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Araştırma Makalesi / Research Article

A Note on Stabilized Finite Element Methods for Predator-Prey Systems

Ali ŞENDUR[1]

[1] *Alanya Alaaddin Keykubat University, Department of Mathematics Education, Alanya, Antalya, Turkey.*

e-posta: ali.sendur@alanya.edu.tr ORCID ID: https://orcid.org/0000-0001-8628-5497

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Predator-prey systems; Convection-diffusion-reaction; Stabilized finite element method; Multiscale Methods.

Abstract

A numerical method that will improve and produce effective results for solving mathematical model for the system of predator-prey interactions which is defined by convection-diffusion-reaction problem is studied herein. We consider the Pseudo Residual-free Bubble (PRFB) method which is based on augmenting the finite element space by appropriate functions for the space discretization. The method is applied on different test problems and the numerical solutions are in good agreement with the result available in literature. The numerical results depict that the algorithm is efficient and feasible.

Av-Avcı Problemleri için Kararlı Sonlu Eleman Yöntemleri Üzerine Bir Not

Anahtar kelimeler
Av-avcı denklem sistemleri; Konveksiyon-difüzyon-reaksiyon; Kararlı Sonlu Eleman Yöntemi; Çok-ölçekli Yöntemler.

Öz

Bu çalışmada, konveksiyon-difüzyon-reaksiyon problemleri ile modellenebilen av-avcı denklem sistemlerinin simülasyonunda kullanılan sayısal çözüm tekniklerini iyileştirecek ve daha etkin sonuçlar üretecek sayısal bir yöntem önerilmiştir. Uzay ayrıklaştırması için, sonlu elemanlar metodunu uygularken seçilen polinom baz fonksiyonlarına ilaveten fonksiyon uzayının özel tip fonksiyonlarla (residual-free bubbles) zenginleştirilmesine dayanan Pseudo Residual-free Bubble (PRFB) yöntemi kullanılmıştır. Söz konusu yöntem, çeşitli test örneklerine uygulanmış olup elde edilen sayısal çözümlerin, literatürde mevcut olan sonuçlar ile iyi bir uyum içinde olduğu gözlemlenmiştir. Sayısal sonuçlar, önerilen yöntemin verimli ve uygulanabilir olduğunu göstermektedir.

1. Introduction

The mathematical model for the predator-prey interactions is an important subject in mathematical biology and ecology (Allen 2007, Murray 2003). There are many approaches to describe the predator-prey systems that have been used to model the process of dispersal and its ecological effects and evolution (Chong *et al.* 2005, Meyer *et al.* 1997). One common approach contains interactions between reaction diffusion and convection processes, which can be modeled in terms of the convection-diffusion-reaction (CDR) problems (Cosner 2014, Medvinsky *et al.* 2002). Here, the convection (C) term can be interpreted as the movement of species; the diffusion (D) describes the dispersion of the species throughout the physical domain of the problem and the reaction (R) defines the interaction process through the generated / consumed species involved in the phenomenon. Analytical solutions of those problems can only be obtained under specific circumstances, therefore efficient and feasible algorithms are needed for the numerical solutions of those problems. However, the numerical solution of those problems might become a challenge as it is well known that the discrete solutions generated by standard numerical methods is usually globally polluted by non-physical oscillations in the whole domain. Therefore, effective algorithms for the numerical solutions of those problems has captured the interest of a number of many researchers (Dimitrov and Kojouharov 2006, Garzon *et al.* 2012, Stefano *et al.* 2013).

A considerable amount of research works has been devoted for discretizing predator-prey systems among them, the finite difference method is frequently used. However, those methods are inefficient and accurate solutions having the same qualitative features as the continuous problem could not be obtained for higher dimensions. Dimitrov and Kojouharo used non-standard techniques to construct stability-preserving Elementary Stable Non-standard (ESN) schemes for arbitrary time step-sizes (Dimitrov and Kojouharov 2007, Mickens 1994). Nevertheless, the need for a positive discrete solution for all positive initial values is considered as a drawback of the ESN method. Recently many researchers have worked on positive and elementary stable nonstandard (PESN) algorithms for predator-prey systems (Dimitrov and Kojouharov 2006, Moghadas *et al.* 2004).

The second approach to get effective approximations for treating CDR problems is the stabilized finite element method (FEM). Among to that class, the Residual-Free Bubbles (RFB) method which is based on enriching the finite element space with special "residual-free bubble" functions could be mentioned first (Brezzi *et al.* 1997, Brezzi and Russo 1994). Although the residual-free bubbles produce effective discretization, the drawback of this methodology resides in that it requires to solve locally defined differential equation which has similar characteristic behavior to the original one (Franca *et al.* 1998). Motivating by that observation, the Pseudo Residual-free Bubble (PRFB) method has been introduced. In this strategy, the residual-free bubbles are approximated by piecewise linear functions on a suitable sub-grid to represent accurately the fine scale-effect of the exact solution in the coarse scale numerical approximation (Brezzi *et al.* 1998, Brezzi *et al.* 2005, Sendur and Nesliturk 2012, Sendur *et al.* 2014). It seems that the PRFB strategy is quite robust and effective method for the numerical solutions of the CDR problems.

In this study, we aim to discover the potential of the Pseudo Residual-free Bubble method as an algorithm for the numerical solution of predator-prey interactions. The organization of the paper is as follows: In Section 2, we introduce the mathematical models for the process of predator-prey dynamics in 1D. We describe the details of the numerical method in Section 3. Finally, we perform the numerical experiments and draw conclusions in Section 4.

2. Model Description

We consider the following CDR problem as a mathematical model for a predator-prey system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \epsilon_1 \dfrac{\partial^2 u}{\partial x^2} - \alpha_1 \dfrac{\partial u}{\partial x} + f(u,v) \\ \dfrac{\partial v}{\partial t} = \epsilon_2 \dfrac{\partial^2 v}{\partial x^2} - \alpha_2 \dfrac{\partial u}{\partial x} + g(u,v) \quad \text{in } \Omega \times (0,T] \end{cases} \tag{1}$$

where $f(\cdot), g(\cdot)$ are assumed to be a $C^2\big([0,T]; L^2(\Omega)\big)$ functions. We will focus on the following two specific type functions,

Kinetics (i) $f(u,v) = u(r_1 - u) - \gamma_1 uv$ and $g(u,v) = v(r_2 - v) + \gamma_2 uv.$

Kinetics (ii) $f(u,v) = u(1-u) - \frac{uv}{u+\alpha}$ and $g(u,v) = \beta \frac{uv}{u+\alpha} - \gamma v.$

Here, the 1-dimensional bounded domain where the problem is to be solved has been denoted by $\Omega \subset \mathbb{R}$, the boundary by $\partial\Omega$ and the time interval by $(0,T]$. The system parameters are as follows: u and v define the populations of prey and predators, ϵ_1 and ϵ_2 are diffusion constants, α_1 and α_2 are the convection rates, r_1 and r_2 are the growth rates, γ_1 is the predation rate and γ_2 is the conversion rates of the prey and the predator, respectively. Here, ϵ_1, ϵ_2, α_1, α_2, r_1, r_2, γ_1, γ_2 are positive constants. The equation (1) will be supplied with the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \quad \text{on } \partial\Omega, \quad t \in (0,T]$$

and with an initial condition of the form

$$u = u^0, \quad v = v^0 \quad \text{in } \Omega, \quad t = 0$$

The qualitative properties of the model (1) including the theoretical aspects are studied in detail in (Garvie 2007, Garvie *et al.* 2015, Hilker and Lewis 2010, Zhang and Jin 2017).

3. Motivation for the numerical method

We will first recall the variational formulation of problem (1):

$$\begin{cases} \text{Find } u \in L^2((0,T);V) \cap C^0([0,T];L^2(\Omega)) \textit{ such that} \\ (\dot{u}(t), w) + a(u(t), w; \epsilon_1, \alpha_1) = (f(u,v), w), \quad \forall w \in H_0^1(\Omega) \\ (\dot{v}(t), w) + a(v(t), w; \epsilon_2, \alpha_2) = (g(u,v), w), \quad \forall w \in H_0^1(\Omega) \\ u(x,0) = u^0, \; v(x,0) = v^0 \end{cases} \tag{2}$$

where $V = H_0^1(\Omega)$. Here, the superposed dot denotes the time differentiation and $(\cdot,\cdot)$ is the $L^2(\Omega)$ inner product and the bilinear operator is

$$a(u(t), w; \epsilon, \beta) = \epsilon \int_\Omega u_x \, w_x + \int_\Omega \beta \, u_x \, w \tag{3}$$

Let $0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1$ and $\mathcal{T}_h = \{K\}$ be a decomposition of Ω into subintervals $K = (x_{k-1}, x_k)$ where $k = 1, \dots, N$ and $\{0 = t_0 < t_1 \dots < t_M = T\}$ be a uniform partition of time interval with $\Delta t_m = t_{m+1} - t_m$. Moreover, the time discretization of the continuous reaction kinetics $f(u,v)$ and $g(u,v)$ are defined by:

Kinetics (i)

$$\hat{f}(u^n, v^n) := \beta_1 u^n (1 - u^{n-1}) - \gamma_1 u^n v^{n-1} \approx f(u,v)$$

$$\hat{g}(u^n, v^n) := \beta_2 v^n (r - v^{n-1}) + \gamma_2 u^{n-1} v^n \approx g(u,v)$$

Kinetics (ii)

$$\hat{f}(u^n, v^n) := \quad u^n (1 - u^{n-1}) - \frac{u^n v^{n-1}}{u^{n-1} + \alpha} \approx f(u,v)$$

$$\hat{g}(u^n, v^n) := \quad \beta \frac{u^{n-1} v^n}{u^{n-1} + \alpha} - \gamma v^n \approx g(u,v)$$

Here, the finite element solutions u^n and v^n are approximations of the continuous solutions u and v at t_n. The problem (1) can be integrated in time by using semi-implicit time-stepping scheme to get the following discretized problem:

$$\begin{cases} \frac{u^n - u^{n-1}}{\Delta t_n} = \epsilon_1 \frac{\partial^2 u^n}{\partial x^2} - \alpha_1 \frac{\partial u^n}{\partial x} + \hat{f} \\ \frac{v^n - v^{n-1}}{\Delta t_n} = \epsilon_2 \frac{\partial^2 v^n}{\partial x^2} - \alpha_2 \frac{\partial v^n}{\partial x} + \hat{g}, \; n = 1, .., M. \end{cases} \tag{4}$$

Then, the corresponding bilinear form is defined as,

$$\begin{cases} \frac{(u^n, w) - (u^{n-1}, w)}{\Delta t_n} + a(u^n, w; \epsilon_1, \alpha_1) = (\hat{f}, w) \\ \frac{(v^n, w) - (v^{n-1}, w)}{\Delta t_n} + a(v^n, w; \epsilon_2, \alpha_2) = (\hat{g}, w), n = 1, \dots, M. \end{cases} \tag{5}$$

for any $w \in H_0^1$.

In order to discretize the problem (5) in space, we use an economical form of the RFB method (Brezzi *et al.* 1992, Hughes 1995) which is designed for the stationary problem and its explicit description is given in (Sendur and Nesliturk 2012). Here, the RFB functions are replaced by pseudo RFBs, which retain the same qualitative behavior as the RFBs, and they are computed by using appropriate sub-grid inside each element K. Once the subgrid points are calculated, we can treat the resulting scheme as doing plain Galerkin on the new mesh that results adding the sub-grid to the original one. We now apply the strategy proposed in Sendur and Nesliturk (2012) to the problem (5), that is:
Find $u_h^n, v_h^n \in V_h$ such that $\forall w_h \in V_h$

$$\begin{cases} \frac{(u_h^n, w_h) - (u_h^{n-1}, w_h)}{\Delta t_n} + a(u_h^n, w_h; \epsilon_1, \alpha_1) = (\hat{f}, w_h) \\ \frac{(v_h^n, w_h) - (v_h^{n-1}, w_h)}{\Delta t_n} + a(v_h^n, w_h; \epsilon_2, \alpha_2) = (\hat{g}, w_h), n = 1, \dots, M. \end{cases} \tag{6}$$

where $V_h \subset H_0^1(I)$ is the finite-dimensional space (see Sendur and Nesliturk (2012) for details). It is also possible to express finite dimensional problem (6) as linear algebraic equations in the following block matrix form for each time step:

$$\begin{pmatrix} A_1^{n-1} & B_1^{n-1} \\ 0 & C_1^{n-1} \end{pmatrix} \begin{pmatrix} u^n \\ v^n \end{pmatrix} = \begin{pmatrix} u^{n-1} \\ v^{n-1} \end{pmatrix} + \begin{pmatrix} \Phi_1^{n-1} \\ \Psi_1^{n-1} \end{pmatrix}$$

where $A_1^{n-1}, B_1^{n-1}, C_1^{n-1}, \Phi_1^{n-1}$ and Ψ_1^{n-1} are some matrices, depending on the solution at time level t_{n-1}.

4. Numerical Results

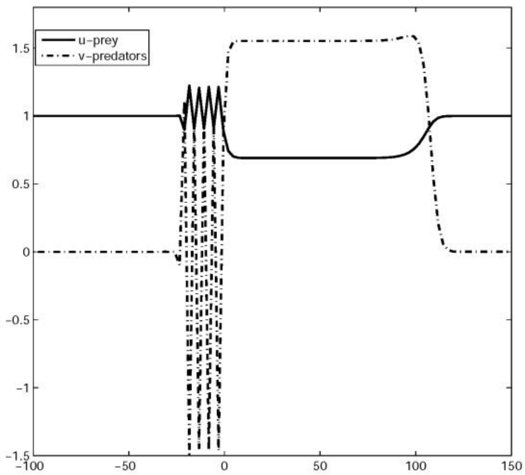
We present experimental results demonstrating the performance of the present algorithm (PRFB) for the nonlinear convection-diffusion-reaction systems modeling predator-prey interactions.

4.1. Experiment 1: Model (1) with kinetics (i)

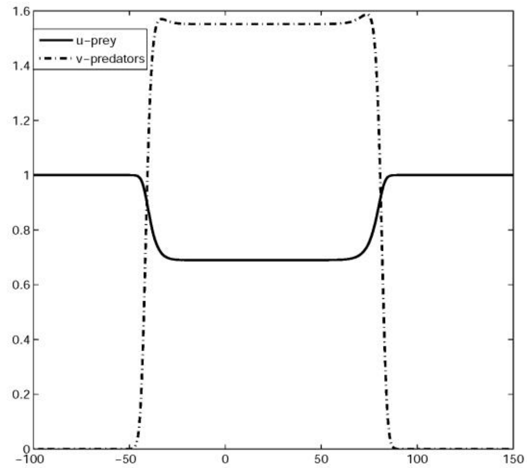
We will first consider the following test problem (see Zhang and Jin (2017)) in a bounded domain $\Omega = (-100,150)$:

$$\begin{cases} \frac{\partial u}{\partial t} = \epsilon_1 \frac{\partial^2 u}{\partial x^2} - 0.3 \ \frac{\partial u}{\partial x} + u(1-u) - 0.2 \ uv \\ \frac{\partial v}{\partial t} = \epsilon_2 \frac{\partial^2 v}{\partial x^2} - 0.8 \ \frac{\partial u}{\partial x} + v(1-v) + 0.8 \ uv \quad \text{in } \Omega \times (0,25] \\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \quad \text{on } \partial\Omega, \ \ t \in (0,25] \\ u(x,0) = 1, \quad v(x,0) = \begin{cases} 0.2, & x \in [-1,1] \\ 0, & \text{otherwise} \end{cases} \end{cases} \tag{7}$$

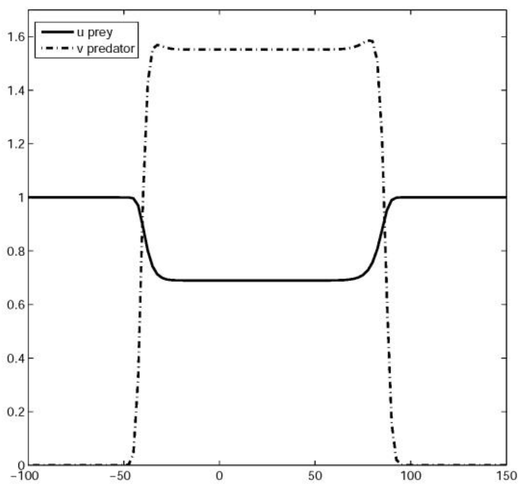
We first set $\epsilon_1 = \epsilon_2 = 1$ and decompose the domain into subintervals of length $N = 100$ and present the solutions obtained with the standard Galerkin method on both coarse / fine mesh (as reference solution) and the present formulation in Figure 1. We observe that the present method is stable and produces a solution that is very close to the exact solution while the approximations obtained by the Galerkin method exhibit non-physical oscillations.



(a) Numerical solution (Gal) for $N = 100$.



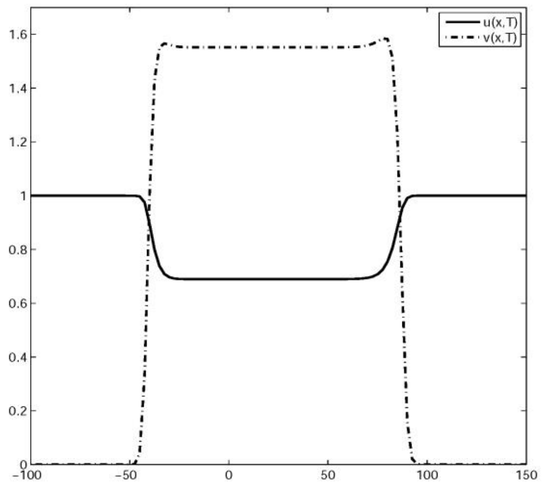
(b) Numerical solution (Gal) for $N = 4000$.



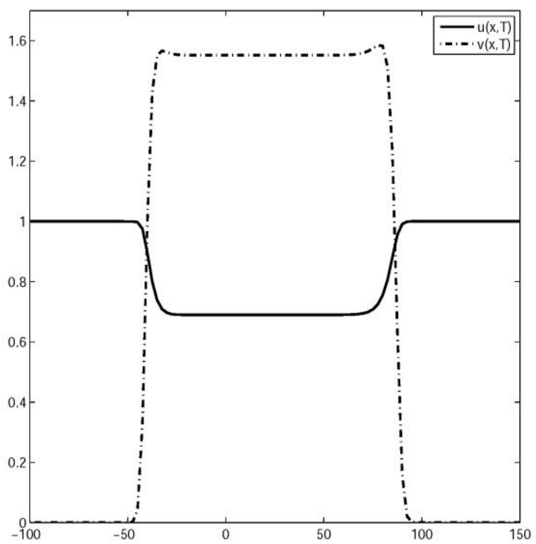
(c) Numerical solution (PRFB) for $N = 100$.

Figure 1: The numerical solution when $\epsilon_1 = \epsilon_2 = 1$ and $\Delta t = 0.001$.

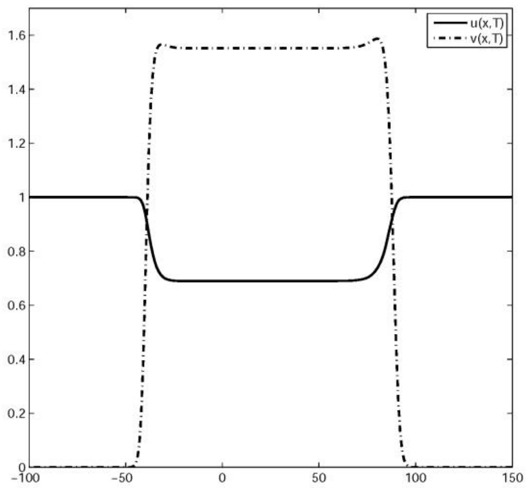
In Figure 2, we present the numerical solutions when $\epsilon_2 = 1$ and $N = 100$ for various intensities of diffusion ($\epsilon_1 = 0.01, 0.001, 0.0001$). The results illustrate the importance of the augmented grid strategy to capture the details of the solution. We note that the results with finer meshes (not presented here) show better numerical approximations to the exact solution as Δt and Δx decrease. The results are qualitatively similar to the one in Zhang and Jin (2017).



(a) $\epsilon_1 = 0.01$



(b) $\epsilon_1 = 0.001$



(c) $\epsilon_1 = 0.0001$

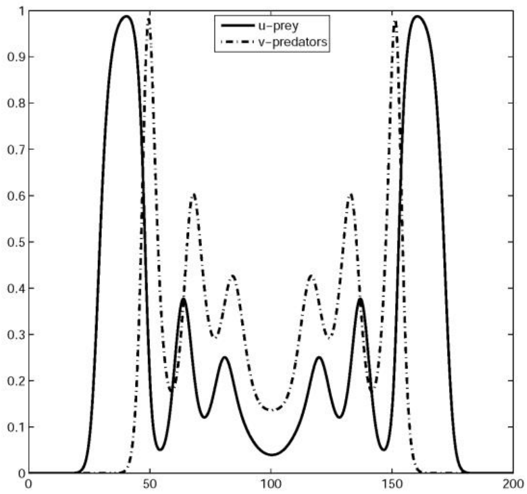
Figure 2: The numerical solutions (PRFB) when $\epsilon_2 = 1$, $\Delta t = 0.001$, $N = 100$.

4.2. Experiment 2: Model (1) with kinetics (ii) with different conditions

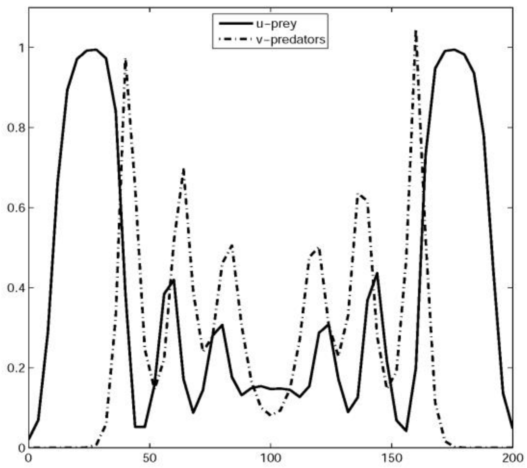
Next, we consider the following test case in a bounded domain $\Omega = (0,4000)$:

$$\begin{cases} \frac{\partial u}{\partial t} = \epsilon_1 \frac{\partial^2 u}{\partial x^2} - \alpha_1 \frac{\partial u}{\partial x} + u(1-u) - \frac{uv}{u+\frac{33}{80}} \\ \frac{\partial v}{\partial t} = \epsilon_2 \frac{\partial^2 v}{\partial x^2} - \alpha_2 \frac{\partial u}{\partial x} + 2\frac{uv}{u+\frac{33}{80}} - 0.8\ v \quad \text{in } \Omega \times (0,40] \\ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \quad \text{on } \partial\Omega, \quad t \in (0,600] \\ u(x,0) = \frac{11}{40} + 10^{-8}(x-1200)(x-2800), v(x,0) = \frac{319}{640} \end{cases} \tag{8}$$

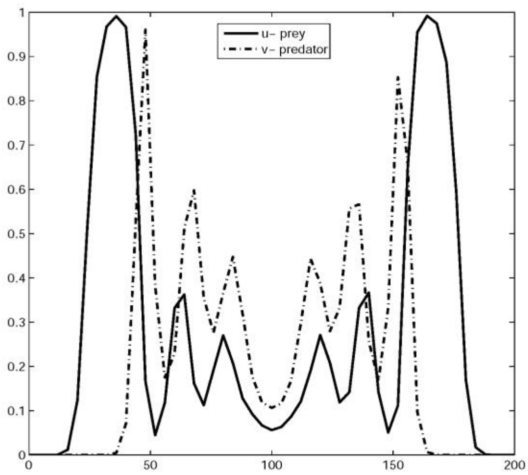
We first set $\epsilon_1 = \epsilon_2 = 1$ and decompose the domain into subintervals of length $N = 50$ and present the numerical solutions obtained with the standard Galerkin method on both coarse and fine mesh (reference solution) and the present formulation. Comparing the results in Figure 3, we conclude that the present method performs very well even on coarse mesh and produces a solution that is very close to the exact solution while the approximation obtained by the Galerkin method is not satisfactory.



(a) Numerical solution (Gal) for $N = 4000$.



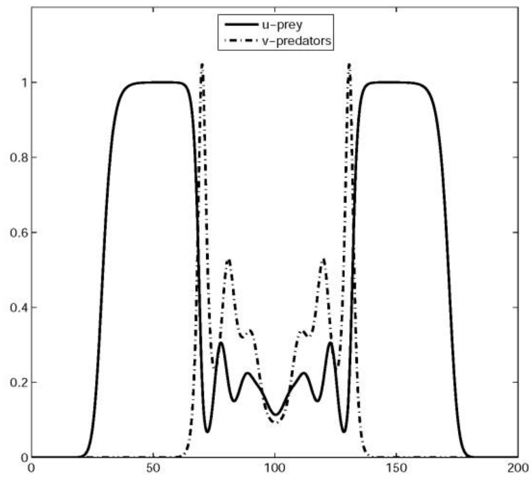
(b) Numerical solution (Gal) for $N = 50$.



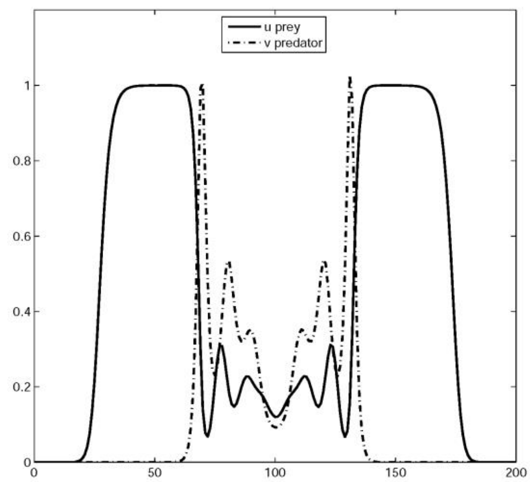
(c) Numerical solution (PRFB) for $N = 50$.

Figure 3: The numerical solutions when $\epsilon_1 = \epsilon_2 = 1$ and $\Delta t = 0.001$.

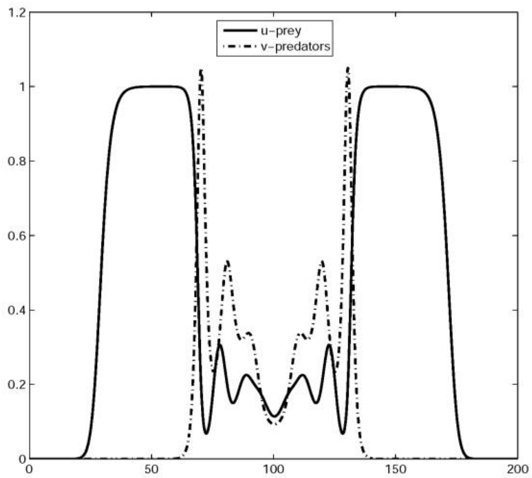
In Figure 4, we present the numerical solutions when $\epsilon_2 = 1$ and $N = 100$ for various intensities of diffusion ($\epsilon_1 = 0.01, 0.0001$). The results show the present formulation is convenient to capture the exact characteristics of the solution for a wide range of problem configurations.



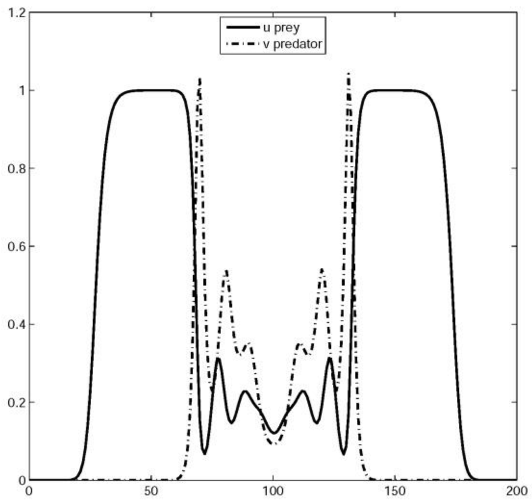
(a) Numerical solution (Gal) for $\epsilon_1 = 0.01,\ N = 4000$.



(b) Numerical solution (PRFB) for $\epsilon_1 = 0.01,\ N = 100$.



(c) Numerical solution (Gal) for $\epsilon_1 = 0.0001,\ N = 4000$.



(d) Numerical solution (PRFB) for $\epsilon_1 = 0.0001, N = 100$.

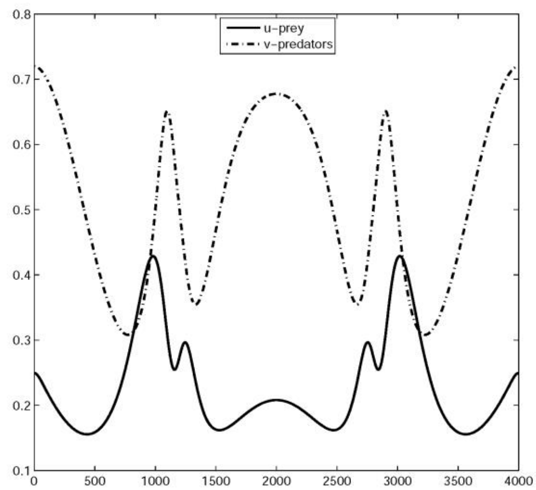
Figure 4: The numerical solution when $\epsilon_2 = 1$ and $\Delta t = 0.001$.

4.3. Experiment 3: Model (1) with kinetics (ii)

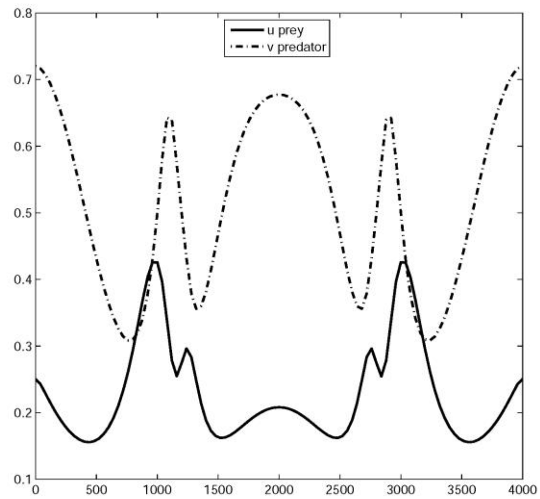
Finally, we take the following test case (see Garvie (2007)) in a bounded domain $\Omega = (0,200)$ with different conditions:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \epsilon_1 \dfrac{\partial^2 u}{\partial x^2} - \alpha_1 \dfrac{\partial u}{\partial x} + u(1-u) - \dfrac{uv}{u+0.3} \\ \dfrac{\partial v}{\partial t} = \epsilon_2 \dfrac{\partial^2 v}{\partial x^2} - \alpha_2 \dfrac{\partial u}{\partial x} + 2\dfrac{uv}{u+\alpha} - 0.8\ v \quad \text{in } \Omega \times (0,40] \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = 0 \quad \text{on } \partial\Omega, \quad t \in (0,40] \\ u(x,0) = e^{-\frac{(x-100)^2}{5}}, \quad v(x,0) = \dfrac{2}{5} \end{cases} \tag{9}$$

In Figures 5-6, we set $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_1 = 0.0001$, $\epsilon_2 = 1$, respectively, decompose the domain into $N = 100$ subintervals and plot the numerical approximations obtained with the standard Galerkin method on fine mesh (as reference solution) and the present method. Figures 5-6 demonstrate that we achieve a perfect match between the numerical and reference solutions. We also note that the plots are qualitatively similar to the one in Garvie (2007).

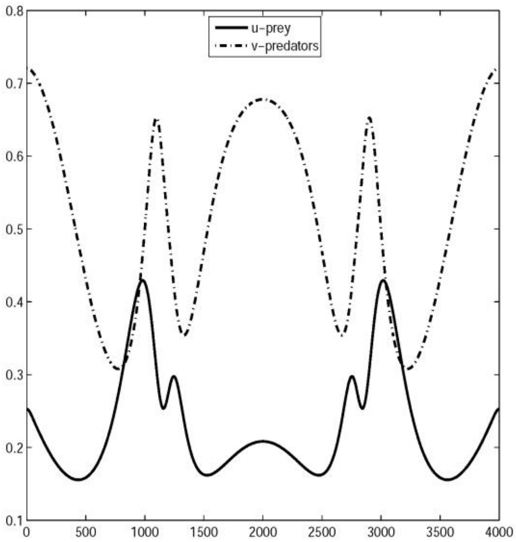


(a) Numerical solution (Gal) for $N = 4000$.

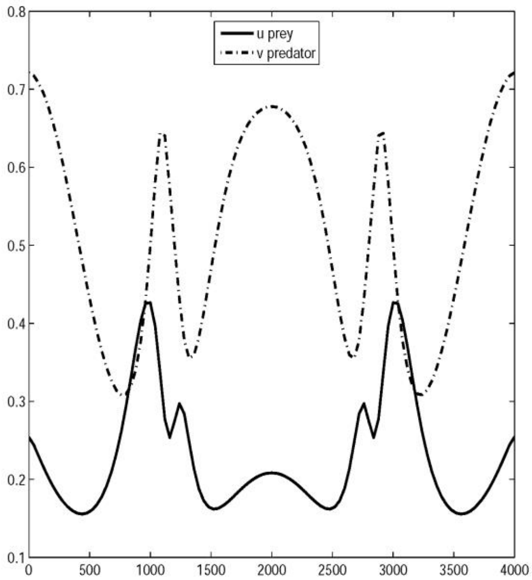


(b) Numerical solution (PRFB) for $N = 100$.

Figure 5: The numerical solution when $\epsilon_1 = \epsilon_2 = 1$ and $\Delta t = 0.001$.



(a) Numerical solution (Gal) for $N = 4000$.



(b) Numerical solution (PRFB) for $N = 100$.

Figure 6: The numerical solution when $\epsilon_1 = 0.0001$ $\epsilon_2 = 1$ and $\Delta t = 0.001$.

5. Conclusion

In this paper, numerous benchmark problems are employed to validate the performance and the robustness of the PRFB method for the numerical solution of predator-prey interactions. Numerical experiments cover a wide range of problem configurations and the results illustrate the good performance of the PRFB method.

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