Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi
Afyon Kocatepe University Journal of Science and Engineering

AKÜ FEMÜBİD **21** (2021) 061301 (1324-1329) AKU J. Sci. Eng. **21** (2021) 061301 (1324-1329)
DOI: 10.35414/akufemubid.977234

Araştırma Makalesi/Research Article

Energies with Constant Mean Curvature of Tubular Surfaces by Bishop Frame

Filiz ERTEM KAYA[1]
[1] *Nigde Ömer Halisdemir University, Faculty of Science-Art, Department of Mathematics, Nigde/TURKEY*

Sorumlu Yazar e-posta: fertem@ohu.edu.tr ORCID ID: http://orcid.org/0000-0003-1538-9154

Geliş Tarihi: 01.08.2021 Kabul Tarihi: 28.12.2021

Keywords
Tubular surfaces; Curvature; Bishop frame; Willmore energy

Abstract

In this paper Euclidean metric induced by µ : M → G be the mean curvature of Tubular surfaces by Bishop frame are computed and the curvatures in differential geometry that are so important for computing the Helfrich and Willmore energy of the tubular surfaces by bishop frame and giving some theorems are seen. Even though these calculations are very important to prove that the curvatures are very important in differential geometry, in actually these calculations are clearly important as mathematical physics.

Sabit Ortalama Eğrilikli Bishop Çatılı Tubular Yüzeylerin Enerjileri

Anahtar kelimeler
Tubular yüzeyler; Eğrilikler; Bishop çatısı; Willmore enerji

Öz

Bu makalede µ : M → G Öklid metrikli Bishop çatılı Tubular yüzeylerin Helfrich ve Willmore enerjileri hesaplandı and Bishop çatılı Tubular yüzeylerin Helfrich ve Willmore enerjilerinin bulunmasında ve bazı teoremlerin verilmesinde diferensiyel geometrideki eğriliklerinin ne denli önemli olduğu görüldü. Bu hesaplamalar her ne kadar diferensiyel geometri için eğriliklerin önemli olduğunu ispatlasa da aslında bu hesaplamalar matematiksel fizik için de önem arz etmektedir.

1. Introduction

Energy is the most important thing for people, animals, plant and all living things. For example the sun is the source of the energy. It is naturel energy for the world. But the other type of energies are artificial. these are very special for physicians and mathematicians.

For $a, b \geq 0$ and $c \in R$, the Helfrich energy of a surface Σ is

$$\mathrm{E}(\Sigma) = \int_{\Sigma} a + bH^2 + cKds$$

$$= aA(\Sigma) + \int_{\Sigma} bH^2 + cKds \qquad (1)$$

The Willmore energy is found by Willmore as follows;
The Willmore energy $W(M)$ of M is given by

$$W(M) = \int_M H^2 dA \qquad (2)$$

(Gray 2005, Willmore 1959, Willmore 1965).

In this paper we use Helfrich energy and Willmore energy formulae and we calculate the Helfrich energy and Willmore energy of the Tubular Surfaces by Bishop frame.

2. Preliminaries

In differential geometry a canal surface is defined as the envelope of a family of some special spheres that have specific characters about their curvatures (Kişi 2017).

If the radius function is constant, then the canal surface is called a tube or tubular surface (Doğan and Yaylı 2011, Kişi 2017).

Let α be a unit speed curve in Euclidean space. If $N_2(s)$ orthogonal to both $T(s)$ and $N_1(s)$, then

$N_2(s) = T(s) \times N_1(s)$ (positively oriented-we say about these according to the righ hand rule or the clockwise in physics) according to this it means $\{T(s), N_1(s), N_2(s)\}$ an orthonormal frame is called Bishop frame. All we have the following formulaes from the references (Doğan and Yaylı 2011, Doğan and Yaylı 2012):

$$T'(s) = k_1 N_1 + k_2 N_{21} \quad (3)$$

$$N_1' = -k_1 T \quad (4)$$

$$N_1' = -k_2 N_2 \quad (5)$$

$$k_1 = \kappa \cos \alpha \quad (6)$$

$$k_2 = \kappa \sin \alpha \quad (7)$$

$$\tau = \gamma' \quad (8)$$

3. Mean and Gauss Curvatures of a Canal Surface

The mean H and Gauss K curvatures of tube surface are computed in (Doğan and Yaylı 2011) by 3 situation. These are,

(i) If the center curve is given with Frenet frame, then the tube is around the center curve α becomes $Y(s,\theta) = \alpha(s) + r(\cos\theta N + \sin\theta B)$, using derivative formulas of this and

$$K = \frac{-\kappa \cos\theta}{r(1 - r\kappa\cos\theta)}, \quad (9)$$

$$H = \frac{1}{2}\left[\frac{1}{r} + \kappa r\right] \quad (10)$$

are obtained (Doğan and Yaylı 2011, Doğan and Yaylı 2012).

(ii)If the center curve is given with Bishop frame, then the tube around the center curve α becomes $Z(s,\theta) = \alpha(s) + r(\cos\theta N_1 + \sin\theta N_2)$, using derivative formulas of this and

$$K = \frac{-k_1\cos\theta + k_2\sin\theta}{r(r(k_1\cos\theta + k_2\sin\theta) - 1)} \quad (11)$$

$$H = rK - \frac{K}{2(k_1\cos\theta + k_2\sin\theta)} \quad (12)$$

are obtained (Doğan and Yaylı 2011, Doğan and Yaylı 2012).

(iii)Let W be a regular surface with a unit normal V and the center curve $\alpha : I \subset R \to W$ be a unit speed. In case the tube around the center curve α becomes $X(s,\theta) = \alpha(s) + r(\cos\theta P + \sin\theta S)$ using derivative formulas of this (P is normal and S is the binormal of the curve)

$$K = \frac{-k_g\cos\theta + k_n\sin\theta}{r(1 - rk_g\cos\theta + rk_n\sin\theta)} \quad (13)$$

$$H = \frac{2r(k_g\cos\theta + k_n\sin\theta) - 1}{2r(rk_g\cos\theta + rk_n\sin\theta - 1)} \quad (14)$$

are obtained (Doğan and Yaylı 2011, Doğan and Yaylı 2012).

From (Doğan and Yaylı 2011), let the center are obtained in $\alpha(s)$ be a spherical curve with order α^2. According to (Bishop 1975), if a space curve is a spherical curve, then the pair (k_1, k_2) lies on not passing throug a line $gx + hy + 1 = 0$. For the $s-$parameter curves $g = \cos\theta$ and $h = \sin\theta$,

$$k_1\cos\theta + k_2\sin\theta + 1 = 0 \quad (15)$$

So

$$K = \frac{-k_1\cos\theta + k_2\sin\theta}{r(r(k_1\cos\theta + k_2\sin\theta) - 1)} = \frac{1}{r(r+1)} \quad (16)$$

$$H = rK - \frac{K}{2(k_1\cos\theta + k_2\sin\theta)} = \frac{2r+1}{2r(r+1)} \quad (17)$$

(Liu 2005, Doğan and Yaylı 2011).

4. Willmore Function on Curvatures of Tubular Surfaces

In 1965 Willmore (Willmore 1959, Willmore 1965) proposed the study of the functional as follows:

By using (2) we compute the Willmore of the Tube surface for the situation (i), so we obtain theorem as below:

Theorem 4.1. In this notation, the Willmore energy $W_i(Y_{tube})$ of M is given by Willmore energy of the Tube surface for the situation (i) as follows:

$$W_i(Y_{tube}) = \int_{Y_{tube}} \left[\begin{array}{l} \dfrac{1}{4r^2} - \dfrac{1}{2r}\dfrac{\kappa\cos\theta}{(1-r\kappa\cos\theta)} \\ + \dfrac{\kappa^2\cos^2\theta}{4(1-r\kappa\cos\theta)^2} \end{array} \right] dA \tag{18}$$

Proof. Let the mean H curvature of the tube surface and $W_i(Y_{tube})$ Willmore energy of the tube surface for the situation (i), so we take

$$\begin{aligned} W_i(Y_{tube}) &= \int_{Y_{tube}} H^2 dA \\ &= \int_{Y_{tube}} \left[\frac{1}{2}\left[\frac{1}{r} + \kappa r\right]\right]^2 dA \\ &= \int_{Y_{tube}} \frac{1}{4}\left[\frac{1}{r} + \frac{-\kappa\cos\theta}{r(1-r\kappa\cos\theta)} r\right]^2 dA \\ &= \int_{Y_{tube}} \left[\begin{array}{l} \dfrac{1}{4r^2} + \dfrac{1}{4}2\dfrac{1}{r}\dfrac{-\kappa\cos\theta}{r(1-r\kappa\cos\theta)} r \\ + \dfrac{-\kappa^2\cos^2\theta}{4(1-r\kappa\cos\theta)^2} \end{array} \right]^2 dA \\ &= \int_{Y_{tube}} \left[\begin{array}{l} \dfrac{1}{4r^2} - \dfrac{1}{2r}\dfrac{\kappa\cos\theta}{(1-r\kappa\cos\theta)} \\ + \dfrac{\kappa^2\cos^2\theta}{4(1-r\kappa\cos\theta)^2} \end{array} \right] dA \end{aligned} \tag{19}$$

This is the proof of the theorem.

Theorem 4.2. Let take H the mean curvature of the Tube Surface. Thus we can obtain the Willmore energy of the Tube surface for the situation (ii) on the side:

$$W_{ii}(Z_{tube}) = \int_{Z_{tube}} \left[\begin{array}{l} \dfrac{k_1^2\cos^2\theta - k_1k_2\sin 2\theta + k_2^2\sin^2\theta}{[r(k_1\cos\theta + k_2\sin\theta) - 1]^2} \\ - r\dfrac{\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}\right]^2}{k_1\cos\theta + k_2\sin\theta} \\ + \dfrac{\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}\right]^2}{2(k_1\cos\theta + k_2\sin\theta)^2} \end{array} \right] dA \tag{20}$$

Proof. Let substituting (17) in (2), so we obtain,

$$\begin{aligned} W_{ii}(Z_{tube}) &= \int_M H^2 dA \\ &= \int_M \left[rK - \frac{K}{2(k_1\cos\theta + k_2\sin\theta)} \right]^2 dA \\ &= \int_{Z_{tube}} \left[\begin{array}{l} r^2\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}\right]^2 \\ - r\dfrac{\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}\right]^2}{k_1\cos\theta + k_2\sin\theta} \\ + \dfrac{\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}\right]^2}{2(k_1\cos\theta + k_2\sin\theta)^2} \end{array} \right] dA \\ &= \int_{Z_{tube}} \left[\begin{array}{l} r^2\dfrac{k_1^2\cos^2\theta - k_1k_2\sin 2\theta + k_2^2\sin^2\theta}{r^2[r(k_1\cos\theta + k_2\sin\theta) - 1]^2} \\ - r\dfrac{\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}\right]^2}{k_1\cos\theta + k_2\sin\theta} \\ + \dfrac{\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}\right]^2}{2(k_1\cos\theta + k_2\sin\theta)^2} \end{array} \right] dA \end{aligned}$$

$$= \int_{Z_{tube}} \left[\begin{array}{l} \dfrac{k_1^2 \cos^2\theta - k_1 k_2 \sin 2\theta + k_2^2 \sin^2\theta}{\left[r(k_1\cos\theta + k_2\sin\theta) - 1\right]^2} \\ - r\dfrac{\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r\left[r(k_1\cos\theta + k_2\sin\theta) - 1\right]}\right]^2}{k_1\cos\theta + k_2\sin\theta} \\ + \dfrac{\left[\dfrac{-k_1\cos\theta + k_2\sin\theta}{r\left[r(k_1\cos\theta + k_2\sin\theta) - 1\right]}\right]^2}{2(k_1\cos\theta + k_2\sin\theta)^2} \end{array} \right] dA$$

This completes the proof and we have (20).

Corollary 4.3. In addition we can write

$$W_{i\,ii}(X_{tube}) = \int_M H^2 dA$$

$$= \int_{X_{tube}} \left[\frac{2r(k_g \cos\theta + k_n \sin\theta) - 1}{2r(rk_g\cos\theta + rk_n\sin\theta - 1)} \right]^2 dA \tag{21}$$

Corollary 4.4. If we compute the Willmore energy for , then we obtain

$$W_{i\,ii}(S_{tube}) = \int_{S_{tube}} H^2 dA$$

$$= \int_{S_{tube}} \frac{(2r+1)^2}{4r^2(r+1)^2} dA \tag{22}$$

Since r is constant, Willmore energy of S_{tube} is taken as

$$W_{i\,ii}(S_{tube}) = \frac{(2r+1)^2}{4r^2(r+1)^2} \int_{S_{tube}} dA$$

$$= \frac{(2r+1)^2}{4r^2(r+1)^2} A + c \tag{23}$$

5. Helfrich Energy for Tube Surfaces

Defintion 5.1. For $a, b \geq 0$ and $c \in R$, From (Karsten 2012) we can give the Helfrich energy of a surface Σ is

$$\mathrm{E}(\Sigma) = \int_\Sigma a + bH^2 + cKds$$

$$= aA(\Sigma) + \int_\Sigma bH^2 + cKds \tag{24}$$

Now let compute the Helfrich energies for tube surfaces according to the 3 situation

Theorem 5.1. If we compute the Willmore energy $E_i(Y_{tube})$ of M is given by Helfrichenergy of the Tube surfaces for the situation (i), then we take as follows:

$$E_i(Y_{tube}) = aA(Y_{tube}) + \int_E b\left[\frac{1}{2}\left[\frac{1}{r} + Kr\right]\right]^2 + c\frac{-\kappa\cos\theta}{r(1 - r\kappa\cos\theta)} ds \tag{25}$$

Proof. Let take the H **and** K be the mean and gauss curvature of the Tube surfaces by Bishop frame. The Helfrich energy of the Tube surface $E_i(Y_{tube})$ of M is given by for the situation (i), so we have

$$\mathrm{E}(Y_{tube}) = aA(Y_{tube}) + \int_{Y_{tube}} bH^2 + cKds$$

$$E_i(Y_{tube}) = aA(Y_{tube}) + \int_{Y_{tube}} b\left[\frac{1}{2}\left[\frac{1}{r} + Kr\right]\right]^2 + c\frac{-\kappa\cos\theta}{r(1 - r\kappa\cos\theta)} ds$$

It is the proof of the theorem and we obtain (25).

Corollary 5.1. Let calculate the Helfrich energy of the Tube surface by Bishop frame for situation (ii). So we have

$$E_i(Z_{tube}) = aA(Z_{tube}) + \int_{Z_{tube}} b\left[rK - \frac{K}{2(k_1\cos\theta + k_2\sin\theta)}\right]^2 + c\frac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}ds \quad (26)$$

Corollary 5.2. Let calculate the Helfrich energy of the Tube surface by Bishop frame for situation (iii).

$$E_i(X_{tube}) = aA(X_{tube}) +$$

So we obtain $\int_{X_{tube}} b\left[rK - \frac{K}{2(k_1\cos\theta + k_2\sin\theta)}\right]^2 + c\frac{-k_1\cos\theta + k_2\sin\theta}{r[r(k_1\cos\theta + k_2\sin\theta) - 1]}ds$ (27)

Corollary 5.3. For $k_1\cos\theta + k_2\sin\theta + 1 = 0$,
If r is constant, then we obtain, the Helfrich energy is written as;

$$E_i(S_{tube}) = aA(S_{tube}) + \int_{S_{tube}} b\frac{(2r+1)^2}{4r^2(r+1)^2} + c\frac{1}{r(r+1)}ds \quad (28)$$

Exercise 5.4. Let compute the Willmore energy of the tube surface for the situation (i), with the curve $\alpha(t) = (t\cos t, t\sin t, t)$ at the point .

Solution. Firstly we have to find the curvatures of the curve. So, we obtain

$\alpha'(t) = (\cos t - \sin t, \sin t + \cos t, 1)$

$\alpha''(t) = (-2\sin t - t\cos t, 2\cos t - t\sin t, 0)$

$\alpha'''(t) = (-3\cos t + t\sin t, -3\sin t - t\cos t, 0)$

and

$\|\alpha'(t)\| = \sqrt{2 + t^2}$.

At the point $t = 0$, we can find values of the differentials of the curve α as follows:

$\alpha'(0) = (1,0,1)$,

$\alpha''(0) = (0,2,0)$,

$\alpha'''(0) = (-3,0,0)$.

So, we have the values f the Frenet frame elements

$T(0) = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$,

$B(0) = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$,

$N(0) = (0,-1,0)$.

After these calculations we obtain the curvatures of the curve as follows:

$K(0) = 1$

$\tau(0) = \frac{3}{4}$,

Willmore energy of the tube surface for the situation (i) can be obtained, so we have

$$W_i(Y_{tube}) = \int_{Y_{tube}} \left[\frac{1}{4r^2} - \frac{1}{2r}\frac{\kappa\cos\theta}{(1 - r\kappa\cos\theta)} + \frac{\kappa^2\cos^2\theta}{4(1 - r\kappa\cos\theta)^2}\right]dA$$

$$= \int_{Y_{tube}} \left[\frac{1}{4r^2} - \frac{1}{2r}\frac{1\cos\theta}{(1 - r1\cos\theta)} + \frac{1^2\cos^2\theta}{4(1 - r1\cos\theta)^2}\right]dA$$

$$= \int_{Y_{tube}} \left[\frac{1}{4r^2} - \frac{1}{2r}\frac{\cos\theta}{(1 - r\cos\theta)} + \frac{\cos^2\theta}{4(1 - r\cos\theta)^2}\right]dA.$$

Special Case:

If radius $r = 1$ and the angle $\theta = 30$, then Willmore energy of the tube surface for the situation (i) can be obtained as follows:

$$W_i(Y_{tube}) = -\frac{7}{4}\int_{Y_{tube}} dA$$

$$W_i(Y_{tube}) = -\frac{7}{4}A\Big|_{Y_{tube}}$$

6. Conclusions

In this work we study and compute the Helfrich and Willmore energies of the Tubular surfaces We obtain the mean and the Gauss curvatures are so important for calculating the energies of the tubular surfaces by Bishop frame.

If we compare the energy of the tube surface with Bishop frame and energy of the the tube surface with Frenet frame, then we obtain that the energy of tube surface with Bishop frame is less than the energy of the tube surface with Frenet frame. Because K and H of the tube surface with Bishop frame is less than K and H of the tube surface with Frenet frame.

7. References

Bishop, R. L., 1975. There is no more than one way to frame a curve, *American Mathematical Monthly*, **82** (3), 246-251.

Doğan, F. and Yaylı, Y., 2011. On the tubular surface with Bishop frame, *Commun. Fac. Sci. Univ. Ank. SeriesA1*, **60** (1), 59-69.

Doğan, F. 2012. A note on Tubes*, International Journal of physical and Mathematical Sciences,* **3** (1), 98-105.

Doğan, F. and Yaylı, Y. 2012. Tubes with darboux frame, *Int J. Contemp. Math. Sciences,* **7** (16), 751-758.

Gray, A., 2005. Modern Differential Geometry of Curves and Surfaces with Matematica, Third Edition by Elsa Abbana and Simon Salamon, 821-927.

Karsten, G.-B., 2012. Triply periodic Minimal and Constant Mean curvature Surfaces, *Interfaces Focus*; **2**, 582-588.

Kişi İ., Öztürk, G., 2017. A new approach to canal surface with parallel transport frame, *International Journal of Geometric Methods in Modern Physics*, **14**(2), 1750026.

Liu, Bing, Sclabassi, R.J., Liu, Qiang, Kassam, A., Li, ChingChung, Sun, Mingui, 2005. Detection of region of interest in neurosurgical video used for telemedicine, *2005 IEEE International Conference on Information Acquisition.*

Willmore, T. J., 1959. Introduction to Differential Geometry, Clarendon, Oxford, 128.

Willmore, T. J., 1965. Note on embedded surfaces, *An. Sti. Univ. Al. Cusa Iasi, N. Ser., Sect. la Mat. 11B*, 493-496.