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Araştırma Makalesi / Research Article

On Various Types w −Compatible Mappings in Modular A −Metric Spaces

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w −compatible maps; w −compatible maps of type (α); w −compatible maps of type (β); Modular A −metric spaces

Abstract

In this article, we have established three new types of compatible mappings named w −compatible maps, w −compatible maps of type $(\alpha)$and w −compatible maps of type (β)in modular A −metric space. Then, by using these definitions we examine the relations between them. The results in this article generalize many known results in the existing literature.

Modüler A −Metrik Uzaylarda Çeşitli Tipte w −Bağdaşabilir Dönüşümler

Anahtar Kelimeler

w −bağdaşabilir dönüşümler; (α) tip w −bağdaşabilir dönüşümler; (β) tip w −bağdaşabilir dönüşümler; Modüler A −metrik uzaylar

Öz

Bu makalede, modüler A −metrik uzaylar üzerinde w −bağdaşabilir dönüşümler, (α) tip w −bağdaşabilir dönüşümler ve (β) tip w −bağdaşabilir dönüşümler olarak adlandırdığımız üç yeni tip bağdaşabilir dönüşüm tanımladık. Bu tanımları kullanarak aralarındaki ilişkileri araştırdık. Bu makalede elde edilen sonuçlar literatürde var olan ve iyi bilinen birçok sonucu genelleştirmesidir.

1. Introduction

In the last quarter of the 20th century, the problem of finding common fixed points of mappings that satisfied the contraction condition began to attract a lot of attention. In 1986, Jungck generalized the Banach contraction theorem using compatible maps and obtained the common fixed point theorem (Jungck 1986). After this, Jungck et al. (1993) gave the concept compatible maps of type (A) and Pathak et al. (1996) also gave the notion compatible maps of type (P) in metric spaces. The fixed point theory for these mappings in metric spaces and different generalized metric spaces was extensively studied by many mathematicians (Cho et al. 1998, Kutukcu and Sharma 2009). On the other hand, Aydin and Kutukcu (2017) introduced the modular $A-$metric space by a generalization of the concepts of the modular metric (Chistyakov 2010)and the $A-$metric space (Abbas et al. 2015). We also introduce the notions of $w-$compatible maps, $w-$compatible maps of type (α) and $w-$compatible maps of type (β) in modular $A-$metric space which is important for fixed point theory. And then, we examine some basic relationships between these maps. We believe that it will create an important infrastructure for researchers who want to work in fixed point theory.

2. Material and Method

Now, we will give some basic properties of modular $A-$ metric spaces.

Definition 2.1 Let χ be a nonempty set. A function $A_\lambda : (0,\infty)\times\chi^n \to [0,\infty]$ is said to be a modular $A-$ metric on χ if it satisfies the following conditions

$MA1)$ $A_\lambda(u_1,u_2,u_3,...,u_{n-1},u_n)\geq 0,$

$MA2)$ $A_\lambda(u_1,u_2,u_3,...,u_{n-1},u_n)=0$ if and only if $u_1=u_2=...=u_{n-1}=u_n,$

$MA3)$

$$\begin{aligned}&A_{\lambda_1+\lambda_2+...+\lambda_n}(u_1,u_2,u_3,...,u_{n-1},u_n)\\&\leq A_{\lambda_1}(u_1,u_1,...,(u_1)_{n-1},a)\\&+A_{\lambda_2}(u_2,u_2,...,(u_2)_{n-1},a)\\&\vdots\\&+A_{\lambda_n}(u_n,u_n,...,(u_n)_{n-1},a)\end{aligned}$$

for all $u_i,a\in\chi,$ $\lambda_i>0$, $i=\overline{1,n}$ and $\lambda>0$. A nonempty set χ together with a modular $A-$ metric is called a modular $A-$ metric space (Aydin and Kutukcu 2017).

Lemma2.2 Let (χ,A) be a modular $A-$ metric space. If for all $u_1,...,u_n\in\chi$, the mapping $A(\cdot,u_1,u_2,...,u_n):(0,\infty)\to[0,\infty]$ is continuous, then the following equality is true

$$A_\lambda(u,u,...,u,v)=A_\lambda(v,v,...,v,u)$$

for each $u,v\in\chi$ and $\lambda>0$ (Kaplan 2021).

Theorem 2.3 For each $u_1,u_2,...,u_n\in\chi$, the mapping $A(\cdot,u_1,u_2,...,u_n):(0,\infty)\to[0,\infty]$ is continuous in (χ,A) modular $A-$ metric space. Then, there are the following inequalities such that

$$\begin{aligned}A_\lambda(u,u,...,u,w)\ &\leq(n-1)A_{\frac{\lambda}{n}}(u,u,...,u,v)\\&+A_{\frac{\lambda}{n}}(w,w,...,w,v)\end{aligned}$$

and

$$\begin{aligned}A_\lambda(u,u,...,u,w)\ &\leq(n-1)A_{\frac{\lambda}{n}}(u,u,...,u,v)\\&+A_{\frac{\lambda}{n}}(v,v,...,v,w)\end{aligned}$$

for each $u,v,w\in\chi$ (Kaplan, 2021).

Proposition 2.4 For each $u_1,u_2,...,u_n\in\chi$, the mapping $A(\cdot,u_1,u_2,...,u_n):(0,\infty)\to[0,\infty]$ is continuous in (χ,A) modular $A-$ metric space. Then, the following inequality

$$A_\lambda(u,u,...,u,v)\leq A_{\frac{\lambda}{n}}(u,u,...,u,v)\leq A_{\frac{\lambda}{n^2}}(u,u,...,u,v)$$

is satisfied for $\frac{\lambda}{n^2}\leq\frac{\lambda}{n}\leq\lambda$ (Kaplan 2021).

Example 2.5 Let $\chi=R.$ A function $A_\lambda : (0,\infty)\times\chi^n\to[0,\infty]$ defined by

$$\begin{aligned}A_\lambda(u_1,...,u_{n-1},u_n)&=\frac{\lambda}{n}[|u_1-u_2|+|u_1-u_3|+...+|u_1-u_n|\\&+|u_2-u_3|+|u_2-u_4|+...+|u_2-u_n|\\&\vdots\\&+|u_{n-2}-u_{n-1}|+|u_{n-2}-u_n|\\&+|u_{n-1}-u_n|]\\&=\frac{\lambda}{n}\sum_{i=1}^{n}\sum_{i<j}|u_i-u_j|\end{aligned}$$

for each $\lambda>0$ and $u_1,u_2,...,u_n\in\chi.$
Then, (χ,A) is a usual modular A -metric space.

Example 2.6 Let $\chi=[0,6]$ and define a modular $A-$ metric $A_\lambda : (0,\infty)\times\chi^n\to[0,\infty]$ by

$$A_\lambda(u_1,u_2,...,u_n)=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{|e^{u_i}-e^{u_j}|}{\lambda}$$

for each $u_1,u_2,...,u_n\in\chi.$

Thus, (χ,A) is a modular $A-$ metric space.

Definition 2.7 The open ball of center $u_0 \in \chi$ and radius $r > 0$ in a modular $A-$metric space (χ, A) is the subset

$$B_{A_\lambda}(u_0, r) = \{ v \in \chi : A_\lambda(v, v, ..., v, u_0) < r \}$$

(Kaplan 2021).

Definition 2.8 Let (χ, A) be a modular $A-$metric space, let $u \in \chi$ be a point and let $\{ u_k \}_{k \in IN} \subset \chi$ be sequence. We say that

i. $\{u_k\}$ converges to u if $A_\lambda(u_k, u_k, u_k, ..., u_k, u) \to 0$ as $k \to \infty$ for all $\lambda > 0$. That is, for each $\varepsilon > 0$, there exists $k_0(\varepsilon) \in IN$ satisfying $A_\lambda(u_k, u_k, u_k, ..., u_k, u) \le \varepsilon$ for all $k \ge k_0$.

ii. $\{u_k\}$ is said to be a Cauchy sequence if $A_\lambda(u_k, u_k, u_k, ..., u_k, u_m) \to 0$ as $k, m \to \infty$ for all $\lambda > 0$. That is, for all $\varepsilon > 0$, there exists $k_0(\varepsilon) \in IN$ such that for all $k, m \ge k_0$, $A_\lambda(u_k, u_k, u_k, ..., u_k, u_m) \le \varepsilon$.

iii. (χ, A) is complete if every Cauchy sequence in χ is convergent in χ (Kaplan 2021).

Theorem 2.9 The limit of a convergent sequence in a modular $A-$metric space is unique (Kaplan 2021)

Theorem2.10 Every convergent sequence in a modular $A-$metric space is a Cauchy sequence (Kaplan 2021)

3. Results

Definition3.1 Let (χ, A) be a modular $A-$metric space and let $\mu, \eta : \chi \to \chi$ be two self-mappings. We say that the pair (μ, η) is a $w-$compatible if we have that

$$\lim_{k\to\infty} A_\lambda(\mu\eta x_k, ..., \mu\eta x_k, \eta\mu x_k) = 0$$

where $\{x_k\}$ is a sequence in χ such that $\lim_{k\to\infty} \mu x_k = \lim_{k\to\infty} \eta x_k = t$ for some point $t \in \chi$ and $\lambda > 0$.

Example 3.2 Let $\chi = IR$ and define a function $A_\lambda : (0,\infty) \times \chi^n \to [0,\infty]$ by

$$A_\lambda(x_1, x_2, ..., x_n) = \frac{\lambda}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n} |x_i - x_j|$$

for all $\lambda > 0$ and $x_1, x_2, ..., x_n \in \chi$. Then, (χ, A) is a modular $A-$metric space. Let $\mu, \eta : \chi \to \chi$ be defined on χ by $\mu(x) = x^2$ and $\eta(x) = x^3$ for each $x \in \chi$. Take $\{x_k\}$ such that $x_k = \frac{1}{k}$, $k = 1, 2, ...$ In this case, we have

$$\lim_{k\to\infty} \mu x_k = \lim_{k\to\infty} \frac{1}{k^2} = 0$$

$$\lim_{k\to\infty} \eta x_k = \lim_{k\to\infty} \frac{1}{k^3} = 0.$$

Moreover, $\mu\eta x_k = \frac{1}{k^6}$ and $\eta\mu x_k = \frac{1}{k^6}$. Thus, the pair (μ, η) is $w-$compatible maps since

$$\begin{aligned} &\lim_{k\to\infty} A_\lambda(\mu\eta x_k, ..., \mu\eta x_k, \eta\mu x_k) \\ &= \lim_{k\to\infty} A_\lambda\left(\frac{1}{k^6}, ..., \frac{1}{k^6}, \frac{1}{k^6}\right) \\ &= \lim_{k\to\infty} \frac{\lambda}{n}\left[\left|\frac{1}{k^6} - \frac{1}{k^6}\right| + \left|\frac{1}{k^6} - \frac{1}{k^6}\right| + ... + \left|\frac{1}{k^6} - \frac{1}{k^6}\right|\right] \\ &= 0. \end{aligned}$$

Example 3.3 Let $\chi = [0, 2]$ and define a function $A_\lambda : (0,\infty) \times \chi^n \to [0,\infty]$ by

$$A_\lambda(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{|e^{x_i} - e^{x_j}|}{\lambda}$$

for all $\lambda > 0$ and $x_1, x_2, ..., x_n \in \chi$. Then, (χ, A) is a modular $A-$metric space. Then, (χ, A) is a modular $A-$metric space. Also, define $\mu,\eta:\chi\to\chi$ by

$$\mu(x)=\begin{cases}1, & x\in[0,1]\\ \dfrac{x+1}{2}, & x\in(1,2]\end{cases}$$

$$\eta(x)=\begin{cases}2, & x=1\\ \dfrac{x+3}{4}, & x\neq 1\end{cases}$$

for each $x\in\chi$. Take $\{x_k\}$ such that $x_k=1-\dfrac{1}{k},k=1,2,\ldots$ In this case, we have

$$\mu x_k=1 \qquad \mu\eta x_k=1$$
$$\eta x_k=\frac{4k-1}{4k} \quad \eta\mu x_k=2.$$

Thus, the pair (μ,η) is not $w-$compatible maps since

$$\begin{aligned}&\lim_{k\to\infty}A_\lambda(\mu\eta x_k,\ldots,\mu\eta x_k,\eta\mu x_k)\\ &=\lim_{k\to\infty}A_\lambda(1,\ldots,1,2)\\ &=\lim_{k\to\infty}\left[\frac{|e-e|}{\lambda}+\ldots+\frac{|e-e^2|}{\lambda}\right]\\ &=\lim_{k\to\infty}(n-1)\frac{|e-e^2|}{\lambda}\\ &\neq 0\end{aligned}$$

Definition 3.4 Let (χ,A) be a modular $A-$metric space and let $\mu,\eta:\chi\to\chi$ be two self mappings. We say that the pair (μ,η) is a $w-$compatible maps of type (α) if we have that

$$\lim_{k\to\infty}A_\lambda(\mu\eta x_k,\eta\eta x_k\ldots,\eta\eta x_k)=0$$
$$\lim_{k\to\infty}A_\lambda(\eta\mu x_k,\mu\mu x_k\ldots,\mu\mu x_k)=0$$

where $\{x_k\}$ is a sequence in χ such that $\lim_{k\to\infty}\mu x_k=\lim_{k\to\infty}\eta x_k=t$ for some point $t\in\chi$ and $\lambda>0$.

Example 3.5 Let $\chi=[0,6]$ and define a function

$A_\lambda:(0,\infty)\times\chi^n\to[0,\infty]$ by

$$A_\lambda(x_1,x_2,\ldots,x_n)=\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{|e^{x_i}-e^{x_j}|}{\lambda}$$

for all $\lambda>0$ and $x_1,x_2,\ldots,x_n\in\chi$. Then, (χ,A) is a modular $A-$metric space. Also, define $\mu,\eta:\chi\to\chi$ by

$$\mu(x)=\begin{cases}x, & x\in[0,3)\\ 6, & x\in[3,6]\end{cases} \qquad \eta(x)=\begin{cases}6-x, & x\in[0,3)\\ 6, & x\in[3,6]\end{cases}$$

for each $x\in\chi$. Take $\{x_k\}$ such that $x_k=3-\dfrac{1}{k^2},k=1,2,\ldots$ In this case, we have

$$\lim_{k\to\infty}\mu x_k=\lim_{k\to\infty}3-\frac{1}{k^2}=3$$
$$\lim_{k\to\infty}\eta x_k=\lim_{k\to\infty}6-3+\frac{1}{k^2}=3$$

and

$$\mu\eta x_k=6 \qquad \eta\mu x_k=6-3+\frac{1}{k^2}=3+\frac{1}{k^2}$$
$$\eta\eta x_k=6 \qquad \mu\mu x_k=3-\frac{1}{k^2}$$

Thus, the pair (μ,η) is $w-$compatible of type (α) since

$$\begin{aligned}&\lim_{k\to\infty}A_\lambda(\mu\eta x_k,\eta\eta x_k,\ldots,\eta\eta x_k)\\ &=\lim_{k\to\infty}A_\lambda(6,6,\ldots,6)\\ &=\lim_{k\to\infty}\frac{|e^6-e^6|+|e^6-e^6|+\ldots+|e^6-e^6|}{\lambda}\\ &=0\end{aligned}$$

$$\lim_{k\to\infty} A_\lambda(\eta\mu x_k,\mu\mu x_k,\ldots,\mu\mu x_k)$$
$$=\lim_{k\to\infty} A_\lambda\left(3+\frac{1}{k^2},3-\frac{1}{k^2},\ldots,3-\frac{1}{k^2}\right)$$
$$=\lim_{k\to\infty}\frac{\left|e^{3+\frac{1}{k^2}}-e^{3-\frac{1}{k^2}}\right|+\ldots+\left|e^{3-\frac{1}{k^2}}-e^{3-\frac{1}{k^2}}\right|}{\lambda}$$
$$=0$$

But, the pair (μ,η) is not $w-$compatible maps since

$$=\lim_{k\to\infty} A_\lambda\left(6,6,\ldots,6,3+\frac{1}{k^2}\right)$$
$$=\lim_{k\to\infty}\left[\frac{\left|e^6-e^6\right|}{\lambda}+\frac{\left|e^6-e^6\right|}{\lambda}+\ldots+\frac{\left|e^6-e^{3+\frac{1}{k^2}}\right|}{\lambda}\right]$$
$$=\lim_{k\to\infty}\frac{\left|e^6-e^{3+\frac{1}{k^2}}\right|}{\lambda}$$
$$\neq 0.$$

Example 3.6 Let $\chi=IR$ and A be a function on χ defined by

$$A_\lambda(x_1,x_2,\ldots,x_n)=\frac{\lambda}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left|x_i-x_j\right|$$

for all $\lambda>0$ and $x_1,x_2,\ldots,x_n\in\chi$. Then, (χ,A) is a modular $A-$metric space. Also, define $\mu,\eta:\chi\to\chi$ by

$$\mu(x)=\begin{cases}2, & x=0\\ \dfrac{1}{x^3}, & x\neq 0\end{cases}\qquad \eta(x)=\begin{cases}3, & x=0\\ \dfrac{1}{x^5}, & x\neq 0\end{cases}$$

for each $x\in\chi$. Take $\{x_k\}$ such that $x_k=k,\ k=1,2,\ldots$ In this case, we have

$$\lim_{k\to\infty}\mu x_k \quad =\lim_{k\to\infty}\frac{1}{k^3}\quad =0$$
$$\lim_{k\to\infty}\eta x_k \quad =\lim_{k\to\infty}\frac{1}{k^5}\quad =0.$$

Also,

$$\mu\eta x_k=k^{15},\eta\mu x_k=k^{15},\eta\eta x_k=k^{25},\mu\mu x_k=k^9$$

Consequently, the pair (μ,η) is $w-$compatible maps since

$$\lim_{k\to\infty} A_\lambda(\mu\eta x_k,\ldots,\mu\eta x_k,\eta\mu x_k)$$
$$=\lim_{k\to\infty} A_\lambda\left(k^{15},\ldots,k^{15},k^{15}\right)$$
$$=\lim_{k\to\infty}\frac{\lambda}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left|x_i-x_j\right|$$
$$=\lim_{k\to\infty}\frac{\lambda}{n}\left[\left|k^{15}-k^{15}\right|+\ldots+\left|k^{15}-k^{15}\right|\right]$$
$$=0$$

But, the pair (μ,η) is not a $w-$compatible maps of type (α) since

$$\lim_{k\to\infty} A_\lambda(\mu\eta x_k,\eta\eta x_k,\ldots,\eta\eta x_k)$$
$$=\lim_{k\to\infty} A_\lambda\left(k^{15},k^{25},\ldots,k^{25}\right)$$
$$=\lim_{k\to\infty}\frac{\lambda}{n}\left[\left|k^{15}-k^{25}\right|+\left|k^{15}-k^{25}\right|+\ldots+\left|k^{25}-k^{25}\right|\right]$$
$$=\lim_{k\to\infty}\frac{\lambda}{n}(n-1)\left|k^{15}-k^{25}\right|$$
$$=\infty$$
$$\lim_{k\to\infty} A_\lambda(\eta\mu x_k,\mu\mu x_k,\ldots,\mu\mu x_k)$$
$$=\lim_{k\to\infty} A_\lambda\left(k^{15},k^{9},\ldots,k^{9}\right)$$
$$=\lim_{k\to\infty}\frac{\lambda}{n}\left[\left|k^{15}-k^{9}\right|+\left|k^{15}-k^{9}\right|+\ldots+\left|k^{9}-k^{9}\right|\right]$$
$$=\lim_{k\to\infty}\frac{\lambda}{n}(n-1)\left|k^{15}-k^{9}\right|$$
$$=\infty.$$

Proposition 3.7 Let $\mu,\eta:\chi\to\chi$ be two continuous maps in (χ,A) modular $A-$metric space. If the pair (μ,η) is $w-$compatible maps, then it is $w-$compatible maps of type (α).

Proof : Assume that μ and η are $w-$compatible maps and $\{\mu x_k\}$ and $\{\eta x_k\}$ converge to same $t \in X$ for a sequence of $\{x_k\}$ in the χ. We have

$$\lim_{k\to\infty} \mu\mu x_k = \lim_{k\to\infty} \mu\eta x_k = \mu t$$
$$\lim_{k\to\infty} \eta\mu x_k = \lim_{k\to\infty} \eta\eta x_k = \eta t$$

since the maps μ and η are continuous. Moreover, we obtain

$$\lim_{k\to\infty} A_\lambda\left(\mu\eta x_k, \mu\eta x_k, \ldots, \mu\eta x_k, \eta\mu x_k\right) = 0$$

since the pair (μ,η) is $w-$compatible maps.

$$\begin{aligned} A_\lambda\left(\eta\mu x_k, \mu\mu x_k, \ldots, \mu\mu x_k\right) &= A_\lambda\left(\eta\mu x_k, \ldots, \eta\mu x_k, \mu\mu x_k\right) \\ &\le A_{\frac{\lambda}{n}}\left(\eta\mu x_k, \ldots, \eta\mu x_k, \mu\eta x_k\right) \\ &\vdots \\ &+ A_{\frac{\lambda}{n}}\left(\eta\mu x_k, \ldots, \eta\mu x_k, \mu\eta x_k\right) \\ &+ A_{\frac{\lambda}{n}}\left(\mu\mu x_k, \ldots, \mu\mu x_k, \mu\eta x_k\right) \end{aligned}$$

from the above inequality,

$$\lim_{k\to\infty} A_\lambda\left(\eta\mu x_k, \mu\mu x_k, \ldots, \mu\mu x_k\right) = 0$$

is obtained. In a similar vein,

$$\lim_{k\to\infty} A_\lambda\left(\mu\eta x_k, \eta\eta x_k, \ldots, \eta\eta x_k\right) = 0.$$

Consequently, the pair (μ,η) is a $w-$compatible maps of type (α).

Proposition 3.8 Let $\mu,\eta : \chi \to \chi$ be $w-$compatible maps of type (α) in (χ, A) modular $A-$metric space. If at least one of μ and η are continuous maps, then they are $w-$compatible maps.

Proof : Assume that the map η be a continuous map and $\{\mu x_k\}$ and $\{\eta x_k\}$ converge to same $t \in \chi$. In this case, we have

$$\lim_{k\to\infty} \eta\mu x_k = \lim_{k\to\infty} \eta\eta x_k = \eta t$$

since the map η is continuous. We can write the following inequality from the condition (MA3)

$$\begin{aligned} & A_\lambda\left(\mu\eta x_k, \ldots, \mu\eta x_k, \eta\mu x_k\right) \\ & \le A_{\frac{\lambda}{n}}\left(\mu\eta x_k, \ldots, \mu\eta x_k, \eta\eta x_k\right) \\ & \vdots \\ & + A_{\frac{\lambda}{n}}\left(\mu\eta x_k, \ldots, \mu\eta x_k, \eta\eta x_k\right) \\ & + A_{\frac{\lambda}{n}}\left(\mu\mu x_k, \ldots, \eta\mu x_k, \eta\eta x_k\right) \end{aligned}$$

Moreover, we obtain

$$\lim_{k\to\infty} A_{\frac{\lambda}{n}}\left(\eta\mu x_k, \mu\mu x_k, \ldots, \mu\mu x_k\right) = 0$$
$$\lim_{k\to\infty} A_{\frac{\lambda}{n}}\left(\mu\eta x_k, \eta\eta x_k, \ldots, \eta\eta x_k\right) = 0$$

since the pair (μ,η) is a $w-$compatible maps of type (α). From the above equalities,

$$\lim_{k\to\infty} A_\lambda\left(\mu\eta x_k, \mu\eta x_k, \ldots, \mu\eta x_k, \eta\mu x_k\right) = 0$$

and the pair (μ,η) is $w-$compatible maps.

Corollary 3.9 Let $\mu,\eta : \chi \to \chi$ be two continuous maps in (χ, A) modular $A-$metric space. If the pair (μ,η) is $w-$compatible maps if and only if it is a $w-$compatible maps of type (α).

Definition 3.10 Let (χ, A) be a modular $A-$metric space and let $\mu,\eta : \chi \to \chi$ be two self mappings. We say that (μ,η) is a $w-$compatible maps of type (β) if we have that

$$\lim_{k\to\infty} A_\lambda\left(\mu\mu x_k, \ldots, \mu\mu x_k, \eta\eta x_k\right) = 0$$

where $\{x_k\}$ is a sequence in χ such that $\lim_{k\to\infty} \mu x_k = \lim_{k\to\infty} \eta x_k = t$ for some point $t \in \chi$ and $\lambda > 0$.

Example 3.11 Let $\chi = IR$ and define a function $A_\lambda : (0,\infty)\times\chi^n \to [0,\infty]$ by

$$A_\lambda(x_1,x_2,...,x_n) = \frac{\lambda}{n}\sum_{i=1}^{n}\sum_{j=i+1}^{n}|x_i - x_j|$$

for all $\lambda > 0$ and $x_1,x_2,...,x_n \in \chi$. Then, (χ, A) is a modular $A-$metric space. Also, define $\mu,\eta : \chi \to \chi$ by

$$\mu(x) = \begin{cases} 5, & x = 0 \\ \frac{1}{x^2}, & x \neq 0 \end{cases} \qquad \eta(x) = \begin{cases} 1, & x = 0 \\ \frac{1}{x^3}, & x \neq 0 \end{cases}$$

for each $x \in \chi$. Take $\{x_k\}$ such that $x_k = k$, $k = 1,2,...$. In this case, we have

$$\lim_{k\to\infty}\mu x_k = \lim_{k\to\infty}\frac{1}{k^2} = 0$$
$$\lim_{k\to\infty}\eta x_k = \lim_{k\to\infty}\frac{1}{k^3} = 0$$

and $\mu\eta x_k = k^6, \eta\mu x_k = k^6, \eta\eta x_k = k^9, \mu\mu x_k = k^4$. Thus, the pair (μ,η) is $w-$compatible maps since

$$\begin{aligned}&\lim_{k\to\infty}A_\lambda(\mu\eta x_k,...,\mu\eta x_k,\eta\mu x_k)\\ &= \lim_{k\to\infty}A_\lambda(k^6,...,k^6,k^6)\\ &= \lim_{k\to\infty}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{|x_i - x_j|}{\lambda}\\ &= \lim_{k\to\infty}\frac{1}{\lambda}\left[|k^6 - k^6| + ... + |k^6 - k^6|\right]\\ &= 0\end{aligned}$$

But, the pair (μ,η) is not a $w-$compatible maps of type (β) since

$$\begin{aligned}&\lim_{k\to\infty}A_\lambda(\mu\mu x_k,...,\mu\mu x_k,\eta\eta x_k)\\ &= \lim_{k\to\infty}A_\lambda(k^4,...,k^4,k^9)\\ &= \lim_{k\to\infty}\frac{1}{\lambda}\left[|k^4 - k^4| + |k^4 - k^4| + ... + |k^4 - k^9|\right]\\ &= \lim_{k\to\infty}\frac{1}{\lambda}(n-1)|k^4 - k^9|\\ &= \infty.\end{aligned}$$

Example 3.12 Let's take modular $A-$metric space and the pair (μ,η) is given at the Example 3.3. It is not $w-$compatible maps and $w-$compatible maps of type (α) but it is $w-$compatible maps of type (β). Really, define $\mu,\eta : \chi \to \chi$ by

$$\mu(x) = \begin{cases} 1, & x \in [0,1] \\ \frac{x+1}{2}, & x \in (1,2] \end{cases}$$

$$\eta(x) = \begin{cases} 2, & x = 1 \\ \frac{x+3}{4}, & x \neq 1 \end{cases}$$

for each $x \in X$. Take $\{x_k\}$ such that $x_k = 1 - \frac{1}{k}, k = 1,2,...$ In this case, we have

$$\mu x_k = 1 \qquad \eta x_k = \frac{4k-1}{4k}$$
$$\mu\mu x_k = 1 \quad \eta\eta x_k = \frac{16k-1}{16k}$$
$$\mu\eta x_k = 1 \quad \eta\mu x_k = 2$$

The Example 3.3 shows that the pair (μ,η) is not $w-$compatible maps. Now, let's show that it is not a $w-$compatible of type (α) but it is a $w-$compatible of type (β). From Definition 3.4, the pair (μ,η) is not a $w-$compatible of type (α) since

$$\lim_{k\to\infty} A_\lambda\left(\mu\eta x_k,\eta\eta x_k,...,\eta\eta x_k\right)$$
$$=\lim_{k\to\infty} A_\lambda\left(1,\frac{16k-1}{16k},...,\frac{16k-1}{16k}\right)$$
$$=\lim_{k\to\infty}\frac{\left|e-e^{\frac{16k-1}{16k}}\right|+\left|e-e^{\frac{16k-1}{16k}}\right|+...+\left|e^{\frac{16k-1}{16k}}-e^{\frac{16k-1}{16k}}\right|}{\lambda}$$
$$=\lim_{k\to\infty}\frac{(n-1)}{\lambda}e.\left(1-e^{-\frac{1}{16k}}\right)$$
$$=0$$
$$\lim_{k\to\infty} A_\lambda\left(\eta\mu x_k,\mu\mu x_k,...,\mu\mu x_k\right)$$
$$=\lim_{k\to\infty} A_\lambda\left(2,1,...,1\right)$$
$$=\lim_{k\to\infty}\frac{\left|e^2-e\right|+\left|e^2-e\right|+...+\left|e-e\right|}{\lambda}$$
$$=\lim_{k\to\infty}\frac{(n-1)}{\lambda}\left|e^2-e\right|$$
$$=\frac{(n-1)}{\lambda}\left|e^2-e\right|$$

From Definition 3.10, the pair (μ,η) is $w-$compatible of type (β) since

$$\lim_{k\to\infty} A_\lambda\left(\mu\mu x_k,...,\mu\mu x_k,\eta\eta x_k\right)$$
$$=\lim_{k\to\infty} A_\lambda\left(1,1,...,1,\frac{16k-1}{16k}\right)$$
$$=\lim_{k\to\infty}\left[\frac{|e-e|}{\lambda}+\frac{|e-e|}{\lambda}+...+\frac{\left|e-e^{\frac{16k-1}{16k}}\right|}{\lambda}\right]$$
$$=\lim_{k\to\infty}\frac{(n-1)}{\lambda}\left|e-e^{\frac{16k-1}{16k}}\right|$$
$$=0$$

Proposition 3.13 Let (χ,A) be a modular $A-$metric space and $\mu,\eta:\chi\to\chi$ be two continuous maps. If the pair (μ,η) is $w-$compatible maps, then it is $w-$compatible of type (β).

Proof: Assume that the maps μ and η are $w-$compatible maps and $\{\mu x_k\}$ and $\{\eta x_k\}$ converge to same $t\in\chi$ for a sequence $\{x_k\}$ in the χ. We have

$$\lim_{k\to\infty}\mu\mu x_k=\lim_{k\to\infty}\mu\eta x_k=\mu t$$
$$\lim_{k\to\infty}\eta\mu x_k=\lim_{k\to\infty}\eta\eta x_k=\eta t$$

since the maps μ and η are continuous. Also, we get

$$\lim_{k\to\infty} A_\lambda\left(\mu\eta x_k,...,\mu\eta x_k,\eta\mu x_k\right)=0$$

since the pair (μ,η) is $w-$compatible maps. Thus, from the following inequality

$$A_\lambda\left(\mu\mu x_k,...,\eta\eta x_k\right)\le A_{\frac{\lambda}{n}}\left(\mu\mu x_k,...,\mu\eta x_k\right)$$
$$+A_{\frac{\lambda}{n}}\left(\mu\mu x_k,...,\mu\eta x_k\right)+...+A_{\frac{\lambda}{n}}\left(\eta\eta x_k,...,\mu\eta x_k\right)$$
$$\le A_{\frac{\lambda}{n}}\left(\mu\mu x_k,...,\mu\eta x_k\right)+...+A_{\frac{\lambda}{n}}\left(\mu\mu x_k,...,\mu\eta x_k\right)$$
$$+A_{\frac{\lambda}{n^2}}\left(\eta\eta x_k,...,\eta\mu x_k\right)+...+A_{\frac{\lambda}{n^2}}\left(\mu\eta x_k,...,\eta\mu x_k\right)$$

we obtain

$$\lim_{k\to\infty} A_\lambda\left(\mu\mu x_k,\mu\mu x_k,...,\eta\eta x_k\right)=0.$$

Consequently, the pair (μ,η) is a $w-$compatible maps of type (β).

Proposition 3.14 Let $\mu,\eta:\chi\to\chi$ be $w-$compatible maps of type (β) in (χ,A) modular $A-$metric space. If the maps μ and η are continuous maps, then they are $w-$compatible maps.

Proof: Assume that the pair (μ,η) is a $w-$compatible maps of type (β) and $\{\mu x_k\}$ and $\{\eta x_k\}$ converge to same $t\in\chi$ for a sequence of $\{x_k\}$ in χ. We have

$$\lim_{k\to\infty} \mu\mu x_n = \lim_{k\to\infty} \mu\eta x_n = \mu t$$
$$\lim_{k\to\infty} \eta\mu x_n = \lim_{k\to\infty} \eta\eta x_n = \eta t$$

since the maps μ and η are continuous. Thus, we know that

$$\begin{aligned} & A_\lambda(\mu\eta x_n,\ldots,\eta\mu x_n) \\ & \leq A_{\frac{\lambda}{n}}(\mu\eta x_k,\ldots,\mu\mu x_k)\ldots + A_{\frac{\lambda}{n}}(\eta\mu x_k,\ldots,\mu\mu x_k) \\ & \leq A_{\frac{\lambda}{n}}(\mu\eta x_k,\ldots,\mu\mu x_k) + \ldots + A_{\frac{\lambda}{n}}(\mu\eta x_k,\ldots,\mu\mu x_k) \\ & + A_{\frac{\lambda}{n^2}}(\eta\mu x_k,\ldots,\eta\eta x_k) + \ldots + A_{\frac{\lambda}{n^2}}(\eta\mu x_k,\ldots,\eta\eta x_k) \\ & + A_{\frac{\lambda}{n^2}}(\mu\mu x_k,\ldots,\eta\eta x_k) \end{aligned}$$

from condition (MA3). Thus, we obtain

$$\lim_{k\to\infty} A_\lambda(\mu\eta x_k,\ldots,\mu\eta x_k,\eta\mu x_k) = 0.$$

Then, the pair (μ,η) is $w-$compatible maps.

Corollary 3.15 Let $\mu,\eta : \chi \to \chi$ be two continuous maps in (χ,A) modular $A-$metric space. If the pair (μ,η) is $w-$compatible maps if and only if it is a $w-$compatible maps of type (β).

Proposition 3.16 Let (χ,A) be a modular $A-$metric space and $\mu,\eta : \chi \to \chi$ be two continuous maps. If the pair (μ,η) is a $w-$compatible maps of type (β), then it is a $w-$compatible maps of type (α).

Proof: Assume that the maps μ and η are $w-$compatible maps of type (β) and $\{\mu x_k\}$ and $\{\eta x_k\}$ converge to same $t \in \chi$ for a sequence of $\{x_k\}$ in the χ. In this case, we have

$$\lim_{k\to\infty} \mu\mu x_k = \lim_{k\to\infty} \mu\eta x_k = \mu t$$
$$\lim_{k\to\infty} \eta\mu x_k = \lim_{k\to\infty} \eta\eta x_k = \eta t$$

since the maps μ and η are continuous. Also, we get

$$\lim_{k\to\infty} A_\lambda(\mu\mu x_k,\ldots,\mu\mu x_k,\eta\eta x_k) = 0$$

since the pair (μ,η) is a $w-$compatible maps of type (β). Thus, we obtain

$$\lim_{k\to\infty} A_\lambda(\mu\eta x_k,\ldots,\mu\eta x_k,\eta\eta x_k) = 0$$

from the following inequality

$$\begin{aligned} & A_\lambda(\mu\eta x_k,\ldots,\mu\eta x_k,\eta\eta x_k) \\ & \leq A_{\frac{\lambda}{n}}(\mu\eta x_k,\ldots,\mu\eta x_k,\mu\mu x_k) \\ & + A_{\frac{\lambda}{n}}(\mu\eta x_k,\ldots,\mu\eta x_k,\mu\mu x_k) \\ & + \ldots + A_{\frac{\lambda}{n}}(\eta\eta x_k,\ldots,\eta\eta x_k,\mu\mu x_k) \end{aligned}$$

In the similar vein, we have

$$\lim_{k\to\infty} A_\lambda(\eta\mu x_k,\ldots,\eta\mu x_k,\mu\mu x_k) = 0$$

Then, the pair (μ,η) is a $w-$compatible of type (α).

Proposition 3.17 Let $\mu,\eta : \chi \to \chi$ be $w-$compatible maps of type (α) in (χ,A) modular $A-$metric space. If at least one of μ and η are continuous maps, then they are $w-$compatible maps of type (β).

Proof: Suppose that the map μ be a continuous map and $\{\mu x_k\}$ and $\{\eta x_k\}$ converge to same $t \in \chi$. In this case, we have

$$\lim_{k\to\infty} \mu\mu x_k = \lim_{k\to\infty} \mu\eta x_k = \mu t$$

since the map μ is continuous. We know that

$$A_{\lambda}\left(\mu\mu x_k,...,\mu\mu x_k,\eta\eta x_k\right)$$
$$\leq A_{\frac{\lambda}{n}}\left(\mu\mu x_k,...,\mu\mu x_k,\mu\eta x_k\right)$$
$$\vdots$$
$$+A_{\frac{\lambda}{n}}\left(\mu\mu x_k,...,\mu\mu x_k,\mu\eta x_k\right)$$
$$+A_{\frac{\lambda}{n}}\left(\eta\eta x_k,...,\eta\eta x_k,\mu\eta x_k\right)$$

from condition (MA3). We have

$$\lim_{k\to\infty} A_{\frac{\lambda}{n}}\left(\eta\mu x_k,\mu\mu x_k,...,\mu\mu x_k\right)=0$$
$$\lim_{k\to\infty} A_{\frac{\lambda}{n}}\left(\mu\eta x_k,\eta\eta x_k,...,\eta\eta x_k\right)=0$$

since the pair (μ,η) is a $w-$compatible maps of type (α). Also, $\lim_{k\to\infty} A_{\frac{\lambda}{n}}\left(\eta\eta x_k,\eta\mu x_k,...,\eta\mu x_k\right)=0.$ From the last inequality, we know that

$$\lim_{k\to\infty} A_{\lambda}\left(\mu\mu x_k,...,\mu\mu x_k,\eta\eta x_k\right)=0.$$

Then, the pair (μ,η) is a $w-$compatible maps of type (β).

Corollary 3.18 Let $\mu,\eta:\chi\to\chi$ be two continuous maps in (χ,A) modular $A-$metric space. If the pair (μ,η) is a $w-$compatible maps of type (α) if and only if it is a $w-$compatible maps of type (β).

4. Conclusion

This study introduces the concepts of $w-$compatible mappings, $w-$compatible maps of type (α) and $w-$compatible maps of type (β) on modular $A-$metric spaces. The relationships between these mappings are examined and necessary inverse examples are presented. Different types of compatible mappings concepts can be defined and with the help of these definitions, common fixed point theorems can be proved.

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