence of sets is Kuratowski $\mathcal{I}-$convergent and we write

$$\mathcal{I}-\liminf_{n\to\infty} A_n = \mathcal{I}-\limsup_{n\to\infty} A_n = \mathcal{I}-\lim_{n\to\infty} A_n.$$

Moreover, it's clear from the inclusion $\mathcal{N}_{\mathcal{I}} \subset \mathcal{N}_{\mathcal{I}}^{\#}$ that

$$\mathcal{I} - \liminf_{n\to\infty} A_n \subseteq \mathcal{I} - \limsup_{n\to\infty} A_n$$

so that in fact, $\mathcal{I} - \lim_{n\to\infty} A_n = A$ if and only if

$$\mathcal{I} - \limsup_{n\to\infty} A_n \subseteq A \subseteq \mathcal{I} - \liminf_{n\to\infty} A_n.$$

Remark 3.2. *$\mathcal{I} - \lim_{n\to\infty} A_n = A$ if and only if the following conditions are satisfied:*

(i) for every $x \in A$ and for every $\varepsilon > 0$ we have $\{k \in \mathbb{N} : B(x,\varepsilon) \cap A_k \neq \emptyset\} \in \mathcal{F}(\mathcal{I})$;

(ii) for every $x \in X \setminus A$ there exists $\varepsilon > 0$ such that $\{k \in \mathbb{N} : B(x,\varepsilon) \cap A_k = \emptyset\} \in \mathcal{F}(\mathcal{I})$.

We give some examples of ideals and corresponding $\mathcal{I}-$convergence.

(I) Put $\mathcal{I}_0 = \{\emptyset\}$. $\mathcal{I}_0$ is the minimal ideal in $\mathbb{N}$. Then for a sequence (A_n) of closed sets we have

$$\mathcal{I}_0 - \liminf_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n \quad \text{and} \quad \mathcal{I}_0 - \limsup_{n\to\infty} A_n = \text{cl} \bigcup_{n=1}^{\infty} A_n,$$

where $\text{cl}(A)$ denotes the closure of the set A in the metric space (X,d). A sequence (A_n) is Kuratowski $\mathcal{I}_0-$convergent if and only if it is constant set.

(II) Let $M \subseteq \mathbb{N}$, $M \neq \mathbb{N}$. Put $\mathcal{I}_M = 2^M$. Then $\mathcal{I}_M$ is a nontrivial ideal in $\mathbb{N}$. Then for a sequence (A_n) of closed sets we have

$$\mathcal{I}_M - \liminf_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}\setminus M} A_n \quad \text{and} \quad \mathcal{I}_M - \limsup_{n\to\infty} A_n = \text{cl} \bigcup_{n\in\mathbb{N}\setminus M} A_n.$$

A sequence (A_n) is Kuratowski $\mathcal{I}_M-$convergent if and only if it is constant set on $\mathbb{N} \setminus M$, i.e. there is a closed set A such that $A_n = A$ for each $n \in \mathbb{N} \setminus M$.

(III) Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and Kuratowski $\mathcal{I}_f-$convergence coincides with the usual Kuratowski convergence.

(IV) Denote by $\mathcal{I}_\delta$ the class of all $A \subset \mathbb{N}$ with $\delta(A) = 0$. Then $\mathcal{I}_\delta$ is non-trivial admissible ideal and Kuratowski $\mathcal{I}_\delta-$convergence coincides with the Kuratowski statistical convergence.

Note that if $\mathcal{I}$ is an admissible, then $\mathcal{I}_f \subseteq \mathcal{I}$. It is clear that

$$\liminf_{n\to\infty} A_n \subseteq \mathcal{I} - \liminf_{n\to\infty} A_n \subseteq \mathcal{I} - \limsup_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n.$$

Hence every Kuratowski convergent sequence is Kuratowski $\mathcal{I}-$convergent, i.e.,

$$\lim_{n\to\infty} A_n = A \text{ implies } \mathcal{I} - \lim_{n\to\infty} A_n = A.$$

But, the converse of this claim does not hold in general.

Example 3.3. *Let $X = \mathbb{R}^2$ (with the usual Euclidean metric). We decompose the set $\mathbb{N}$ into countably many disjoint sets*

$$N_j = \{2^{j-1}(2s-1) : s \in \mathbb{N}\}, \ (j = 1,2,3,...).$$

It is obvious that $\mathbb{N} = \bigcup_{j=1}^{\infty} N_j$ and $N_i \cap N_j = \emptyset$ for $i \neq j$. Denote by $\mathcal{I}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of N_j. It is easy to see that $\mathcal{I}$ is an admissible ideal. Define (A_n) as follows: for $n \in N_j$ we put

$$A_n = \left\{(x,y) \in \mathbb{R}^2 : \frac{1}{(j+1)^2} \leq x^2 + y^2 \leq \frac{1}{j^2}\right\} \ (j = 1,2,3,...).$$

Let $\varepsilon > 0$. Choose $p \in \mathbb{N}$ such that $\frac{1}{p} < \varepsilon$. Then

$$\left\{n \in \mathbb{N} : A_n \cap B(0,\varepsilon) = \emptyset\right\} \subseteq N_1 \cup N_2 \cup \cdots \cup N_p.$$

Thus $\left\{n \in \mathbb{N} : A_n \cap B(0,\varepsilon) = \emptyset\right\} \in \mathcal{I}$ *i.e.,* $\left\{n \in \mathbb{N} : A_n \cap B(0,\varepsilon) \neq \emptyset\right\} \in \mathcal{F}(\mathcal{I})$. *So* $\mathcal{I} - \lim_{n\to\infty} A_n = \{0\}$. *However*

$$\liminf_{n\to\infty} A_n = \emptyset \quad \textit{and} \quad \limsup_{n\to\infty} A_n = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

Therefore (A_n) is not Kuratowski convergent.

In what follows $\mathcal{I}$ denotes a non-trivial admissible ideal of subsets of $\mathbb{N}$.

Proposition 3.4. *Let (A_n) be a sequence of closed subsets of X. Then*

$$\mathcal{I} - \liminf_{n\to\infty} A_n = \bigcap_{N \in \mathcal{N}_{\mathcal{I}}^{\#}} cl \bigcup_{n \in N} A_n \quad \textit{and} \quad \mathcal{I} - \limsup_{n\to\infty} A_n = \bigcap_{N \in \mathcal{N}_{\mathcal{I}}} cl \bigcup_{n \in N} A_n.$$

Proof. We prove only the first equality because the proof of the second one is similar to the first one. Let $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$ be arbitrary and $N \in \mathcal{N}_{\mathcal{I}}^{\#}$ be arbitrary. For every $\varepsilon > 0$ there exists $N_1 \in \mathcal{N}_{\mathcal{I}}$ such that for every $n \in N_1$

$$A_n \cap B(x,\varepsilon) \neq \emptyset.$$

From Lemma 2.2 we have $N \cap N_1 \notin \mathcal{I}$. So there exists $n_0 \in N \cap N_1$ such that $A_{n_0} \cap B(x,\varepsilon) \neq \emptyset$. Therefore,

$$\left(\bigcup_{n \in N} A_n\right) \cap B(x,\varepsilon) \neq \emptyset.$$

This means that $x \in \mathrm{cl} \bigcup_{n \in N} A_n$. This holds for any $N \in \mathcal{N}_{\mathcal{I}}^{\#}$. Consequently,

$$x \in \bigcap_{N \in \mathcal{N}_{\mathcal{I}}^{\#}} \mathrm{cl} \bigcup_{n \in N} A_n.$$

For the reverse inclusion, suppose that $x \notin \mathcal{I} - \liminf_{n\to\infty} A_n$. Then, there exists $\varepsilon > 0$ such that

$$N = \left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\right\} \notin \mathcal{I},$$

i.e., $N \in \mathcal{N}_{\mathcal{I}}^{\#}$. Thus

$$\left(\bigcup_{n \in N} A_n\right) \cap B(x,\varepsilon) = \emptyset.$$

This means that $x \notin \mathrm{cl} \bigcup_{n \in N} A_n$. This completes the proof. □

As a consequence of Proposition 3.4, for any given sequence (A_n) the sets $\mathcal{I} - \liminf_{n\to\infty} A_n$ and $\mathcal{I} - \limsup_{n\to\infty} A_n$ are closed.

Proposition 3.5. *Let (A_n) be a sequence of closed subsets of X. Then*

$$\begin{aligned} \mathcal{I} - \liminf_{n\to\infty} A_n &= \left\{x \mid \mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0\right\}, \\ \mathcal{I} - \limsup_{n\to\infty} A_n &= \left\{x \mid \mathcal{I} - \liminf_{n\to\infty} d(x, A_n) = 0\right\}. \end{aligned}$$

Proof. For any closed set A we have

$$d(x,A) \geq \varepsilon \Leftrightarrow A \cap B(x,\varepsilon) = \emptyset. \tag{3}$$

Suppose that $\mathcal{I} - \lim_{n\to\infty} d(x,A_n) = 0$. Then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x,A_n) \geq \varepsilon\right\} \in \mathcal{I}.$$

By (3), for every $\varepsilon > 0$ we obtain

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\right\} \in \mathcal{I}.$$

This means that

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) \neq \emptyset\right\} \in \mathcal{F}(\mathcal{I}).$$

That is, $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$.

Now, we show the reverse inclusion. Let $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. Then for every $\varepsilon > 0$ there exists $N \in \mathcal{N}_{\mathcal{I}}$ such that $A_n \cap B(x,\varepsilon) \neq \emptyset$ for every $n \in N$. Since

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\right\} \subseteq \mathbb{N} \setminus N$$

we have

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\right\} \in \mathcal{I}.$$

By (3)

$$\left\{n \in \mathbb{N} : d(x,A_n) \geq \varepsilon\right\} \in \mathcal{I}.$$

That is, $\mathcal{I} - \lim_{n\to\infty} d(x,A_n) = 0$.

Similarly, for any closed set A we have

$$d(x,A) < \varepsilon \quad \Leftrightarrow \quad A \cap B(x,\varepsilon) \neq \emptyset. \tag{4}$$

Suppose that $\mathcal{I} - \liminf_{n\to\infty} d(x,A_n) = 0$. Then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x,A_n) < \varepsilon\right\} \notin \mathcal{I}.$$

By (4), for every $\varepsilon > 0$ we obtain

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) \neq \emptyset\right\} \notin \mathcal{I}.$$

This means that $x \in \mathcal{I} - \limsup_{n\to\infty} A_n$.

Now, we show the reverse inclusion. Let $x \in \mathcal{I} - \limsup_{n\to\infty} A_n$. Then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) \neq \emptyset\right\} \notin \mathcal{I}.$$

By (4) and Lemma 2.7, we have $\mathcal{I} - \liminf_{n\to\infty} d(x,A_n) = 0$. □

Proposition 3.6. *Let (A_n) be a sequence of closed subsets of X. Then*

$$\mathcal{I} - \liminf_{n\to\infty} A_n \quad = \quad \left\{x \mid \forall n \in \mathbb{N},\ \exists y_n \in A_n : \mathcal{I} - \lim_{n\to\infty} y_n = x\right\}. \tag{5}$$

Proof. Let $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$ be arbitrary. By Proposition 3.5,

$$\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0.$$

For every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x, A_n) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}.$$

Since A_n is closed, for $n \in \mathbb{N}$, there exists $y_n \in A_n$ such that $d(x, y_n) \leq 2d(x, A_n)$. Now, we define the sequence $\{y_n \mid y_n \in A_n,\ n \in \mathbb{N}\}$. Then $\mathcal{I} - \lim_{n\to\infty} y_n = x$.

On the contrary, assume that x belongs to the right-hand side set of the equality (5). Then, there exist $\{y_n \mid y_n \in A_n, n \in \mathbb{N}\}$ such that $\mathcal{I} - \lim_{n\to\infty} y_n = x$. Then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x, y_n) \geq \varepsilon\right\} \in \mathcal{I}.$$

The inequality $d(x, y_n) \geq d(x, A_n)$ yields the inclusion

$$\{n \in \mathbb{N} : d(x, A_n) \geq \varepsilon\big\} \subseteq \{n \in \mathbb{N} : d(x, y_n) \geq \varepsilon\big\}.$$

So,

$$\{n \in \mathbb{N} : d(x, A_n) \geq \varepsilon\big\} \in \mathcal{I}.$$

This means that $\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0$. By Proposition 3.5 we have $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. □

The following result is well known in the theory of Kuratowski convergence. $x \in \liminf_{n\to\infty} A_n$ if and only if there exist $N \in \mathcal{N} = \mathcal{N}_{\mathcal{I}_f}$ and $x_n \in A_n$ for all $n \in N$ such that $\lim_{n\in N} x_n = x$. For Kuratowski $\mathcal{I}$−convergence, if $\mathcal{I}$ has property (AP), then this fact holds.

Corollary 3.7. *Let $\mathcal{I}$ be an admissible ideal. If the ideal $\mathcal{I}$ has property (AP) then*

$$\mathcal{I} - \liminf_{n\to\infty} A_n = \left\{x \mid \exists N \in \mathcal{N}_{\mathcal{I}}, \forall n \in N,\ \exists y_n \in A_n : \lim_{n\in N} y_n = x\right\}. \tag{6}$$

Proof. Suppose that $\mathcal{I}$ satisfies condition (AP). Let $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. Then $\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0$. By condition (AP) we have $\mathcal{I}^* - \lim_{n\to\infty} d(x, A_n) = 0$. Then there is a set $M \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{m\in M} d(x, A_m) = 0.$$

Since A_n is closed, for $m \in M$, there exists $y_m \in A_m$ such that $d(x, y_m) \leq 2d(x, A_m)$. Now, we define the sequence $\{y_m \mid y_m \in A_m,\ m \in M\}$. Then $\lim_{m\in M} y_m = x$.

On the contrary, assume that x belongs to the right-hand side set of the equality (6). Let us define

$$z_n = \begin{cases} y_n, & \text{if } n \in N, \\ \text{arbitrary element of } A_n, & \text{if } n \notin N. \end{cases}$$

Then $\mathcal{I}^* - \lim_{n\to\infty} z_n = x$. So $\mathcal{I} - \lim_{n\to\infty} z_n = x$. By Proposition 3.6, we have $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. □

Remark 3.8. *In Corollary 3.7 the property (AP) can not be dropped. Let $X = \mathbb{R}$ (with the usual Euclidean metric) and $\mathcal{I}$ be the ideal introduced in Example 3.3. Define (A_n) as follows: for $n \in N_j$ we put $A_n = \{\frac{1}{j}\}$ $(j = 1, 2, 3, ...)$. Then the sequence $\{y_n \mid y_n \in A_n, n \in \mathbb{N}\}$ can be defined as follows: for $n \in N_j$ we put $y_n = \frac{1}{j}$ $(j = 1, 2, 3, ...)$. Clearly, $\mathcal{I} - \lim_{n\to\infty} y_n = 0$. So $\mathcal{I} - \liminf_{n\to\infty} A_n = \{0\}$.*

Suppose in contrary that 0 belongs to the right-hand side set of the equality (6). Then there is a set $M \in \mathcal{F}(\mathcal{I})$ such that for $m \in M$, there exists $y_m \in A_m$ and

$$\lim_{m\in M} y_m = 0. \tag{7}$$

By the definition of $\mathcal{F}(\mathcal{I})$ *we have* $M = \mathbb{N} \setminus H$, *where* $H \in \mathcal{I}$. *By the definition of* $\mathcal{I}$ *there is a* $p \in \mathbb{N}$ *such that*

$$H \subseteq N_1 \cup N_2 \cup ... \cup N_p.$$

But then M contains the set N_{p+1} *and so* $y_m = \frac{1}{p+1}$ *for infinitely many m's from M. This contradicts (7).*

Corollary 3.9. *Let X be a normed linear space and* (A_n) *be a sequence of subsets of X. If the ideal* $\mathcal{I}$ *has property (AP) and there is a set* $K \in \mathcal{F}(\mathcal{I})$ *such that* A_n *is convex for each* $n \in K$, *then* $\mathcal{I} - \liminf_{n\to\infty} A_n$ *is convex and so, when it exists, is* $\mathcal{I} - \lim_{n\to\infty} A_n$.

Proof. Let $\mathcal{I} - \liminf_{n\to\infty} A_n = A$. If x_1 and x_2 belong to A, by Corollary 3.7, we can find for all $n \in N$ in some set $N \in \mathcal{F}(\mathcal{I})$ points y_n^1 and y_n^2 in A_n such that

$$\lim_{n\in N} y_n^1 = x_1 \quad \text{and} \quad \lim_{n\in N} y_n^2 = x_2.$$

Since $K \in \mathcal{F}(\mathcal{I})$, we have $M \in \mathcal{F}(\mathcal{I})$ with $M = N \cap K$. Then for arbitrary $\lambda \in [0,1]$ and $n \in M$, let us define

$$y_n^\lambda := (1-\lambda)y_n^1 + \lambda y_n^2 \quad \text{and} \quad x_\lambda := (1-\lambda)x_1 + \lambda x_2.$$

Then

$$\lim_{n\in M} y_n^\lambda = x_\lambda.$$

By Corollary 3.7, we obtain $x_\lambda \in A$. This means that A is convex. □

Proposition 3.10. *Let* (A_n) *be a sequence of closed subsets of X. Then*

$$\mathcal{I} - \limsup_{n\to\infty} A_n = \left\{ x \mid \exists N \in \mathcal{N}_{\mathcal{I}}^{\#},\ \forall n \in N,\ \exists y_n \in A_n : x \in \mathcal{I}(\Gamma_y) \right\}. \tag{8}$$

Proof. Let $x \in \mathcal{I} - \limsup_{n\to\infty} A_n$ be arbitrary. By Proposition 3.5,

$$\mathcal{I} - \liminf_{n\to\infty} d(x, A_n) = 0.$$

By Lemma 2.7, for every $\varepsilon > 0$ we have

$$\left\{ n \in \mathbb{N} : d(x, A_n) < \frac{\varepsilon}{2} \right\} \notin \mathcal{I}.$$

Since A_n is closed, for $n \in \mathbb{N}$, there exists $y_n \in A_n$ such that $d(x, y_n) \le 2d(x, A_n)$. Now, we define the sequence $\{y_n \mid y_n \in A_n,\ n \in \mathbb{N}\}$. Then

$$\left\{ n \in \mathbb{N} : d(x, y_n) < \varepsilon \right\} \notin \mathcal{I}.$$

Therefore $x \in \mathcal{I}(\Gamma_y)$.

On the contrary, assume that x belongs to the right-hand side set of the equality (8). Then there exist $N \in \mathcal{N}_{\mathcal{I}}^{\#}$ a the sequence $\{y_n \mid y_n \in A_n, n \in N\}$ such that $x \in \mathcal{I}(\Gamma_y)$. That is, for every $\varepsilon > 0$

$$\left\{ n \in \mathbb{N} : d(x, y_n) < \varepsilon \right\} \notin \mathcal{I}.$$

The inequality $d(x, y_n) \ge d(x, A_n)$ yields the inclusion

$$\{ n \in \mathbb{N} : d(x, y_n) < \varepsilon \Big\} \subseteq \{ n \in \mathbb{N} : d(x, A_n) < \varepsilon \Big\}.$$

So, the set

$$N' = \{ n \in \mathbb{N} : d(x, A_n) < \varepsilon \Big\} \notin \mathcal{I}.$$

That is, $N' \in \mathcal{N}_{\mathcal{I}}^{\#}$. By (4), for every $n \in N'$ we obtain $A_n \cap B(x, \varepsilon) \ne \emptyset$. This means that $x \in \mathcal{I} - \limsup_{n\to\infty} A_n$. □

Remark 3.11. *In Proposition 3.10 the set of $\mathcal{I}-$cluster points can not be replaced by the set of $\mathcal{I}-$limit points. Let (A_n) and (y_n) be the sequences introduced in Remark 3.8. Let us take $\mathcal{I} = \mathcal{I}_\delta$. It can be easily shown that $\delta(N_j) = 1\backslash 2^j$. From Example 2.1 of [6] we have $0 \in \mathcal{I}_\delta(\Gamma_y)$ but $0 \notin \mathcal{I}_\delta(\Lambda_y)$. So, $0 \in \mathcal{I}_\delta - \limsup_{n\to\infty} A_n$. However*

$$0 \notin \left\{x \mid \exists N \in \mathcal{N}_{\mathcal{I}}^{\#},\ \forall n \in N,\ \exists y_n \in A_n : \lim_{n\in N} y_n = x\right\}.$$

By Proposition 3.6 and Proposition 3.10, note that $\mathcal{I} - \liminf_{n\to\infty} A_n$ is the set of $\mathcal{I}-$limits of sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in A_n$ and $\mathcal{I} - \limsup_{n\to\infty} A_n$ is the set of $\mathcal{I}-$cluster points of sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in A_n$.

Lemma 3.12. *Let (A_n) and (B_n) be two sequences of closed subsets of X. If there is a set $K \in \mathcal{N}_{\mathcal{I}}$ such that $A_n \subseteq B_n$ for each $n \in K$, then the inclusions*

$$\mathcal{I} - \liminf_{n\to\infty} A_n \subseteq \mathcal{I} - \liminf_{n\to\infty} B_n \quad \text{and} \quad \mathcal{I} - \limsup_{n\to\infty} A_n \subseteq \mathcal{I} - \limsup_{n\to\infty} B_n$$

hold.

Proof. To prove the first inclusion suppose that there exists $K \in \mathcal{N}_{\mathcal{I}}$ such that for each $n \in K$ the inclusion $A_n \subseteq B_n$ holds. In this case for each $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$, we obtain

$$d(x, B_n) \leq d(x, A_n). \tag{9}$$

By Proposition 3.5, we have

$$\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0. \tag{10}$$

Consequently, combining (9) and (10), we have $\mathcal{I} - \lim_{n\to\infty} d(x, B_n) = 0$. Namely $x \in \mathcal{I} - \liminf_{n\to\infty} B_n$.

The proof of second inclusion is analogous to that of the first one and so we omit the details. □

Corollary 3.13. *Let (A_n) and (B_n) be two sequences of closed subsets of X. Then, the following statements hold:*

1. $\mathcal{I} - \limsup_{n\to\infty}(A_n \cap B_n) \subseteq \mathcal{I} - \limsup_{n\to\infty} A_n \cap \mathcal{I} - \limsup_{n\to\infty} B_n$.
2. $\mathcal{I} - \liminf_{n\to\infty}(A_n \cap B_n) \subseteq \mathcal{I} - \liminf_{n\to\infty} A_n \cap \mathcal{I} - \liminf_{n\to\infty} B_n$.
3. $\mathcal{I} - \limsup_{n\to\infty}(A_n \cup B_n) = \mathcal{I} - \limsup_{n\to\infty} A_n \cup \mathcal{I} - \limsup_{n\to\infty} B_n$.
4. $\mathcal{I} - \liminf_{n\to\infty}(A_n \cup B_n) \supseteq \mathcal{I} - \liminf_{n\to\infty} A_n \cup \mathcal{I} - \liminf_{n\to\infty} B_n$.

Proof. For each $n \in \mathbb{N}$, the inclusions $A_n \cap B_n \subseteq A_n$, $A_n \cap B_n \subseteq B_n$, $A_n \subseteq A_n \cup B_n$ and $B_n \subseteq A_n \cup B_n$ hold. Now, the proof is immediate by Lemma 3.12. □

Definition 3.14. *A sequence (A_k) is said to be $\mathcal{I}-$monotonic increasing, if there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \in F(\mathcal{I})$ such that $A_{k_n} \subseteq A_{k_{n+1}}$ for every $n \in \mathbb{N}$. Similarly, sequence (A_k) is said to be $\mathcal{I}-$monotonic decreasing, if there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \in F(\mathcal{I})$ such that $A_{k_n} \supseteq A_{k_{n+1}}$ for every $n \in \mathbb{N}$.*

Theorem 3.15. *Suppose that (A_k) is $\mathcal{I}-$monotonic increasing sequence of closed subsets of X. Then $\mathcal{I} - \lim_{k\to\infty} A_k$ exists and*

$$\mathcal{I} - \lim_{k\to\infty} A_k = cl \bigcup_{n\in\mathbb{N}} A_{k_n}.$$

Proof. Let (A_k) is a $\mathcal{I}-$monotonic increasing sequence of closed subsets of X and $A = cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. Then, $A_{k_n} \subseteq A$ for every $n \in \mathbb{N}$. If $A = \emptyset$, then $A_{k_n} = \emptyset$ for every $n \in \mathbb{N}$. So, $\mathcal{I} - \lim A_k = \emptyset$. Let $A \neq \emptyset$ and $x \in cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. In this case, for every $\varepsilon > 0$

$$B(x, \varepsilon) \cap \bigcup_{n\in\mathbb{N}} A_{k_n} \neq \emptyset.$$

Then there exists $n_0 \in \mathbb{N}$ such that $B(x,\varepsilon) \cap A_{k_{n_0}} \neq \emptyset$. Since (A_{k_n}) is an increasing sequence, $A_{k_{n_0}} \subseteq A_{k_n}$ for all $n \geq n_0$. Define the set M

$$M = \{m \mid m = k_n,\ n \geq n_0,\ n \in \mathbb{N}\}.$$

Then $M \in F(\mathcal{I})$ and $B(x,\varepsilon) \cap A_m \neq \emptyset$ for all $m \in M$. Consequently, we obtain $x \in \mathcal{I} - \liminf_{k\to\infty} A_k$.

Now we show that $\mathcal{I} - \limsup_{k\to\infty} A_k \subseteq A$. Let $x \in \mathcal{I} - \limsup_{k\to\infty} A_k$ be arbitrary. Then for every $\varepsilon > 0$ there exists $N \in \mathcal{N}_{\mathcal{I}}^{\#}$ such that for every $k \in N$ we have $A_k \cap B(x,\varepsilon) \neq \emptyset$. By Lemma 2.2, since $K \in F(\mathcal{I})$ and $N \notin \mathcal{I}$, we have $K \cap N \notin \mathcal{I}$. So, there exists $k_{n_0} \in K \cap N$ such that

$$B(x,\varepsilon) \cap A_{k_{n_0}} \neq \emptyset.$$

Therefore we obtain

$$B(x,\varepsilon) \cap \bigcup_{n\in\mathbb{N}} A_{k_n} \neq \emptyset.$$

This means that $x \in cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. This step concludes the proof. □

Theorem 3.16. *Suppose that (A_k) is an $\mathcal{I}-$monotonic decreasing sequence of closed subsets of X. Then $\mathcal{I}-\lim_{k\to\infty} A_k$ exists and*

$$\mathcal{I} - \lim_{k\to\infty} A_k = \bigcap_{n\in\mathbb{N}} A_{k_n}.$$

Proof. Let $A = \bigcap_{n\in\mathbb{N}} A_{k_n}$. Clearly if $x \in A$, then $x \in A_{k_n}$ for every $n \in \mathbb{N}$. Define $M = \{m \mid m = k_n, n \in \mathbb{N}\}$. Then $M \in F(\mathcal{I})$. Also for all $\varepsilon > 0$ and $m \in M$ we have $B(x,\varepsilon) \cap A_m \neq \emptyset$. This means that $x \in \mathcal{I} - \liminf_{k\to\infty} A_k$.

Now we show that $\mathcal{I} - \limsup_{k\to\infty} A_k \subseteq A$. Let $x \in \mathcal{I} - \limsup_{k\to\infty} A_k$ be arbitrary. Then, for every $\varepsilon > 0$ there exists $N \notin \mathcal{I}$ such that for every $m \in N$, $A_m \cap B(x,\varepsilon) \neq \emptyset$. Since $\mathcal{I}$ is an admissible, N is infinite. So for every $n \in \mathbb{N}$ there exists $m \in N$ such that $k_n \leq m$. Since the sequence (A_k) is decreasing, the inclusion $A_{k_n} \supseteq A_m$ holds and consequently $B(x,\varepsilon) \cap A_{k_n} \neq \emptyset$. This means that $x \in clA_{k_n}$. Since A_{k_n} is closed, $x \in A_{k_n}$. Therefore $x \in \bigcap_{n\in\mathbb{N}} A_{k_n}$. This step concludes the proof. □

In the next section we introduce Hausdorff $\mathcal{I}-$convergence of closed sets. Then, we compare Hausdorff $\mathcal{I}-$convergence and Kuratowski $\mathcal{I}-$convergence of the sequence of closed sets.

4. Hausdorff $\mathcal{I}-$Convergence

The Hausdorff distance $h(E,F)$ between the subsets E and F of X is defined as follows:

$$h(E,F) = \max\{D(E,F), D(F,E)\},$$

where

$$D(E,F) = \sup_{x\in E} d(x,F) = \inf\{\varepsilon > 0 : E \subseteq B(F,\varepsilon)\}$$

unless both E and F are empty in which case $h(E,F) = 0$. Note that if only one of the two sets is empty then $h(E,F) = \infty$.

It is known, for a long time (see [3, 14]), that

$$h(E,F) \quad = \quad \sup_{x\in X} |d(x,E) - d(x,F)|.$$

Definition 4.1. *Let (A_n) be a sequence of closed subsets of X. We say that the sequence (A_n) is Hausdorff $\mathcal{I}-$convergent to a closed subset A of X if*

$$\mathcal{I} - \lim_{n\to\infty} h(A_n, A) = 0. \tag{11}$$

In this case, we write $A = H_{\mathcal{I}} - \lim_{n\to\infty} A_n$.

Lemma 4.2. *Suppose that* $\{A; A_n, n \in \mathbb{N}\}$ *is a family of closed subsets of* X. *Then* $A = H_{\mathcal{I}} - \lim_{n\to\infty} A_n$ *if and only if either there exists* $M \in F(\mathcal{I})$ *such that* A *and* A_n *are empty for all* $n \in M$ *or for any* $\varepsilon > 0$ *the sets*

$$\left\{n \in \mathbb{N} : A \not\subseteq B(A_n, \varepsilon)\right\} \quad \textit{and} \quad \left\{n \in \mathbb{N} : A_n \not\subseteq B(A, \varepsilon)\right\} \tag{12}$$

belong to $\mathcal{I}$.

Proof. If $A = \emptyset$, then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon\right\} = \left\{n \in \mathbb{N} : A_n \neq \emptyset\right\}.$$

Thus $\left\{n \in \mathbb{N} : A_n \neq \emptyset\right\} \in \mathcal{I}$. Namely, $\left\{n \in \mathbb{N} : A_n = \emptyset\right\} \in F(\mathcal{I})$.

Conversely, there exists $M \in F(\mathcal{I})$ such that A_n is empty for all $n \in M$. Then, for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : h(A_n, \emptyset) \geq \varepsilon\right\} \in \mathcal{I}.$$

So $A = \emptyset$.

On the other hand if $A \neq \emptyset$, then (11) holds if and only if for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon\right\} \in \mathcal{I}$$

or equivalently,

$$\left\{n \in \mathbb{N} : h(A_n, A) < \varepsilon\right\} \in F(\mathcal{I}).$$

By the definition of Hausdorff metric,

$$\left\{n \in \mathbb{N} : A \subseteq B(A_n, \varepsilon) \text{ and } A_n \subseteq B(A, \varepsilon)\right\} \in F(\mathcal{I}).$$

Consequently,

$$\left\{n \in \mathbb{N} : A \not\subseteq B(A_n, \varepsilon)\right\} \cup \left\{n \in \mathbb{N} : A_n \not\subseteq B(A, \varepsilon)\right\} \in \mathcal{I}.$$

This completes the proof. □

The next theorem answers a natural question about relationships between Hausdorff $\mathcal{I}-$convergence and Kuratowski $\mathcal{I}-$convergence.

Theorem 4.3. *Suppose that* $\{A; A_n, n \in \mathbb{N}\}$ *is a family of closed subsets of* X *with* $A \neq \emptyset$. *Then Hausdorff* $\mathcal{I}-$*convergence implies Kuratowski* $\mathcal{I}-$*convergence, i.e.,*

$$H_{\mathcal{I}} - \lim_{n\to\infty} A_n = A \textit{ implies } \mathcal{I} - \lim_{n\to\infty} A_n = A.$$

Proof. Take $x \in A$. By (12), for any $\varepsilon > 0$

$$M = \left\{n \in \mathbb{N} : A \subseteq B(A_n, \varepsilon)\right\} \in F(\mathcal{I}).$$

Then, for $n \in M$ we have $B(x, \varepsilon) \cap A_n \neq \emptyset$. So condition (i) in Remark 3.2 is provided.

Conversely, $x \notin A$. Then, there exists $\varepsilon > 0$ such that $x \notin B(A, \varepsilon)$, i.e., $d(x, A) > \varepsilon$. By (12)

$$K = \left\{n \in \mathbb{N} : A_n \subseteq B(A, \varepsilon)\right\} \in F(\mathcal{I}).$$

Take $\delta = d(x, A) - \varepsilon$. Then, for $n \in K$ we obtain $B(x, \delta) \cap A_n = \emptyset$. So condition (ii) in Remark 3.2 is provided. From conditions (i) and (ii) in Remark 3.2 we have $\mathcal{I} - \lim_{n\to\infty} A_n = A$. □

Definition 4.4. *The sequence* (A_n) *is said to be* $\mathcal{I}-$*bounded if there exists a compact set* K *such that*

$$\left\{n \in \mathbb{N} : A_n \not\subseteq K\right\} \in \mathcal{I}.$$

Now, our aim is to show that, for a $\mathcal{I}-$bounded closed set, Kuratowski $\mathcal{I}-$convergence is equivalent to Hausdorff $\mathcal{I}-$convergence.

Theorem 4.5. *Let (A_n) be a $\mathcal{I}-$bounded sequence of closed subsets of X. If $\mathcal{I}-\lim_{n\to\infty} A_n = A$ with $A \neq \emptyset$, then $H_{\mathcal{I}}-\lim_{n\to\infty} A_n = A$.*

Proof. Let (A_n) be a $\mathcal{I}-$bounded sequence of closed subsets of X. Then there is a compact subset K of X such that

$$M = \left\{n \in \mathbb{N} : A_n \subseteq K\right\} \in F(\mathcal{I}).$$

By Lemma 3.12, $\mathcal{I}-\lim_{n\to\infty} A_n = A \subseteq K$. So, the closed set A is compact. Then given $\varepsilon > 0$, A has a finite cover with open balls of radius ε; i.e., there exists $\{x_1, x_2, x_3, \dots, x_n\}$ with $x_i \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{\varepsilon}{2}\right).$$

Since $\mathcal{I}-\lim_{n\to\infty} A_n = A$ and $x_i \in A$ for $i \in \{1, 2, \dots, n\}$, we obtain $\mathcal{I}-\lim_{n\to\infty} d(x_i, A_n) = 0$. Therefore, for each i

$$\{n \in \mathbb{N} : d(x_i, A_n) < \varepsilon/2\} \in F(\mathcal{I}).$$

Let us define

$$N = \bigcap_{i=1}^{n} \{n \in \mathbb{N} : d(x_i, A_n) < \varepsilon/2\}.$$

Then $N \in F(\mathcal{I})$. Thus, we obtain

$$d(y, A_n) \leq d(y, x_i) + d(x_i, A_n) < \varepsilon$$

for any $y \in A$ and $n \in N$. So, $A \subseteq B(A_n, \varepsilon)$ for every $n \in N$. This means that $\left\{n \in \mathbb{N} : A \nsubseteq B(A_n, \varepsilon)\right\} \in \mathcal{I}$.

Now, suppose that $C = \left\{n \in \mathbb{N} : A_n \nsubseteq B(A, \varepsilon)\right\} \notin \mathcal{I}$ for some $\varepsilon > 0$. Then, there exists a sequence $\{y_k \mid y_k \in A_k \backslash B(A, \varepsilon),\ k \in C\}$. By Lemma 2.2, $M \cap C \notin \mathcal{I}$. Hence, $\{k \mid y_k \in K\} \notin \mathcal{I}$. By Lemma 2.5, the sequence (y_n) has at least $\mathcal{I}-$cluster point that belongs to $\mathcal{I}-\limsup_{n\to\infty} A_n = A$ but does not belong to $B(A, \varepsilon) \supseteq A$, which leads to a contradiction. So we have shown that $\left\{n \in \mathbb{N} : A_n \nsubseteq B(A, \varepsilon)\right\} \in \mathcal{I}$. This completes the proof. □

5. Conclusion

In this paper we give the definitions and some properties of $\mathcal{I}-$outer and $\mathcal{I}-$inner limits for a sequence of closed sets. We have also introduced two kinds of $\mathcal{I}-$convergence for sequences of closed sets which are called Kuratowski $\mathcal{I}-$convergence and Hausdorff $\mathcal{I}-$convergence. We prove that Hausdorff $\mathcal{I}-$convergence implies Kuratowski $\mathcal{I}-$convergence. Additionally, for a $\mathcal{I}-$bounded sequence of closed sets, we show that these convergences are equivalent.

Continuity properties of a set-valued mapping can be defined on the basis of Kuratowski convergence or Hausdorff convergence (see Chapter 1 in [1], Chapter 3 in [8] and Chapter 5 in [20]). In the light of the main results of our paper, one can define $\mathcal{I}-$continuity for a set-valued mapping and get $\mathcal{I}$ analogues of continuity properties.

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References

[1] J. P. Aubin, H. Frankowska, Set-Valued Analysis, Birkhäuser, Boston, 1990.
[2] M. Balcerzak, K. Dems, A. Komisarski, Statistical convergence and ideal convergence for sequences of functions, Journal of Mathematical Analysis and Applications 328 (2007) 715–729.
[3] M. Baronti, P. Papini, Convergence of sequences of sets, Methods of functional analysis in approximation theory, ISNM 76 Birkhäuser, Basel, (1986) 135–155.
[4] G. Beer, Topologies on closed and closed convex sets, Kluwer Academic, Dordrecht, 1993.
[5] G. Beer, On convergence of closed sets in a metric space and distance functions, Bulletin of the Australian Mathematical Society 31 (1985) 421–432.
[6] J. Cincura, T. Šalát, M. Sleziak, V. Toma, Sets of statistical cluster points and $\mathcal{I}-$cluster points, Real Analysis Exchange 30(2) (2004/2005) 565–580.
[7] K. Demirci, $\mathcal{I}-$limit superior and limit inferior, Mathematical Communications 6 (2001) 165–172.
[8] A. L. Dontchev, R. T. Rockafellar, Implicit functions and solution mappings, A view from variational analysis, Springer, 2009.
[9] H. Fast, Sur la convergence statistique, Colloquium Mathematicum 2 (1951) 241–244.
[10] A. Komisarski, Pointwise $\mathcal{I}-$convergence and $\mathcal{I}-$convergence in measure of sequences of functions, Journal of Mathematical Analysis and Applications 340 (2008) 770–779.
[11] P. Kostyrko, T. Šalát, W. Wilczyński, $\mathcal{I}-$convergence, Real Analysis Exchange 26(2) (2000) 669–686.
[12] P. Kostyrko, M. Mačaj, T. Šalát, M. Sleziak, $\mathcal{I}-$convergence and extremal $\mathcal{I}-$limit points, Mathematica Slovaca 55 (2005) 443–464.
[13] V. Kumar, K. Kumar, On the ideal convergence of sequences of fuzzy numbers, Information Sciences 178 (2008) 4670–4678.
[14] C. Kuratowski, Topologie, vol.I, PWN, Warszawa, 1958.
[15] A. Löhne, C. Zalinescu, On convergence of closed convex sets, Journal of Mathematical Analysis and Applications 319 (2006) 617–634.
[16] M. Mursaleen, S. A. Mohiuddine, On ideal convergence in probabilistic normed spaces, Mathematica Slovaca 62 (2012) 49–62.
[17] A. Nabiev, S. Pehlivan, M. Gürdal, On $\mathcal{I}-$Cauchy sequence, Taiwanese Journal of Mathematics 11(2) (2007) 569–576.
[18] F. Nuray, W. H. Ruckle, Generalized statistical convergence and convergence free spaces, Journal of Mathematical Analysis and Applications 245 (2000) 513–527.
[19] F. Nuray, B. E. Rhoades, Statistical convergence of sequences of sets, Fasciculi Mathematici 49 (2012) 87–99.
[20] R. T. Rockafellar, R. J-B. Wets, Variational Analysis, Springer, Berlin, 1998.
[21] T. Šalát, B. C. Tripathy, M. Ziman, On $\mathcal{I}-$convergence field, Italian Journal of Pure and Applied Mathematics 17 (2005) 45–54.
[22] G. Salinetti, R. J-B. Wets, On the convergence of sequences of convex sets in finite dimensions, SIAM Review 21 (1979) 18–33.
[23] I. J. Schoenberg, The integrability of certain functions and related summability methods, The American Mathematical Monthly 66 (1959) 361–375.
[24] Y. Sonntag, C. Zalinescu, Scalar convergence of convex sets, Journal of Mathematical Analysis and Applications 164 (1992) 219–241.
[25] Y. Sonntag, C. Zalinescu, Set convergences. An attempt of classification, Transactions of the American Mathematical Society 340(1) (1993) 199-226.
[26] A. Şahiner, M. Gürdal, S. Saltan, H. Gunawan, Ideal convergence in 2-normed spaces, Taiwanese Journal of Mathematics 11(5) (2007) 1477–1484.
[27] Ö. Talo, Y. Sever, F. Başar, On statistically convergent sequences of closed sets, Filomat 30(6) (2016) 1497-1509.
[28] R. A. Wijsman, Convergence of sequences of convex sets, cones and functions, American Mathematical Society. Bulletin 70 (1964) 186–188.
[29] R. A. Wijsman, Convergence of sequences of convex sets, cones and functions II, Transactions of the American Mathematical Society 123(1) (1966) 32–45.