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Gegenbauer Polinomları İçin Bilineer ve Bilateral Doğurucu Fonksiyonlar

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Anahtar kelimeler
Doğurucu fonksiyon; Gegenbauer polinomları; Multilinear ve multilateral doğurucu fonksiyon.

Özet

Bu çalışma Gegenbauer (veya ultraküresel) polinomları için bilineer ve bilateral doğurucu fonksiyonlarla ilgilidir. Gegenbauer polinomları için bazı özel durumları, çeşitli özelliklerini, multilinear ve multilateral doğurucu fonksiyonları elde edilmiştir.

Bilinear and Bilateral generating functions for the Gegenbauer Polynomials

Keywords
Generating function; Gegenbauer polynomials; Multilinear and multilateral generating function.

Abstract

The present study deals with bilateral and bilinear generating functions for the Gegenbauer (or ultraspherical) polynomials. In this manuscript we obtain some special cases for Gegenbauer polynomials. miscellaneous properties and multilinear and multilateral generating functions.

1. Introduction

Generalized functions oscupy the pride of place in literature on special functions. Their importance which is mounting everyday stems from the fact that they generalize well-know one variable special functions namely Hermite polynomials, Laguerre polynomials, Legendre polynomials, Gegenbauer polynomials, Jacobi polynomials, Rice polynomials, Generalized Sylvester polynomials etc. All these polynomials are closely associated with problems of applied nature. For example, Gegenbauer polynomials are deeply connected with axially symmetric potentials in dimensions and contain the Legendre and Chebyshev polynomials as special cases. The hypergeometric functions of which the Jacobi polynomials is a special case, is important in many cases of mathematics analysis and its applications.

The Gegenbauer (or ultraspherical) polynomials $C_m^{\mu} : \mathbb{C} \to \mathbb{C}$ can be defined by Gauss hypergeometric series as follows (Olver et al. 2010)

$$C_m^{\mu}(z) := \frac{(2\mu)_m}{m!}\, {}_2F_1\left(\begin{matrix} -m, \quad m+2\mu \\ \\ \mu+\frac{1}{2} \end{matrix} ; \frac{1-z}{2}\right), \qquad (1)$$

for $m \in \mathbb{N}_0$ and $\mu \in (-1/2, \infty) \setminus \{0\}$.

The Gauss hypergeometric function ${}_2F_1 : \mathbb{C}^2 \times (\mathbb{C} \setminus \mathbb{N}_0) \times D \to \mathbb{C}$ is defined as

$${}_2F_1\left(\begin{matrix} \alpha, \beta \\ \\ \gamma \end{matrix} ; z\right) := \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{(\gamma)_m} \frac{z^m}{m!},$$

where the Pochammer symbol (rising factorial) $(\cdot)_m : \mathbb{C} \to \mathbb{C}$ is defined by

$$(\alpha)_m := \prod_{s=1}^{m} (\alpha + s - 1),$$

where $m \in \mathbb{N}_0$.

Consider the generating function for Gegenbauer polynomials given by Olver et al. (2010), namely

$$\sum_{m=0}^{\infty} C_m^{\mu}(z) t^m = (1 - 2zt + t^2)^{-\mu}. \qquad (2)$$

We attempt to generalize this expansion using the representation of Gegenbauer polynomials in terms of Jacobi polynomials given by Olver et al. (2010), namely

$$C_m^{\mu}(z) = \frac{(2\mu)_m}{(\mu+\frac{1}{2})_m} P_m^{(\mu-1/2,\, \mu-1/2)}(z)\,. \qquad (3)$$

The Jacobi polynomials $P_m^{(\alpha,\beta)} : \mathbb{C} \to \mathbb{C}$ can be defined by Gauss hypergeometric series as follows (Srivastava and Monocha 1984):

$$P_m^{(\alpha,\beta)}(z) := \frac{(\alpha+1)_m}{m!}\, {}_2F_1\left(\begin{matrix} -m, m+\alpha+\beta+1; \\ \\ \alpha+1; \end{matrix} \frac{1-z}{2}\right),$$

for $m \in \mathbb{N}_0$ and $\alpha, \beta > -1$ such that if $\alpha, \beta \in (-1, 0)$ then $\alpha + \beta + 1 \neq 0.$

Gegenbauer polynomials by suitably specializing the parameters in the corresponding results for the Jacobi polynomials:

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{(\mu)_m} C_m^{\alpha}(z) t^m$$

$$= F_1\left[\lambda, \alpha, \alpha; \mu; (z+\sqrt{z^2-1})t, (z-\sqrt{z^2-1})t\right], \qquad (4)$$

which, for $\mu = 2\alpha,$ yields

$$\sum_{m=0}^{\infty} \frac{(\lambda)_m}{(2\alpha)_m} C_m^{\alpha}(z) t^m$$

$$= \left(1-(z-\sqrt{z^2-1})t\right)^{-\lambda} {}_2F_1\left(\begin{matrix} \lambda, \quad \alpha; \\ \\ 2\alpha; \end{matrix} \frac{2t\sqrt{x^2-1}}{1-(x-\sqrt{x^2-1})t}\right), \qquad (5)$$

or, alternatively,

$$\sum_{m=0}^{\infty}\frac{(\lambda)_m}{(2\alpha)_m}C_m^{\alpha}(z)t^m$$

$$=(1-zt)^{-\lambda}\;{}_2F_1\left(\begin{array}{c}\frac{1}{2}\lambda,\quad \frac{1}{2}\lambda+\frac{1}{2};\\ \qquad\qquad \frac{(z^2-1)t^2}{(1-zt)^2}\\ \alpha+\frac{1}{2};\end{array}\right), \tag{6}$$

view of the quadratic transformation (Srivastava and Monocha 1984):

$${}_2F_1\left(\begin{array}{c}a,b;\\ \qquad 2z\\ 2b;\end{array}\right)=(1-z)^{-\alpha}\;{}_2F_1\left(\begin{array}{c}\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2};\\ \qquad\qquad \left(\frac{z}{1-z}\right)^2\\ b+\frac{1}{2};\end{array}\right).$$

Starting, as usual, from (2) we get the following formula of the type (1) for the polynomials $C_m^{\alpha}(x)$ (Srivastava and Monocha 1984):

$$\sum_{m=0}^{\infty}\binom{r+m}{m}C_{m+r}^{\alpha}(z)t^m=\rho^{-2\alpha-r}C_r^{\alpha}\left(\frac{z-t}{\rho}\right), \tag{7}$$

where ρ is defined by $\rho=(1-2zt+t^2)^{1/2}$, $|\rho|<1$.

The Gegenbauer (or ultraspherical) polynomial $C_m^{\mu}(z)$ $(\mu>-1/2, |z|\le 1)$ are defined by Horadam (1985),

$$C_0^{\mu}(z)=1,\quad C_1^{\mu}(z)=2\mu z,$$

with the recurrence relation

$$mC_m^{\mu}(z)$$
$$=2z(\mu+m-1)C_{m-1}^{\mu}(z)-(2\mu+m-2)C_{m-2}^{\mu}(z)$$
$$(\ m\ge 2).$$

Lemma 1. The following addition formula holds for the Gegenbauer (or ultraspherical) polynomials:

$$C_n^{\mu_1+\mu_2}(z)=\sum_{m=0}^{n}C_{n-m}^{\mu_1}(z)C_m^{\mu_2}(z).$$

(8)

Proof. Replacing μ by $\mu_1+\mu_2$ in (2), we obtain

$$\sum_{n=0}^{\infty}C_n^{\mu_1+\mu_2}(z)t^n$$
$$=(1-2zt+t^2)^{-\mu_1-\mu_2}$$
$$=(1-2zt+t^2)^{-\mu_1}(1-2zt+t^2)^{-\mu_2}$$
$$=\sum_{n=0}^{\infty}C_n^{\mu_1}(z)t^n\sum_{m=0}^{\infty}C_m^{\mu_2}(z)t^m$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}C_{n-m}^{\mu_1}(z)C_m^{\mu_2}(z)t^n$$

From the coefficients of t^n on the both sides of the last equality, one can get the desired result.

2. Multilinear and Multilateral Generating Functions

We aim here at presenting a family of bilinear and bilateral generating relations for the Gegenbauer (or ultraspherical) polynomials $C_m^{\mu}(x)$ which are generated by (2) without using grouptheoretic technique but, with the help of the similar method as given in (Özmen and Erkuş-Duman 2013, 2015, 2018).

Theorem 1. For a non-vanishing function $\Omega_{\eta}(s_1,...,s_r)$ of r complex variables $s_1,...,s_r$ $(r\in \mathrm{N})$ and for $\eta,\gamma\in \mathrm{C}$, $m\in \mathrm{N}$; let

$$\Lambda_{\eta,\gamma}(s_1,...,s_r;\zeta):=\sum_{m=0}^{\infty}a_m\Omega_{\eta+\gamma m}(s_1,...,s_r)\zeta^m$$
$$(a_m\neq 0)$$

and

$$\Upsilon_{n,s}^{\eta,\gamma}(z;s_1,...,s_r;\xi)$$
$$:=\sum_{m=0}^{[n/s]}a_mC_{n-sm}^{\mu}(z)\Omega_{\eta+\gamma m}(s_1,...,s_r)\xi^m.$$

Then, for $s\in \mathrm{N}$, we have

$$\sum_{l=0}^{\infty}\Upsilon_{l,s}^{\eta,\gamma}\left(z;s_1,...,s_r;\frac{\lambda}{w^s}\right)w^l$$
$$=(1-2zw+w^2)^{-\mu}\Lambda_{\mu,\psi}(s_1,...,s_r;\lambda)\quad \left(|w|<1\right). \tag{9}$$

Proof. Let E denote the first member of the assertion (9) of Theorem1.

Then,

$$E=\sum_{l=0}^{\infty}\sum_{m=0}^{[l/s]}a_m C_{l-sm}^{\mu}(z)\Omega_{\eta+\gamma m}(s_1,...,s_r)\lambda^m w^{l-sm}.$$

Replacing l by $l+sm$,

$$E=\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}a_m\, C_l^{\mu}(z)\Omega_{\eta+\gamma m}(s_1,...,s_r)\lambda^m w^l$$

$$=\sum_{l=0}^{\infty}C_l^{\mu}(z)w^l\sum_{m=0}^{\infty}a_m\Omega_{\eta+\gamma m}(s_1,...,s_r)\lambda^m$$

$$=(1-2zw+w^2)^{-\mu}\Lambda_{\eta,\gamma}(s_1,...,s_r;\lambda)\qquad(|w|<1)$$

which completes the proof.

Theorem 2. For a non-vanishing function $\Omega_\eta(y_1,...,y_r)$ of r complex variables $y_1,...,y_r$ $(r\in\mathrm{N})$ and of complex order η, let

$$\Lambda_{\eta,\psi}^{\mu_1+\mu_2}(x;y_1,...,y_r;\omega)$$
$$:=\sum_{s=0}^{[m/p]}a_s C_{m-ps}^{\mu_1+\mu_2}(x)\Omega_{\eta+\psi s}(y_1,...,y_r)\omega^s,$$

where $a_s\neq 0,\ \eta,\psi\in\mathrm{C}$, $m,p\in\mathrm{N}$ and the notation $[m/p]$ means the greatest integer less than or equal m/p .

Then, we get

$$\sum_{k=0}^{m}\sum_{l=0}^{[k/p]}a_l C_{m-k}^{\mu_1}(x)C_{k-pl}^{\mu_2}(x)\Omega_{\eta+\psi l}(y_1,...,y_r)\omega^l$$
$$=\Lambda_{\eta,\psi}^{\mu_1+\mu_2}(x;y_1,...,y_r;\omega).\qquad(10)$$

Proof. Let H denote the right side of the assetion (10). Then, upon subsituting for the Gegenbauer polynomials $C_{m-pk}^{\mu_1+\mu_2}(x)$ from the (8) into the left-hand side of (10), we get

$$H=\sum_{l=0}^{[m/p]}\sum_{k=0}^{m-pl}a_l C_{m-k-pl}^{\mu_1}(x)C_k^{\mu_2}(x)\Omega_{\eta+\psi l}(y_1,...,y_r)\omega^l$$

$$=\sum_{l=0}^{[m/p]}a_l\left(\sum_{k=0}^{m-pl}C_{m-k-pl}^{\mu_1}(x)C_k^{\mu_2}(x)\right)\Omega_{\eta+\psi l}(y_1,...,y_r)\omega^l$$

$$=\sum_{l=0}^{[m/p]}a_l C_{m-pl}^{\mu_1+\mu_2}(x)\Omega_{\eta+\psi l}(y_1,...,y_r)\omega^l$$

$$=\Lambda_{\eta,\psi}^{\mu_1+\mu_2}(x;y_1,...,y_r;\omega).$$

Theorem 3. For a non-vanishing function $\Omega_\mu(s_1,...,s_r)$ of r complex variables $s_1,...,s_r$ $(r\in\mathrm{N})$ and of complex order μ , suppose that

$$\Lambda_{\mu,p,q}^{\alpha}[z;s_1,...,s_r;\omega]$$
$$:=\sum_{m=0}^{\infty}a_m C_{r+qm}^{\alpha}(z)\Omega_{\mu+pm}(s_1,...,s_r)\omega^m$$

where $a_m\neq 0$ and

$$\theta_{\mu,p,q}(s_1,...,s_r;t)$$

$$:=\sum_{k=0}^{[m/q]}a_k\binom{r+m}{m-qk}\Omega_{\mu+pk}(s_1,...,s_r)t^k.$$

Then, for $p,q\in\mathrm{N}$; we obtain

$$\sum_{m=0}^{\infty}C_{m+r}^{\alpha}(z)\theta_{\mu,p,q}(s_1,...,s_r;t)\omega^m$$
$$=\rho^{-2\alpha-r}\Lambda_{\mu,p,q}^{\alpha}\left[\frac{z-\omega}{\rho};s_1,...,s_r;t(\frac{\omega}{\rho})^q\right]\qquad(11)$$

$$\left(\rho=(1-2x\omega+\omega^2)^{1/2},\ |\omega|<1\right).$$

Proof. For the proof of Teorem 3, we find it to be convenient to denote the right side of the assertion (11) by Π.

Then,

$$\Pi=\sum_{m=0}^{\infty}C_{m+r}^{\alpha}(z)\sum_{k=0}^{[m/q]}a_k\binom{r+m}{m-qk}\Omega_{\mu+pk}(s_1,...,s_r)t^k\omega^m.$$

Replacing m by $m+qk$ and then using (7), we may write that

$$\Pi=\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{r+m+qk}{m}C_{m+r+qk}^{\alpha}(z)\times a_k\,\Omega_{\mu+pk}(s_1,...,s_r)t^k\omega^{m+qk}$$

$$=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty}\binom{r+m+qk}{m}C_{m+r+qk}^{\alpha}(z)\omega^m\right)\times a_k\Omega_{\mu+pk}(s_1,...,s_r)(t\omega^q)^k$$

$$=\sum_{k=0}^{\infty}a_k\rho^{-2\alpha-r-qk}C_{r+qk}^{\alpha}\left(\frac{z-\omega}{\rho}\right)\times\Omega_{\mu+pk}(s_1,...,s_r)(t\omega^q)^k$$

$$=\rho^{-2\alpha-r}\sum_{k=0}^{\infty}a_kC_{r+qk}^{\alpha}\left(\frac{z-\omega}{\rho}\right)\times\Omega_{\mu+pk}(s_1,...,s_r)\left(\frac{t\omega^q}{\rho^q}\right)^k$$

$$=\rho^{-2\alpha-r}\Lambda_{\mu,p,q}^{\alpha}\left[\frac{z-\omega}{\rho};s_1,...,s_r;t\left(\frac{\omega}{\rho}\right)^q\right]$$

$$\left(\rho=(1-2x\omega+\omega^2)^{1/2},\ |\omega|<1\right)$$

which completes the proof.

3.Special Cases

We can give promote apposition of the on teorems.
Set

$$\Omega_{\eta+\gamma m}(s_1,...,s_r)=h_{\eta+\gamma m}^{(\beta_1,...,\beta_r)}(s_1,...,s_r)$$

in Theorem 1, where the multivariable extension of the Lagrange-Hermite polynomials $h_{\eta+\gamma m}^{(\beta_1,...,\beta_r)}(s_1,...,s_r)$ (Altın and Erkuş 2006), generated by

$$\prod_{k=1}^{r}\left\{(1-s_kz^k)^{-\beta_k}\right\}=\sum_{m=0}^{\infty}h_m^{(\beta_1,...,\beta_r)}(s_1,...,s_r)z^n. \tag{12}$$

$$\left(\beta\in\mathbb{C};\ |z|<\min\left\{|s_1|^{-1},|s_2|^{-1/2},...,|s_r|^{-1/r}\right\}\right)$$

We obtain, the following result which provides a class of bilateral generating functions for the multivariable extension of the Lagrange-Hermite polynomials $h_{\mu+\psi k}^{(\beta_1,...,\beta_r)}(s_1,...,s_r)$ and the Gegenbauer polynomials $C_l^{\mu}(x)$.

Corollary 1. If

$$\Lambda_{\mu,\psi}(s_1,...,s_r;\zeta):=\sum_{m=0}^{\infty}a_mh_{\eta+\gamma m}^{(\beta_1,...,\beta_r)}(s_1,...,s_r)\zeta^m$$

$$\left(a_m\neq0,\ \eta,\gamma\in\mathbb{C}\right),$$

then, we have

$$\sum_{l=0}^{\infty}\sum_{m=0}^{[l/s]}a_mC_{l-sm}^{\mu}(z)h_{\eta+\gamma m}^{(\beta_1,...,\beta_r)}(s_1,...,s_r)\frac{\lambda^m}{w^{pm}}w^l$$

$$=(1-2zw+w^2)^{-\mu}\Lambda_{\eta,\gamma}(s_1,...,s_r;\lambda). \tag{13}$$

Remark 1. Taking $a_m=1,\ \eta=0,\ \gamma=1$ and using the generating relation (12) for the multivariable polynomials $h_{\eta+\gamma m}^{(\beta_1,...,\beta_r)}(s_1,...,s_r)$ in Corollary 1, we have

$$\sum_{l=0}^{\infty}\sum_{m=0}^{[l/s]}C_{l-sm}^{\mu}(x)h_m^{(\beta_1,\beta_2,...,\beta_r)}(s_1,...,s_r)\lambda^mw^{l-sm}$$

$$=(1-2xw+w^2)^{-\mu}\prod_{j=1}^{r}\left\{(1-s_j\lambda^j)^{-\beta_j}\right\}.$$

$$\left(|\lambda|<\min\left\{|s_1|^{-1},|s_2|^{-1/2},...,|s_r|^{-1/r}\right\},\ |w|<1\right)$$

If we set $r=1,\ \mathrm{y}_1=\mathrm{y}$ and

$$\Omega_{\eta+\psi s}(y)=C_{\eta+\psi s}^{\mu_3}(y)$$

in Theorem 2, we have the following summation expression for the Gegenbauer polynomials.

Corollary 2. If

$$\Lambda_{\eta,\psi}^{\mu_1+\mu_2}(x;y_1,...,y_r;\omega)$$

$$:=\sum_{s=0}^{[m/p]}a_sC_{m-ps}^{\mu_1+\mu_2}(x)C_{\eta+\psi s}^{\mu_3}(y)\omega^s,$$

then, we have

$$\sum_{k=0}^{m}\sum_{l=0}^{[k/p]}a_lC_{m-k}^{\mu_1}(x)C_{k-pl}^{\mu_2}(x)C_{\eta+\psi l}^{\mu_3}(y)\omega^l$$

$$= \Lambda_{\eta,\psi}^{\mu_1+\mu_2}(x; y_1,..., y_r; \omega). \quad (14)$$

provided that each member of (14) exists.

Remark 2. Taking $a_l = 1,\ \eta = 0,\ p = 1,\ \psi = 1\ y = x,\ \omega = 1$ in Corollary 2 and using (8) we have

$$\sum_{k=0}^{m}\sum_{l=0}^{k} C_{m-k}^{\mu_1}(x) C_{k-l}^{\mu_2}(x) C_l^{\mu_3}(x)$$

$$= C_m^{\mu_1+\mu_2+\mu_3}(x).$$

If we set $s = 1, s_1 = y$ and

$$\Omega_{\mu+\psi k}(y_1) = P_{\mu+\psi k}^{(\lambda,\beta)}(y)$$

in Theorem 3, where the classical Jacobi polynomials $P_n^{(\lambda,\beta)}(y)$ is generated by (Erdélyi et al. 1955),

$$\sum_{n=0}^{\infty} P_n^{(\lambda,\beta)}(x) t^n = \frac{2^{\lambda+\beta}}{\rho}(1-t+\rho)^{-\lambda}(1+t+\rho)^{-\beta}$$
$$\left\{\rho = (1-2xt+t^2)^{1/2}, |t| < 1\right\}$$

we get a family of the bilateral generating functions for the classical Jacobi polynomials and the Gegenbauer (or ultraspherical) polynomials $C_m^{\mu}(x)$ as follows:

Corollary 3. If

$$\Lambda_{\mu,p,q}^{\alpha}(z; y; \omega) := \sum_{m=0}^{\infty} a_m C_{r+qm}^{\alpha}(z) P_{\mu+pm}^{(\lambda,\beta)}(y)\omega^k,$$
$$\left(a_m \neq 0,\ r, p, q \in \mathrm{N},\ \mu \in \mathrm{C}\right)$$

and

$$\theta_{\mu,p,q}(y;t) := \sum_{k=0}^{[m/q]} a_k \binom{r+m}{m-qk} P_{\mu+pk}^{(\lambda,\beta)}(y) t^k.$$

Then, we get

$$\sum_{m=0}^{\infty} C_{m+r}^{\alpha}(z)\theta_{\mu,p,q}(y;t)\omega^m$$
$$= \rho^{-2\alpha-r}\Lambda_{\mu,p,q}^{\alpha}\left(\frac{z-\omega}{\rho}; y; t(\frac{\omega}{\rho})^q\right).$$
$$\left(\rho = (1-2x\omega+\omega^2)^{1/2},\ |\omega| < 1\right)$$

Furthermore, for each suitable choice of the coefficients a_m $(m \in \mathrm{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1,..., y_r)$, $r \in \mathrm{N}$, are expressed as an appropriate product of a lot of simpler functions, the assertions of Theorem 1, Teorem 2, Teorem 3 can be applied to yield many different families of the Gegenbauer polynomials.

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