STURM-LIOUVILLE PROBLEMS AND BOOLEAN DIFFERENTIAL EQUATIONS

M.SC. THESIS

Elif NURAY

SUPERVISORS

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DEPARTMENT OF MATHEMATICS

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AFYON KOCATEPE UNIVERSITY
GRADUATE SCHOOLOF NATURAL AND APPLIED SCIENCE

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TEZ ONAY SAYFASI

Elif NURAY tarafından hazırlanan "Sturm-Liouville Problems and Booelan Differential Equations" adlı tez çalışması lisansüstü eğitim ve öğretim yönetmeliğinin ilgili maddeleri uyarınca 24/09/2013 tarihinde aşağıdaki jüri tarafından oy birliği/oy çokluğu ile Afyon Kocatepe Üniversitesi Fen Bilimleri Enstitüsü **Matematik Anabilim Dalı'nda YÜKSEK LİSANS TEZİ** olarak kabul edilmiştir.

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ABSTRACT

M.Sc.Thesis

STURM-LIOUVILLE PROBLEMS AND BOOLEAN DIFFERENTIAL EQUATIONS

Elif Nuray
Afyon Kocatepe University
Graduate School of Natural and Applied Sciences
Department of Mathematics
Supervisors: Prof. Dr. Gheorghe Morosanu
Assist. Prof. Dr. Mehmet Eyüp Kiriş

This thesis consists of five sections. In the first section, the important points of subject which we studied are introduced. In the second section, the biological diffusion process is examined with mathematical model and the solution by method of separation variables. Formulation of the general Sturm-Liouville problem is given in the third section. In the fourth section, main results on Sturm-Liouville problem is given with important properties. In addition to the properties, boundary value problem and Green's function is introduced and relation between Green's function and regular Sturm-Liouville problem via eigenfunction expansion is given. After giving the important theorems, completeness of eigenfunctions of Sturm-Liouville problems are investigated in the fourth section of the thesis. In the final section, Boolean Differential Equations which are very important in Mathematical Logic are introduced. The purpose of this section is to start to investigate getting Boolean form of Sturm-Liouville problems if it is possible.

2013, v+50 pages.

Key Words: Sturm-Liouville Problems, logic function, Boolean Differential Equation, diffusion process on biological systems, the method of separation of variables, Green's function, boundary value problems, eigenvalue and eigenfunction.

ÖZET

Yüksek Lisans Tezi

STURM-LIOUVILLE PROBLEMLERİ VE BOOLEAN DİFERANSİYEL DENKLEMLERİ

Elif Nuray
Afyon Kocatepe Üniversitesi
Fen Bilimleri Enstitüsü
Matematik Anabilim Dalı
Danışmanlar: Prof. Dr. Gheorghe Morosanu
Yrd. Doc. Dr. Mehmet Eyüp Kiriş

Bu tez beş bölümden oluşmaktadır. Birinci bölümde, çalıştığımız konunun önemli noktaları tanıtıldı. İkinci bölümde, biyolojiksel difüzyon sürecinin matematiksel modeli ve değişkenlerine ayrılabilir yöntemi çalışıldı. Sturm-Liouville probleminin formülü üçüncü bölümde ve ana sonuçları ile önemli özellikleri ise dördüncü bölümde verildi. Özelliklere ek olarak, sınır değer problemi ile Green fonksiyonu tanıldı ve özfonksiyon açılımı yardımıyla olan Sturm-Liouville problemi ve Green fonksiyonu arasındaki ilişki anlatıldı. Yine dördüncü bölümde, önemli teoremler verildikten sonra, Sturm-Liouville problemlerinin özfonksiyonlarının tamlığı araştırıldı. Final bölümünde, Matematiksel Lojik'te çok önemli bir yer tutan Boolean Diferansiyel Denklemleri tanıtıldı. Bu bölüm, eğer varsa, Sturm-Liouville problemlerinin Boolean formunu araştırmak için bir ön adım niteliğini taşımaktadır.

2013, v+50 sayfa.
Anahtar Kelimeler: Sturm-Liouville Problemleri, lojik fonksiyonu, Boolean Diferansiyel Denklemi, biyolojiksel sistemlerde difüzyon süreci, değişkenlerine ayırma metodu, Green Fonksiyonu, sınır değer problemleri, özdeğer ve özfonksiyon.

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Elif Nuray

Budapest, 2013

CONTENTS

LIST OF FIGURES

List of Figures

1 INTRODUCTION

Many of problems of the mathematical physics, mathematical biology and engineering are connected to Sturm-Liouville problems. Mathematicians have studied Sturm-Liouville problems for over 200 years. Highly developed theory and remains an active area of interest. A regular Sturm–Liouville equation, named after Jacques Charles François Sturm (1803–1855) and Joseph Liouville (1809–1882).

In this thesis, the main aim is giving the important points and the details about Sturm-Liouville problem and also solving the problems by using several mathematical methods. In addition to this, the other important aim of this study is introducing Boolean Differantial Equation. Biological diffusion process and heat equation examined before discussing about Sturm-Liouville problem. The examined mathematical model was solved by the method of separation of variables. After reaching the formulation of Sturm-Liouville problem, the main results and important properties was given in the related section. In the last section, the basic subject was given about Boolean Differenatial Equation. This study will be a first step for further studies about the relation between Sturm-Liouville problems and Boolean Differential Equations.

2 ON A BIOLOGICAL DIFFUSION PROCESS

In this section, we discuss a biological process that leads to an example of a Sturm-Liouville problem, thus motivating the mathematical theory included in the next sections of this thesis. More precisely we are interested in

2.1 Diffusion Through Membranes. Description and the Mathematical Modeling

If some matter is dissolved in a given fluid we obtain a **solution**. The solution is characterized by its **mass concentration**, say c, which depends on the time t and the space variable x, i.e., $c = c(t, x)$. Thus c represents the mass of dissolved matter per unit volume of liquid. The typical combination is salt plus water, and in general **solute** plus **solvent** gives a solution. The solute molecules move through the solvent from a high concentration to a low concentration region. This process is called **diffusion**. According to the well-known **Fick's law of diffusion**, the solute flux per unit area, say $\mathbf{j}$, is proportional to the rate of change $\frac{\partial c}{\partial x}$, i.e.,

$$\mathbf{j} = -D\frac{\partial c}{\partial x},$$

where D is a positive constant called the **diffusion coefficient**. It depends on the corresponding solution. The minus sign is motivated by the fact that the molecular flow is from a high concentration to a low concentration region.

If the solution occupies a region in $\mathbb{R}^3$, then $c = c(t, x, y, z)$. And the components of the flux $\mathbf{j}$ satisfy the equations

$$j_x = -D\frac{\partial c}{\partial x},\ j_y = -D\frac{\partial c}{\partial y},\ j_z = -D\frac{\partial c}{\partial z}. \tag{2.1.1}$$

Let V be an arbitrary volume with boundary S. By the law of mass conservation

$$\frac{\partial}{\partial t}\int_V c(t, x, y, z)\mathrm{d}V = -\int_S \mathbf{j}.\mathbf{n}\mathrm{d}S, \tag{2.1.2}$$

where $\mathbf{n}$ is the outward unit normal to S.

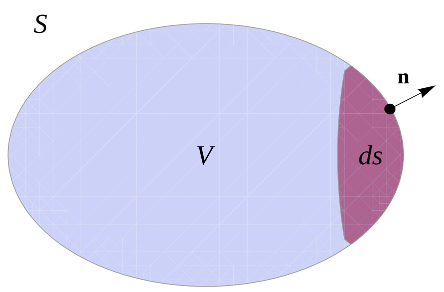

Figure 2.1.1: The conservation of mass.

By making use of (2.1.2) and Gauss' divergence theorem, one obtains

$$\int_V \left(\frac{\partial c}{\partial t} + \nabla\cdot\mathbf{j}\right)\mathrm{d}V = 0. \tag{2.1.3}$$

In equation (2.1.3) $\nabla\cdot\mathbf{j}$ is the divergence of the vector $\mathbf{j}$ and defined as

$$\nabla\cdot\mathbf{j} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}.$$

As a result, equations (2.1.1) and (2.1.3) lead us to the classic **diffusion equation**

$$\frac{\partial c}{\partial t} = D\nabla^2 c. \tag{E}$$

Equation (E) is also the basic equation of heat conduction, so it is also called the **heat equation**. In this case $c = c(t, x, y, z)$ represents the temperature within a body and D is the heat conductivity of the body, which is assumed to be homogeneous.

Now, following the e.g. (Jones et al. 2010), we are going to describe the diffusion of a solute into a cell. The plasma membrane of the cell consists of a double layer of lipid molecules, as shown in the figure below,

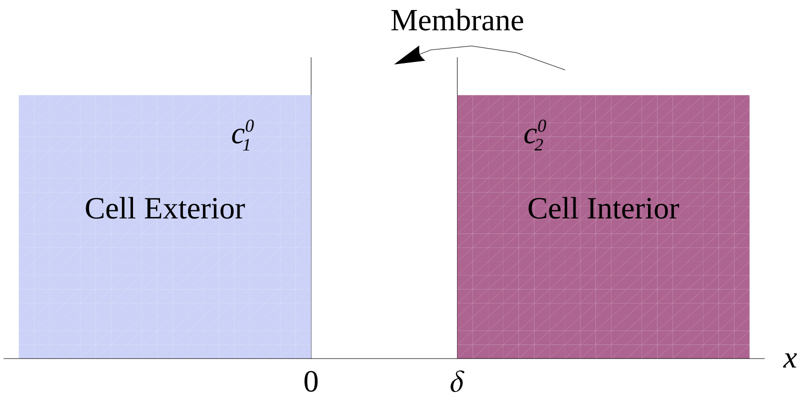

Figure 2.1.2: Idealised cell membrane model.

We consider diffusion through the double lipid layer that has the form of a slab whose thickness is equal to $\delta > 0$. For simplicity, assume that there is no solute in the membrane at the initial time instant., i.e.,

$$c(0, x) = 0, \quad 0 \leq x \leq \delta. \tag{IC1}$$

We also assume

$$c(t, 0) = c_1^0, \quad c(t, \delta) = c_2^0, \tag{BC1}$$

where c_1^0 and c_2^0 are constants, i.e., the concentrations at the two outer walls of the membrane are constant functions. The diffusion equation (E) has the form

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}, \tag{E1}$$

where $c = c(t, x)$, $t \geq 0$, $0 \leq x \leq \delta$.

Therefore the **mathematical model** associated with the diffusion process described above consists of equation (E1), initial condition (IC1) and boundary conditions (BC1).

2.2 Solving the Mathematical Model by the Method of Separation of Variables

The boundary conditions that we denoted by (BC1) are called Dirichlet boundary conditions. In order to solve our problem (E1), (IC1), (BC1), firstly we should homogenize the boundary conditions (BC1) by using the change

$$\begin{aligned}\widetilde{c}(t,x) =& c(t,x) - (1-\frac{x}{\delta})c_1^0 - \frac{x}{\delta}c_2^0 \\ =& c(t,x) + \underbrace{\frac{x}{\delta}(c_1^0 - c_2^0) - c_1^0}_{f(x)}.\end{aligned} \tag{2.2.1}$$

The function $\widetilde{c}$ must satisfy the following equations

$$\frac{\partial \widetilde{c}}{\partial t} = D\frac{\partial^2 \widetilde{c}}{\partial x^2}, \quad 0 < x < \delta, t > 0, \tag{E2}$$

$$\widetilde{c}(0,x) = f(x), \quad 0 \leq x \leq \delta, \tag{IC2}$$

$$\widetilde{c}(t,0) = 0, \quad \widetilde{c}(t,\delta) = 0, \ t > 0. \tag{BC2}$$

The new boundary conditions (BC2) are indeed homogeneous. Instead, a function f occurs in the initial condition, representing the initial distribution of $\widetilde{c}$ in $[0, \delta]$. We seek $\widetilde{c}$ of the form

$$\widetilde{c}(t,x) = u(x)v(t), \tag{2.2.2}$$

i.e., the variables x and t are separated. That is why the method is called the **method of separation of variables**. Substituting (2.2.2) into (E2) yields

$$u(x)v'(t) = Du''(x)v(t),$$

or

$$\frac{u''(x)}{u(x)} = \frac{1}{D}\frac{v'(t)}{v(t)}.$$

Since the left -hand side depends only on x and the right-hand side depends only on t, it is necessary that both sides of the equation above be equal to the same constant, say k, that is

$$u''(x) = ku(x), \tag{2.2.3}$$

$$v'(t) = Dkv(t). \tag{2.2.4}$$

Conditions (BC2) imply

$$u(0) = 0 = u(\delta), \tag{2.2.5}$$

unless $v(t) = 0$ for all t. But this case is excluded because we would obtain $\widetilde{c} \equiv 0$ which is possible only if $f \equiv 0$.

The problem we have to solve is to find those constants $k \in \mathbb{R}$ for which problem (2.2.3), (2.2.5) has no trivial solutions. We can see easily that the only solution of this problem is $u \equiv 0$, for all $k \geq 0$. This means that k must be negative. Let us say $k = -q^2$, where $q > 0$. In this case, the general solution of equation (2.2.3) is given by

$$u(x) = \alpha \cos(qx) + \beta \sin(qx),$$

where $\alpha, \beta \in \mathbb{R}$. From (2.2.5), it is easily seen that $\alpha = 0$ and

$$q = \frac{n\pi}{\delta}, \quad n = 1, 2, \cdots \tag{2.2.6}$$

Consequently

$$u(x) = \beta \sin\left(\frac{n\pi}{\delta}x\right),$$

where $\beta \in \mathbb{R}\backslash\{0\}$.

To complete the solution of our original problem, we have to determine $v = v(t)$. Substituting $k = -q^2 = -\frac{n^2\pi^2}{\delta^2}$ into equation (2.2.4), we obtain

$$v'(t) = -D\frac{n^2\pi^2}{\delta^2}v(t),$$

so that

$$v(t) = \gamma \exp\left\{ - D\frac{n^2\pi^2}{\delta^2}t\right\},$$

where $\gamma \in \mathbb{R}\backslash\{0\}$. Therefore we obtain

$$\widetilde{c}(t,x) = \omega \exp\left\{ - D\frac{n^2\pi^2}{\delta^2}t \right\} \sin\left(\frac{n\pi}{\delta}x\right), \tag{2.2.7}$$

where $\omega \in \mathbb{R}\backslash\{0\}$.

Obviously any $\widetilde{c}$ given by the equation (2.2.7) satisfy (E2) and (BC2). But it does *not* satisfies in general (IC2) because f is not a sin function. The same remark is valid for a finite sum of such functions $\widetilde{c}$. This suggests that we formally extend our search to series

$$\widetilde{c}(t,x) = \sum_{n=1}^{\infty} \gamma_n \exp\left\{ - D\frac{n^2\pi^2}{\delta^2}t \right\} \sin\left(\frac{n\pi}{\delta}x\right).$$

Formally this $\widetilde{c}$ satisfies (E2) and (BC2). In order to satisfy (IC2) we must have

$$\sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n\pi}{\delta}x\right) = f(x),\ 0 \le x \le \delta.$$

Thus γ_n must be the Fourier coefficients of f with respect to the orthogonal system $\{\sin(\frac{n\pi}{\delta}x)\}_{n\in\mathbb{N}}$, i.e.,

$$\gamma_n = \frac{2}{\delta}\int_0^{\delta} f(t) \sin\left(\frac{n\pi}{\delta}t\right) \mathrm{d}t,\ n \in \mathbb{N}.$$

The solution we have provided so far for the mathematical model presented above is a formal one. It is possible to show rigorously that $\widetilde{c}$ is indeed a solution of problem (E2), (BC2) and (IC2), provided that f is square integrable on $[0,\delta]$. Coming back to (2.2.1) we obtain $c(t,x)$.

Comments

In this section, by using the method of separation variables to solve problem (E2), (BC2) and (IC2) we arrived at the problem

$$\begin{cases} u''(x) = ku(x),\ 0 < x < \delta \\ u(0) = 0 = u(\delta). \end{cases} \tag{P}$$

We did show that there exist a sequence of k's, $k_n = -\frac{n^2\pi^2}{\delta^2}$ $(n \in \mathbb{N})$, such that for each n, problem (P) has nontrivial solutions. We also know that $\{u_n\}_{n\in\mathbb{N}}$ is a complete orthogonal system in the space of square integrable functions. This information is useful when we want to expand $f = f(x)$ as a Fourier series.

Usually k_n are called **eigenvalues** and u_n **eigenfunctions** of problem (P). There are many problems similar to (P) that arise when one uses separation of variables to solve problems involving the Laplace operator. Next we will discuss a general class of problems, called Sturm-Liouville problems, including the ones similar to (P). The Sturm-Liouville theory aims to identify common features.

3 FORMULATION OF THE GENERAL STURM-LIOUVILLE PROBLEM

In the previous section we solved a specific mathematical model by using the method of separation of variables. Thus we arrived at the problem

$$\begin{cases} u''(x) = -\lambda u(x), \quad 0 < x < \delta, \\ u(0) = 0 = u(\delta), \end{cases} \tag{3.1}$$

where δ is a positive constant, and λ is a parameter (here we prefer to change k to $-\lambda$). The main question was to find those λ for which problem (3.1) has non-trivial solutions. Such values of λ are called eigenvalues of problem (3.1), and the corresponding non-trivial solutions $u = u(x)$ are the eigenfunctions of this problem. In the following we discuss two more examples.

Example 3.1: Consider the problem

$$\begin{cases} \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \ 0 < x < L, \ t > 0, \\ T(t, 0) = 0 = \frac{\partial T}{\partial x}(t, L), \ t > 0, \\ T(0, x) = f(x), \ 0 \le x \le L. \end{cases} \tag{3.2}$$

This problem is a mathematical model describing the distribution of the heat (or variation in temprature) in a rod of length L. Here $\alpha > 0$ is the thermal diffusivitiy, and $f = f(x)$ is the initial distribution of T. For simplicity, we assume that f is a smooth function. The boundary conditions show that: T is zero at the left endpoint of the rod, and the heat flux is zero at the right endpoint.

Again, we use separation of variables. Thus we seek a solution of the form

$$T(t, x) = u(x)v(t).$$

Inserting this into the above equation, we get

$$u(x)v'(t) = \alpha u''(x)v(t)$$

or

$$\frac{u''(x)}{u(x)} = \frac{1}{\alpha}\frac{v'(t)}{v(t)}.$$

We then have the equations

$$u''(x) = -\lambda u(x) \tag{3.3}$$

and

$$v'(t) = -\alpha\lambda v(t).$$

Here λ is a real parameter. As we will see later the case $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is excluded. From the boundary conditions we easily derive

$$u(0) = 0 = u'(L), \tag{3.4}$$

since we are looking for non-trivial solutions of problem (3.2). We are looking for those $\lambda \in \mathbb{R}$ for which there exist non-trivial solutions of the problem

$$\begin{cases} u''(x) = -\lambda u(x), \; 0 < x < L, \\ u(0) = 0 = u'(L). \end{cases} \tag{3.5}$$

This problem is similar to the one we solved in the preceding section, except for the new condition $u'(L) = 0$. It is easily seen that $\lambda \leq 0$ can not be eigenvalues of problem (3.5). If $\lambda > 0$, i.e., $\lambda = \mu^2$ with $\mu > 0$, then imposing to the general solution of the above differential equation,

$$u(x) = A\cos(\mu x) + B\sin(\mu x),$$

to satisfy the boundary conditions,

$$u(0) = 0 = u'(L),$$

we obtain

$$A = 0 \text{ and } \cos(\mu L) = 0.$$

Therefore, we have the following eigenvalues

$$\lambda = \lambda_n = \frac{(2n+1)^2\pi^2}{4L^2}, \; n = 0, 1, \cdots$$

and the corresponding eigenfunctions

$$u_n(x) = \sin\left(\frac{(2n+1)\pi}{2L}x\right), \; n = 0, 1, \cdots$$

Using these eigenvalues and eigenfunctions, one can proceed as in the previous section to find the solution of problem (3.2) in the form of a series expansion.

Example 3.2: Consider the problem

$$\begin{cases} \rho(x)\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial}{\partial x}\left[p(x)\dfrac{\partial y}{\partial x}\right],\ 0 < x < L,\ t > 0, \\ y(t,0) = 0,\ y(t,L) = 0,\ t > 0, \\ u(0,x) = f(x),\ \dfrac{\partial y}{\partial t}(0,x) = g(x),\ 0 < x < L, \end{cases} \tag{3.6}$$

which describes the vibrations of a string of length L, with space-dependent tension $p(x)$ and variable density $\rho(x)$, with fixed end-points (i.e., the displacement $y = y(t,x)$ is null at $x = 0$ and $x = L$). The initial conditions show the initial position of the string and its initial velocity at point x.

We are looking for solutions of the form

$$y(t,x) = v(t)u(x).$$

Plugging this into the above partial differential equation it follows that

$$\rho(x)v''(t)u(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)u'(x)\right]v(t),$$

or

$$\frac{v''(t)}{v(t)} = \frac{1}{\rho(x)u(x)}\left(p(x)u'(x)\right)' = -\lambda.$$

Therefore

$$v''(t) = -\lambda v(t),\ t > 0,$$

and

$$\left(p(x)u'(x)\right)' = -\lambda\rho(x)u(x). \tag{3.7}$$

We also have

$$u(0) = 0 = u(L). \tag{3.8}$$

In order to solve problem (3.6) we try to find those λ for which the problem (3.7), (3.8) has non-trivial solutions.

Now, having in mind the above examples, let us formulate the general Sturm-Liouville problem

$$\left(p(x)u'(x)\right)' + q(x)u(x) = -\lambda\rho(x)u(x), \quad a \le x \le b, \tag{E3}$$

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0,\ \beta_1 u(b) + \beta_2 u'(b) = 0, \tag{BC3}$$

where $-\infty < a < b < +\infty$.

In Example 3.2 (see (3.1)), we had

$$a = 0,\ b = \delta,\ p(x) \equiv 1,\ q(x) \equiv 0,\ \rho(x) \equiv 1,$$
$$\alpha_1 = 1,\ \alpha_2 = 0,\ \beta_1 = 1,\ \beta_2 = 0.$$

In Example 3.1 above we had (see equations (3.3) and (3.4))

$$a = 0,\ b = L,\ p(x) \equiv 1,\ q(x) \equiv 0,\ \rho(x) \equiv 1,$$
$$\alpha_1 = 1,\ \alpha_2 = 0,\ \beta_1 = 0,\ \beta_2 = 1.$$

In Example 3.2 (see equations (3.7) and (3.8)),

$$a = 0,\ b = L,\ q(x) \equiv 0,$$
$$\alpha_1 = \beta_1 = 1, \quad \alpha_2 = \beta_2 = 0.$$

Throughout in the following we assume that

$$p \in C^1[a, b], p(x) > 0 \text{ for all } x \in [a, b];$$
$$q, \rho \in C[a, b], \rho(x) > 0 \text{ for all } x \in [a, b];$$
$$\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}, \alpha_1^2 + \alpha_2^2 \neq 0, \beta_1^2 + \beta_2^2 \neq 0.$$

Remark 3.3: Consider the general second-order differential equation

$$r(x)u''(x) + s(x)u'(x) + w(x)u(x) = -\lambda\mu(x)u(x),\ a \leq x \leq b, \tag{3.9}$$

where $r, s, w, \mu \in C[a, b]$, $r(x) > 0$ for all $x \in [a, b]$.

We divide the equation by $r(x)$ and then multiply the resulting equation by

$$p(x) = \exp \int_a^x \frac{s(\tau)}{r(\tau)} d\tau$$

to obtain

$$(p(x)u'(x))' + q(x)u(x) = -\lambda\rho(x)u(x),$$

where

$$q(x) = \frac{p(x)w(x)}{r(x)}, \quad \rho(x) = \frac{p(x)\mu(x)}{r(x)}.$$

Therefore, every general second-order differential equation of the form (3.9) can be written in the form (3.1), which is more convenient in what follows. For example, the well-known Hermite equation

$$u''(x) - 2xu'(x) + \lambda u(x) = 0$$

can be written as

$$\left(e^{-x^2}u'(x)\right)' = -\lambda e^{-x^2}u(x).$$

Remark 3.4: There are cases in which the conditions $p > 0$, $\rho > 0$ are not satisfied. Such Sturm-Liouville problems are called singular. We shall investigate only regular Sturm-Liouville problems (i.e., (E3) and (BC3), with $p, \rho > 0$).

4 MAIN RESULTS ON STURM-LIOUVILLE PROBLEM

In the previous section, we discussed a few examples and obtained the formulation of the general Sturm-Liouville problem. Now, we will give some properties related to the eigenvalues and the eigenfunctions of our regular Sturm-Liouville problem (defined by (E3) and (BC3)). Then, we will introduce Lagrange and Green's identity. Thus, we will be able to prove some of properties by using these important identities.

4.1 Properties of Sturm-Liouville Eigenvalue Problems

There are several properties that can be proven for the regular Sturm-Liouville problem. However, we will not prove them all in this thesis. Following the e.g. (Int.Ref.1) we will merely list some of the important facts and focus on a few of the properties.

1. All eigenvalues λ are real.

2. There is an infinite number of eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_{n+1} < \cdots$$

   There is a smallest eigenvalue λ_1 but no largest eigenvalue: $\lambda_n \to \infty$ as $n \to \infty$.

3. Corresponding to each eigenvalue λ_n there is an eigenfunction say $u_n(x)$ which is unique up to an arbitrary multiplicative constant. $u_n(x)$ has $(n-1)$ zeros for $x \in (a, b)$.

4. The eigenfunctions form is a *complete set.* For details see Section 4.8 below.

5. Eigenfunctions associated with distinct eigenvalues are orthogonal relative to the weight function $\rho(x)$. I.e., if $\lambda_m \neq \lambda_n$ $(m \neq n)$

$$\int_a^b \rho(x)\overline{\phi_m(x)}\phi_n(x)\mathrm{d}x = 0.$$

Example 4.1.1: We want to solve the eigenvalue problem

$$\mathcal{L}u = (xu')' + \frac{2}{x}u = -\lambda\rho u$$

subject to a set of boundary conditions. Let us use the boundary conditions

$$u'(1) = 0,\ u'(2) = 0.$$

[Note that we do not know $\rho(x)$ yet, but will choose an appropriate function to obtain solutions.]

Expanding the derivative, we have

$$xu'' + u' + \frac{2}{x}u = -\lambda\rho u.$$

Multiply this equation by x to obtain

$$x^2u'' + xu' + (2 + \lambda x\rho)u = 0.$$

Notice that if we choose $\rho(x) = x^{-1}$, then this equation can be made a Cauchy-Euler type equation. Thus, we have

$$x^2u'' + xu' + (2 + \lambda)u = 0.$$

The characteristic equation is

$$r^2 + \lambda + 2 = 0.$$

We know that all eigenvalues are real numbers. It is easy to see that for $\lambda < -2$ the only solution is $u \equiv 0$. For $\lambda = -2$, the problem has the non-trivial solution $u \equiv 1$. Now, if $\lambda > -2$ the general solution is

$$u(x) = c_1 \cos(\sqrt{\lambda + 2}\ln|x|) + c_2 \sin(\sqrt{\lambda + 2}\ln|x|).$$

Next we apply the boundary conditions. $u'(1) = 0$ forces $c_2 = 0$. So that, we arrive at

$$u(x) = c_1 \cos(\sqrt{\lambda + 2}\ln|x|).$$

The second condition, $u'(2) = 0$, yields

$$\sin(\sqrt{\lambda + 2}\ln 2) = 0.$$

This will give nontrivial solutions when

$$\sqrt{\lambda+2}\ln 2 = n\pi, \quad n = 1, 2, 3, \ldots$$

In summary, the eigenvalues are

$$\lambda_n = 2 + \left(\frac{n\pi}{\ln 2}\right)^2 \text{ for } n = 0, 1, 2, \ldots$$

and the eigenfunctions for this eigenvalue problem are

$$u_n(x) = \cos\left(\frac{n\pi}{\ln 2}\ln x\right), \quad 1 \le x \le 2$$

We note that some of the properties listed in the beginning of the section hold for this example. The eigenvalues are seen to be real, countable and $\lambda \to \infty$. Next, one can find the zeros of each eigenfunction on $[1, 2]$. Then the argument of the cosine, $\frac{n\pi}{\ln 2}\ln x$ takes values 0 to $n\pi$ for $x \in [1, 2]$. The cosine function has $(n-1)$ roots on this interval.

4.2 Lagrange's and Green's Identities

Before turning to the proofs that the eigenvalues of a Sturm-Liouville problem are real and the associated eigenfunctions orthogonal, we will first need to introduce two important identities. For the Sturm-Liouville operator, following the e.g. (Int.Ref.1),

$$\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}}{\mathrm{d}x}\right) + q, \tag{4.2.1}$$

we have two identities:

Lagrange's Identity: $u\mathcal{L}v - v\mathcal{L}u = [p(uv' - vu')]'$;
Green's Identity: $\int_a^b (u\mathcal{L}v - v\mathcal{L}u) = [p(uv' - vu')]|_a^b$.

Proof: The proof of Lagrange's identity follows by a simple manipulation of the operator:

$$\begin{aligned} u\mathcal{L}v - v\mathcal{L}u =& u\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}v}{\mathrm{d}x}\right) + qv\right] - v\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}u}{\mathrm{d}x} + qu\right)\right] \\ =& u\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}v}{\mathrm{d}x}\right) - v\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}v}{\mathrm{d}x}\right) \\ =& u\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}v}{\mathrm{d}x}\right) + p\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x} - v\frac{\mathrm{d}}{\mathrm{d}x}\left(p\frac{\mathrm{d}v}{\mathrm{d}x}\right) - p\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x} \\ =& \frac{\mathrm{d}}{\mathrm{d}x}\left[pu\frac{\mathrm{d}v}{\mathrm{d}x} - pv\frac{\mathrm{d}u}{\mathrm{d}x}\right]. \end{aligned}$$

Green's identity is simply proven by integrating Lagrange's identity. □

4.3 Orthogonality and Reality

We are now ready to prove that the eigenvalues of a Sturm-Liouville problem are real and the corresponding eigenfunctions are orthogonal. These are easily established using Green's identity.

Theorem 4.3.1: *The eigenvalues of the Sturm-Liouville problem are real. (Int.Ref.1)*

Proof: Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n:

$$\mathcal{L}\phi_n(x) = -\lambda_n \rho \phi_n.$$

The complex conjugate of this equation is

$$\mathcal{L}\overline{\phi_n} = -\overline{\lambda}_n \rho \overline{\phi}_n.$$

Now, multiply the first equation by $\overline{\phi_n}$ and the second equation by ϕ_n and then subtract the results. We obtain

$$\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi}_n = (\overline{\lambda}_n - \lambda_n)\rho\phi_n\overline{\phi}_n.$$

Integrate both sides of this equation:

$$\int_a^b \left(\overline{\phi}_n \mathcal{L}\phi_n - \phi_n \mathcal{L}\overline{\phi_n}\right) \mathrm{d}x = (\overline{\lambda}_n - \lambda_n) \int_a^b \rho\phi_n\overline{\phi}_n \mathrm{d}x.$$

Apply Green's identity to the left hand side to find

$$[p(\overline{\phi}_n \phi' - \phi_n \overline{\phi}_n^{\prime})]|_a^b = (\overline{\lambda}_n - \lambda_n) \int_a^b \rho\phi_n\overline{\phi}_n \mathrm{d}x.$$

Using the homogeneous boundary conditions for a self-adjoint operator (see (BC) above), the left side vanishes to give

$$0 = (\overline{\lambda}_n - \lambda_n) \int_a^b \rho \left|\phi_n\right|^2 \mathrm{d}x.$$

The integral is positive, so we must have $\overline{\lambda}_n = \lambda_n$. Therefore, the eigenvalues are real. □

Remark 4.3.2: Obviously, the eigenfunctions ϕ_n corresponding to λ_n can be chosen to be real-valued functions.

Theorem 4.3.3: *The eigenfunctions corresponding to distinct eigenvalues of the Sturm-Liouville problem are orthogonal.*

Proof: We can prove this similar to the previous theorem. Let $\phi_n(x)$ be a solution of the eigenvalue problem associated with λ_n,

$$\mathcal{L}\phi_n = -\lambda_n \rho \phi_n,$$

and let $\phi_m(x)$ be a solution of the eigenvalue problem associated with $\lambda_m \neq \lambda_n$,

$$\mathcal{L}\phi_m = -\lambda_m \rho \phi_m.$$

Now, multiply the first equation by ϕ_m and the second equation by ϕ_n Subtracting the results, we obtain

$$\phi_m \mathcal{L}\phi_n - \phi_n \mathcal{L}\phi_m = (\lambda_m - \lambda_n)\rho \phi_n \phi_m.$$

Similar to the previous prooof, we integrate both sides of the equation and use Green's identity and the boundary conditions for a self-adjoint operator. This leaves

$$0 = (\lambda_m - \lambda_n) \int_a^b \rho \phi_n \phi_m \mathrm{d}x.$$

Since the eigenvalues are distinct, we can divide by $\lambda_m - \lambda_n$, so that we arrive at

$$0 = \int_a^b \rho \phi_n \phi_m \mathrm{d}x.$$

Therefore, the eigenfunctions are orthogonal with respect to the weight function $\rho(x)$. □

Now, let us give two more important theorems.

Theorem 4.3.4: *The eigenvalues of the regular Sturm-Liouville problem are simple, i.e., if λ is an eigenvalue of the regular Sturm-Liouville problem and $\phi_1(x)$ and $\phi_2(x)$ are the corresponding eigenfunctions, then $\phi_1(x)$ and $\phi_2(x)$ are linearly dependent.*

Proof: Since $\phi_1(x)$ and $\phi_2(x)$ both are solutions of (E), we have

$$(p(x)\phi_1')' + q(x)\phi_1 + \lambda\rho(x)\phi_1 = 0 \tag{4.3.1}$$

and

$$(p(x)\phi_2')' + q(x)\phi_2 + \lambda\rho(x)\phi_2 = 0. \tag{4.3.2}$$

Multiplying (4.3.1) by ϕ_2, and (4.3.2) by ϕ_1 and subtracting, we get

$$\phi_2\left(p(x)\phi_1'\right)' - \left(p(x)\phi_2'\right)'\phi_1 = 0. \tag{4.3.3}$$

since

$$\begin{aligned}\left[\phi_2\left(p(x)\phi_1'\right) - \left(p(x)\phi_2'\right)\phi_1\right]' &= \phi_2\left(p(x)\phi_1'\right)' + \phi_2'\left(p(x)\phi_1'\right) - \left(p(x)\phi_2'\right)'\phi_1 - \left(p(x)\phi_2'\right)\phi_1' \\ &= \phi_2\left(p(x)\phi_1'\right)' - \left(p(x)\phi_2'\right)'\phi_1\end{aligned}$$

from (4.3.3) it follows that

$$\left[\phi_2\left(p(x)\phi_1'\right) - \left(p(x)\phi_2'\right)\phi_1\right]' = 0$$

and hence

$$p(x)\left[\phi_2\phi_1' - \phi_2'\phi_1\right] = \text{constant} = C. \tag{4.3.4}$$

To find the value of C, we note that ϕ_1 and ϕ_2 satisfy the boundary conditions (BC), and hence

$$\begin{aligned}\alpha_1\phi_1(a) + \alpha_2\phi_1'(a) &= 0 \\ \alpha_1\phi_2(a) + \alpha_2\phi_2'(a) &= 0.\end{aligned}$$

which implies

$$\phi_1(a)\phi_2'(a) - \phi_2(a)\phi_1'(a) = 0.$$

Thus, from (4.3.4) it follows that

$$p(x)\left[\phi_2\phi_1' - \phi_2'\phi_1\right] = 0 \quad \text{for all } x \in [a,b].$$

Since $p(x) > 0$, we must have $\phi_2\phi_1' - \phi_2'\phi_1 = 0$ for all $x \in [a, b]$. But, this means that ϕ_1 and ϕ_2 are linearly dependent. $\square$

Corollary 4.3.5: *Let λ_1 and λ_2 be two eigenvalues of the regular Sturm-Liouville problem* (E3)*,* (BC3) *and $\phi_1(x)$ and $\phi_2(x)$ be the corresponding eigenfunctions. Then, $\phi_1(x)$ and $\phi_2(x)$ are linearly dependent if and only if $\lambda_1 = \lambda_2$.*

Theorem 4.3.6: *For the regular Sturm-Liouville problem* (E3) *there exists an infinite number of eigenvalues λ_n, $n = 1, 2, \ldots$. These eigenvalues can be arranged as a monotonically increasing sequence $\lambda_1 < \lambda_2 < \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$. Further, eigenfunction $\phi_n(x)$ corresponding to the eigenvalue λ_n has exactly $(n-1)$ zeros in the open interval (a, b).*

4.4 The Eigenfunction Expansion Method

Followinf the e.g. (Int.Ref.1, Int.Ref.2) Let us consider a differential equation

$$\mathcal{L}u = f,$$

where $u(x)$ satisfies given homogenous boundary conditions (see (BC3)). The method makes use of the eigenfunctions satisfying the eigenvalue problem

$$\mathcal{L}\phi_n = -\lambda_n \rho \phi_n$$

subject to the given boundary conditions. Then, one assumes that $u(x)$ can be written as an expansion in the eigenfunctions,

$$u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

and inserts the expansion into the nonhomogeneous equation. This gives

$$f(x) = \mathcal{L}\left(\sum_{n=1}^{\infty} c_n \phi_n(x)\right) = -\sum_{n=1}^{\infty} c_n \lambda_n \rho(x) \phi_n(x).$$

The expansion coefficients are found by making use of the orthogonality of the eigenfunctions. Namely, we multiply the last equation by $\phi_m(x)$ and integrate. We obtain

$$\int_a^b f(x)\phi_m(x)\mathrm{d}x = -\sum_{n=1}^{\infty} c_n \lambda_n \int_a^b \phi_n(x)\phi_m(x)\rho(x)\mathrm{d}x.$$

Orthogonality yields

$$\int_a^b f(x)\phi_m(x)\mathrm{d}x = -c_m\lambda_m \int_a^b \phi_m^2(x)\rho(x)\mathrm{d}x.$$

Solving for c_m, we have

$$c_m = -\frac{\int_a^b f(x)\phi_m(x)\mathrm{d}x}{\lambda_m \int_a^b \phi_m^2(x)\rho(x)\mathrm{d}x}.$$

Example 4.4.1: As an example, we consider the solution of the boundary value problem

$$(xu')' + \frac{u}{x} = \frac{1}{x}, \ x \in [1, \mathrm{e}],$$
$$u(1) = 0 = u(\mathrm{e}).$$

This equation is already in self-adjoint form. So, we know that the associated Sturm-Liouville eigenvalue problem has an orthogonal set of eigenfunctions. We first determine this set. Namely, we need to solve

$$(x\phi')' + \frac{\phi}{x} = -\lambda\rho\phi, \quad \phi(1) = 0 = \phi(\mathrm{e}).$$

Rearranging the terms and multiplying by x, we have that

$$x^2\phi'' + x\phi' + (1 + \lambda\rho x)\phi = 0.$$

This is almost an equation of Cauchy-Euler type. By taking the weight function $\rho(x) = \frac{1}{x}$, we have

$$x^2\phi'' + x\phi' + (1 + \lambda)\phi = 0.$$

This is easy to solve. The characteristic equation is

$$r^2 + (1 + \lambda) = 0.$$

One obtains nontrivial solutions of the eigenvalue problem satisfying the boundary conditions when $\lambda > -1$. The solutions are

$$\phi_n(x) = A\sin(n\pi \ln x), \ n = 1, 2, \cdots$$

where $\lambda_n = n^2\pi^2 - 1$.

It is often useful to normalize the eigenfunctions. This means that one chooses A so that the norm of each eigenfunction is one. Thus, we have

$$\begin{aligned} 1 &= \int_1^e \phi_n^2(x)\rho(x)\mathrm{d}x \\ &= A^2 \int_1^e \sin(n\pi \ln x)\frac{1}{x}\mathrm{d}x \\ &= A^2 \int_0^1 \sin(n\pi u)\mathrm{d}u \\ &= \frac{1}{2}A^2. \end{aligned}$$

Thus $A = \sqrt{2}$.

We now turn towards solving the nonhomogeneous problem, $\mathcal{L}u = \frac{1}{x}$. We first expand the unknown solution in terms of the eigenfunctions,

$$u(x) = \sum_{n=1}^{\infty} c_n \sqrt{2} \sin(n\pi \ln x).$$

Inserting the solution into the differential equation, we have

$$\frac{1}{x} = \mathcal{L}u = -\sum_{n=1}^{\infty} c_n \lambda_n \sqrt{2} \sin(n\pi \ln x)\frac{1}{x}.$$

Next, we make use of orthogonality. Multiplying both sides by $\phi_m(x) = \sqrt{2}\sin(m\pi \ln x)$ and integrating, gives

$$\lambda_m c_m = \int_1^e \sqrt{2}\sin(m\pi \ln x)\frac{1}{x}\mathrm{d}x = \frac{\sqrt{2}}{m\pi}\left[(-1)^m - 1\right].$$

Solving for c_m, we have

$$c_m = \frac{\sqrt{2}}{m\pi}\frac{\left[(-1)^m - 1\right]}{m^2\pi^2 - 1}.$$

Finally, we insert our coefficients into the expansion for $u(x)$.The solution is then

$$u(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\frac{\left[(-1)^n - 1\right]}{n^2\pi^2 - 1}\sin(m\pi \ln(x)).$$

Remark 4.4.2: The eigenfunction expansion method is useful to solve particular nonhomogenous boundary value problems.

4.5 Boundary Value Problem and Green's Function

Firstly, following the e.g. (Int.Ref.2) let us consider a simple problem here in order to motivate the idea of a Green's function. Consider the equation

$$u''(x) = f(x),$$

where $x \in [0, L]$ and the homogeneous boundary conditions $u(0) = 0 = u(L)$. This problem is in fact the *steady state heat equation, i.e., the heat* equation without any time dependence.

In order to solve this problem, we note that the complementary function satisfies

$$u_c''(x) = 0,$$

and if we integrate the problem, we obtain 1 and x as fundamental solutions. Now, we choose linear combinations, to obtain

$$u_1(x) = x, \quad u_2(x) = L - x,$$

satisfying the left and right boundary condition respectively.

By using the method of variation of parameters, since $W = u_1 u_2^{'} - u_1^{'} u_2 = x(-1) - (1)(L - x) = -L$, we find that

$$\begin{aligned} v_1(x) &= \frac{1}{L} \int_0^x f(x_0)(L - x_0) \mathrm{d}x_0, \\ v_2(x) &= -\frac{1}{L} \int_0^x f(x_0) x_0 \mathrm{d}x_0. \end{aligned}$$

Therefore we obtain this general solution

$$u(x) = (c_1 + v_1(x))x + (c_2 + v_2(x))(L - x).$$

Now, let us apply the boundary conditions. Setting $x = 0$ means that $c_2 = 0$ and for $x = L$ we find

$$0 = (c_1 + v_1(L))L.$$

So that $c_1 = -v_1(L)$. Then we note that

$$\begin{aligned} c_1 + v_1(x) &= -v_1(L) + v_1(x) \\ &= -\frac{1}{L} \int_0^L f(x_0)(L - x_0) \mathrm{d}x_0 + \frac{1}{L} \int_0^x f(x_0)(L - x_0) \mathrm{d}x_0 \\ &= -\frac{1}{L} \int_x^L f(x_0)(L - x_0) \mathrm{d}x_0. \end{aligned}$$

We can write

$$u(x) = \frac{x}{L}\int_x^L (x_0 - L) f(x_0)\mathrm{d}x_0 + \frac{(x-L)}{L}\int_0^x x_0 f(x_0)\mathrm{d}x_0.$$

Finally this means we can write the solution in the form

$$u(x) = \int_0^L G(x, x_0) f(x_0)\mathrm{d}x_0,$$

where

$$G(x, x_0) = \begin{cases} \frac{x_0}{L}(x - L), & 0 \leq x_0 \leq x, \\ \frac{x}{L}(x_0 - L), & x \leq x_0 \leq L. \end{cases}$$

Now, let us give a few important theorems related to boundary value problem and Green's function.

Theorem 4.5.1: *If*

$$\mathcal{L}u = 0, \quad a < x < b,$$

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \beta_1 u(b) + \beta_2 u'(b) = 0,$$

has only the trivial solution, then the Green's function $G(x, x_0)$ exists and is uniqe. (Lo 2000)

Theorem 4.5.2: *Let x_0 be fixed. Define*

$$G(x, x_0) = \begin{cases} cAu_1(x), & x \leq x_0, \\ Bu_2(x), & x \geq x_0, \end{cases}$$

where A and B are constants to be determined. The unique solution for the nonhomogeneous problem

$$\mathcal{L}u = -f, \quad a < x < b,$$

$$\alpha_1 u(a) + \alpha_2 u'(a) = m, \quad \beta_1 u(b) + \beta_2 u'(b) = n,$$

is given by

$$u(x) = \int_a^b G(x, x_0) f(x_0)\mathrm{d}x_0 + k_1 u_1(x) + k_2 u_2(x),$$

where

$$k_1 = \frac{n}{\beta_1 u_1(b) + \beta_2 u_1'(b)} \quad \text{and} \quad k_2 = \frac{m}{\alpha_1 u_2(a) + \alpha_2 u_2'(a)}.$$

Now, let us give some properties of the Green's function (Int.Ref.4).

1. **Differential Equation:**

$$(p(x)G'(x,x_0))' + q(x)G(x,x_0) = 0, \quad x \neq x_0$$

   For $x < x_0$ we are on the second branch and $G(x,x_0)$ is proportional to $u_1(x)$. Thus, since $u_1(x)$) is a solution of the homogeneous equation, then so is $G(x,x_0)$. For $x > x_0$ we are on the first branch and $G(x,x_0)$ is proportional to $u_2(x)$. So, once again $G(x,x_0)$ is a solution of the homogeneous problem.

2. **Boundary Conditions:** For $x = a$ we are on the second branch and $G(x,x_0)$ is proportional to $u_1(x)$.Thus, whatever condition $u_1(x)$ satisfies, $G(x,x_0)$ will satisfy. A similar statement can be made for $x = b$.

3. **Symmetry or Reciprocity:** $G(x,x_0) = G(x_0,x)$

4.6 Green's Functions for Regular Sturm-Liouville Problems via Eigenfunction Expansions

Remark 4.6.1: For the problem $u'' = -\lambda u$ with $u(0) = 0 = u(L)$ we have
Eigenvalues: $\frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2}, \ldots$

Eigenfunctions: $\sin\left(\frac{\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \ldots$
The eigenfunction expansion of a function $f : [0,L] \to \mathbb{R}$ is

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right), \quad c_n = \frac{\langle f, \sin(n\pi x/L)\rangle}{\langle \sin(n\pi x/L), \sin(n\pi x/L)\rangle} = \frac{\int_0^L f(x)\sin(n\pi x/L)\mathrm{d}x}{\int_0^L \sin^2(n\pi x/L)\mathrm{d}x}.$$

Let us compute the denominator of c_n. We have

$$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{1}{2}\int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)\mathrm{d}x = \frac{L}{2}.$$

Therefore the eigenfunction expansion of $f(x)$ is

$$\sum_{n=1}^{\infty} c_n \sin\sin\left(\frac{n\pi x}{L}\right), \; c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

This particular eigenfunction expansion of $f(x) : [0,L] \to \mathbb{R}$ is called the **Fourier sine series**.

Now, following the e.g. (Int.Ref.2) let us consider again the regular Sturm-Liouville problem of the form

$$\mathcal{L}u = f(x), \tag{4.6.1}$$

where L given by (4.2.1), $x \in [a, b]$. Also consider the related eigenvalue problem

$$\mathcal{L}u = -\lambda \rho(x) u,$$

with some appropriately chosen $\rho(x)$. We can solve (4.6.1) by using an eigenfunction expansion of the form

$$u(x) = \sum_{n=1}^{\infty} c_n \phi_n(x).$$

This can be differentiated term-by-term. So that, applying $\mathcal{L}$ we find

$$\mathcal{L}u(x) = -\sum_{n=1}^{\infty} c_n \lambda_n \rho(x) \phi_n(x) = f(x).$$

Let us multiply by $\phi_m(x)$ and integrate over the domain $x \in [a, b]$. The orthogonality of the eigenfunctions (with respect to the weight $\rho(x)$) allows us to then show that

$$-c_n \lambda_n = \frac{\int_a^b f(x)\phi_n(x)\mathrm{d}x}{\int_a^b \phi_n^2(x)\rho(x)\mathrm{d}x}.$$

Therefore

$$u(x) = \int_a^b f(x_0) \sum_{n=1}^{\infty} \left(\frac{-\phi_n(x)\phi_n(x_0)}{\lambda_n \int_a^b \phi_n^2 \rho(x_1)\mathrm{d}x_1} \right) \mathrm{d}x_0$$

and so we recognize that we can write

$$u(x) = \int_a^b f(x_0) G(x, x_0) \mathrm{d}x_0,$$

where

$$G(x, x_0) = \sum_{n=1}^{\infty} \left(\frac{-\phi_n(x)\phi_n(x_0)}{\lambda_n \int_a^b \phi_n^2 \rho(x_1)\mathrm{d}x_1} \right)$$

which is therefore an eigenfunction expansion of the Green's function.

Example 4.6.2: We shall find the Fourier sine series of the function

$$f(x) = \begin{cases} 2, & 0 < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi. \end{cases}$$

For $n \geq 1$

$$
\begin{aligned}
c_n &= \frac{2}{\pi} \int_0^\pi f(t)\sin(nt)\mathrm{d}t \\
&= \frac{2}{\pi} \int_0^{\pi/2} 2\sin(nt)\mathrm{d}t \\
&= \frac{4}{\pi} \left[\frac{-\cos(n\pi/2) + 1}{n} \right].
\end{aligned}
$$

Thus

$$
f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi} \left[\frac{-\cos(n\pi/2) + 1}{n} \right] \sin nx, \quad 0 < x < \pi.
$$

Example 4.6.3: Consider

$$
\mathcal{L}u = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = f(x)
$$

with $u(0) = u(\delta) = 0$ and the related eigenvalue problem

$$
\frac{\mathrm{d}^2 \phi}{\mathrm{d}x^2} = -\lambda \phi
$$

with $\phi(0) = \phi(\delta) = 0$. We already know from Subsection 2.2 that (here we just prefer to write ϕ_n instead of u_n) $\lambda_n = \left(\frac{n\pi}{\delta}\right)^2$ and $\phi_n(x) = \sin\left(\frac{n\pi x}{\delta}\right)$ with $n = 1, 2, 3, \cdots$. Therefore, $u(x)$ is given by

$$
\begin{aligned}
u(x) &= \sum_{n=1}^{\infty} c_n \phi_n(x), \\
&= \int_o^\delta f(x_0) G(x, x_0) \mathrm{d}x_0,
\end{aligned}
$$

where

$$
G(x, x_0) = -\frac{2}{\delta} \sum_{n=1}^{\infty} \frac{\sin(n\pi x/\delta)\sin(n\pi x_0)}{(n\pi/\delta)^2}.
$$

4.7 Convergence in the Mean

Following the e.g. (Lo 2000), let S be the space of all square integrable functions with positive weight function $\rho(x)$ in $[a, b]$. For any f and g in S, we define their inner product by

$$
(f, g) = \int_a^b \rho(x) f(x) \overline{g(x)} \mathrm{d}x.
$$

We know that the inner product has the following properties

$$(mf + ng, h) = m(f, h) + n(g, h),$$

$$(g, f) = \overline{(f, g)},$$

where m and n are arbitrary numbers.

Two nonzero functions f and g in S are orthogonal if $(f, g) = 0$. For each f which is in S, we define the norm of f as

$$\|f\| = \left(\int_a^b \rho(x) \, |f(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

It follows that

$$\|f\| \geq 0; \quad \|f\| = 0 \Leftrightarrow f = 0.$$

$$\|mf\| = |m| \, \|f\|$$

for any number m. Besides that we have two inequalities:

Schwarz's inequality: $|(f, g)| \leq \|f\| \, \|g\|$.

Triangle inequality: $\|f + g\| \leq \|f\| + \|g\|$.

Let $\{\phi_n(x)\}$ be a sequence of functions in S. The series $\sum_{k=1}^{\infty} a_k \phi_k(x)$ converges in the mean to a function $f \in S$ if for $\varepsilon > 0$, there is an integer $N = N(\varepsilon)$ such that for $n \geq N$,

$$\int_a^b \rho(x) \, |S_n(x) - f(x)|^2 \, \mathrm{d}x < \varepsilon,$$

where $S_n(x) = \sum_{k=1}^{n} a_k \phi_k(x)$.

Now, let $F = \{\phi_n(x)\}$ be a family of orthogonal functions on $[a, b]$ and let $f \in S$. We have

$$\begin{aligned} E_n &= \|f - S_n\|^2 \\ &= \|f\|^2 - \sum_{k=1}^{n} a_k(\phi_k, f) - \sum_{k=1}^{n} \overline{a}_k(f, \phi_k) + \sum_{k=1}^{n} |a_k|^2 \, \|\phi_k\|^2. \end{aligned}$$

Set

$$c_k = \frac{(f, \phi_k)}{\|\phi_k\|^2},$$

and we obtain

$$E_n = \|f\|^2 + \sum_{k=1}^{n} |a_k - c_k|^2 \|\phi_k\|^2 - \sum_{k=1}^{n} |c_k|^2 \|\phi_k\|^2 .$$

Since $En \geq 0$, the minimum of E_n is achieved if we choose $a_k = c_k$. Hence

$$\min E_n = \|f\|^2 - \sum_{k=1}^{n} |c_k|^2 \|\phi_k\|^2 . \tag{4.7.1}$$

The coefficients $\{c_k\}$ are called the Fourier coefficients of f with respect to F and the series $\sum_{k=1}^{\infty} c_k \phi_k(x)$ is called Fourier series of f with respect to F. From (4.7.1), we have

$$\|f\|^2 \geq \sum_{k=1}^{n} |c_k|^2 \|\phi_k\|^2 .$$

Definition 4.7.1: A family $F = \{\phi_n(x)\}$ of orthogonal functions is complete in S if, for any f in S,

$$\|f\|^2 = \sum_{k=1}^{\infty} |c_k|^2 \|\phi_k\|^2 . \tag{4.7.2}$$

(4.7.2) is called Parseval's equation.

Theorem 4.7.2: *Let $\{\phi_n(x)\}$ be a family of orthogonal functions on $[a, b]$. Then the Fourier series of $f \in S$ converges in the mean to f if and only if $\{\phi_n(x)\}$ is complete.*

Proof: We note that

$$\left\| f - \sum_{k=1}^{n} c_k \phi_k \right\|^2 = \|f\|^2 - \sum_{k=1}^{n} |c_k|^2 \|\phi_k\|^2.$$

The theorem is proved if we let $n \to \infty$ in the above identitiy. □

Remark 4.7.3: If $\{\phi_n(x)\}$ is an orthogonal set, i.e., $\{\phi_j, \phi_k\} = \delta_{jk}$, then $c_k = (f, \phi_k)$ and the completeness relation is

$$\|f\|^2 = \sum_{k=1}^{\infty} |c_k|^2 . \tag{4.7.3}$$

Theorem 4.7.4: *Let $\{\phi_n(x)\}$ be a complete orthonormal set on $[a, b]$. Then any continious function f in S such that $(f, \phi_n) = 0$ for all n must be identically zero.*

Proof: From (4.7.3), we have $\|f\|^2 = 0$. Hence $f \equiv 0$. □

Theorem 4.7.5: *Let $\{\phi_n(x)\}$ be a complete orthonormal set on $[a, b]$ and let $\sum_{n=1}^{\infty} c_n\phi_n(x)$ be the Fourier series of f in S. Then*

$$\int_a^b f(x)\mathrm{d}x = \sum_{n=1}^{\infty} c_n \int_a^b \phi_n(x)\mathrm{d}x.$$

Proof:

$$\begin{aligned}\left|\int_a^b f(x)\mathrm{d}x - \sum_{n=1}^{m} c_n \int_a^b \phi_n(x)\mathrm{d}x\right| &\leq \int_a^b \left|f(x) - \sum_{n=1}^{m} c_n\phi_n(x)\right|\mathrm{d}x \\ &\leq \left\|f - \sum_{n=1}^{m} c_n\phi_n\right\| \|1\|\end{aligned}$$

from Schwars's inequality. The last term goes to 0 as $m \to \infty$ since the Fourier series of f converges to f in the mean. □

4.8 Completeness of Eigenfunctions of Sturm-Liouville Problems

In this section, we discuss completeness of eigenfunctions. Our background about Green's function and Fourier series is going to lead to our basic approach in this section. Before giving the details about completeness, we need to give some necessary and important definitions and theorems related to integral operator with continuous kernel.

Following the e.g. (Lo 2000, Agarwal and O'Regan 2009), let Ku be defined by

$$(Ku)(x) = \int_a^b k(x, x_0)u(x_0)\mathrm{d}x_0,$$

where $k(x, x_0)$ is a complex-valued continuous function of x and x_0, such that $k(x, x_0) = \overline{k(x_0, x)}$ but not identically zero on $[a, b] \times [a, b]$ and $u \in C[a, b]$.

Theorem 4.8.1: *The set of functions $\{Ku\}$, with $\|u\| = 1$, is uniformly bounded and equicontinuous in $[a, b]$.*

Theorem 4.8.2: *$(Ku, v) = (u, Kv)$ for u and $v \in C[a, b]$ and (Ku, u) is real. Since $(Ku, v) = (u, Kv)$, K is called a symmetric operator.*

Proof:

$$\begin{aligned}(Ku,v) &= \int_a^b \left[\int_a^b k(x,x_0)u(x_0)\mathrm{d}x_0\right]\overline{v}(x)\mathrm{d}x \\ &= \int_a^b \left[\int_a^b k(x,x_0)\overline{v}(x)\mathrm{d}x\right]u(x_0)dx_0 \\ &= \int_a^b \left[\int_a^b \overline{k(x_0,x)v(x)}\mathrm{d}x\right]u(x_0)dx_0 \\ &= \int_a^b \overline{\left[\int_a^b k(x_0,x)v(x)\mathrm{d}x\right]}u(x_0)dx_0 \\ &= \int_a^b u(x_0)\overline{Kv(x_0)}dx_0 = (u,Kv).\end{aligned}$$

Therefore, $(Ku,u) = (u,Ku) = \overline{(Ku,u)}$. Thus, (Ku,u) is real. □

Remark 4.8.3: All the eigenvalues of K are real and eigenfunctions of K corresponding to distinct eigenvalues are orthogonal on $[a,b]$.

Theorem 4.8.4: *Let $\{f_n(x)\}$ be a sequence of uniformly bounded and equicontinuous functions on $[a,b]$. Then it contains a subsequence $\{f_{nk}(x)\}$ which converges uniformly on $[a,b]$.*

Let μ_i be the eigenvalues of K and χ_i are the corresponding eigenfunction of K.

Theorem 4.8.5: *Let $u \in C[a,b]$. The Fourier series of Ku with respect to $\{\chi_i(x)\}$ converges uniformly to Ku on $[a,b]$.*

Proof: Let $g_m(x) = u(x) - \sum_{i=1}^m (u,\chi_i)\chi_i(x)$. Then, $(g_m,\chi_i) = 0$ for $i = 1,\ldots m$. From extremal principles,

$$\|Kg_m\| \le |\mu_{m+1}| \, \|g_m\| \, .$$

Since $\mu_{m+1} \to 0$, the sequence $\{Kg_m\}$ converges to 0 in the mean. Thus

$$Ku = \sum_{i=1}^{\infty}(u,\chi_i)K\chi_i = \sum_{i=1}^{\infty}\mu_i(u,\chi_i)\chi_i = \sum_{i=1}^{\infty}(u,K\chi_i)\chi_i = \sum_{i=1}^{\infty}(Ku,\chi_i)\chi_i.$$

For any $q > p$,

$$\sum_{i=p}^{q}\mu_i(u,\chi_i)\chi_i = K\left[\sum_{i=p}^{q}(u,\chi_i)\chi_i\right].$$

Since $|Ku| \le M(b-a)^{1/2}\|u\|$, we have

$$\left|\sum_{i=p}^{q}\mu_i(u,\chi_i)\chi_i\right| \le M(b-a)^{1/2}\left[\sum_{i=p}^{q}|(u,\chi_k)|^2\right]^{1/2}$$

which goes to zero as $p, q \to \infty$ by Bessel's inequality. So that,

$$\sum_{i=1}^{\infty}(Ku, \chi_i)\chi_i$$

is uniformly convergent to a continuous function on $[a, b]$, namely, Ku. □

Theorem 4.8.6: *For each nonzero eigenvalue of K, there corresponds at most a finite number of linearly independent eigenfunctions, and the number of nonzero eigenvalues is either finite or infinite with $\mu_i \to 0$.*

Now, we can discuss completeness fo eigenfunctions of Sturm-Liouville problems. Here we assume $\rho \equiv 1$.

Let $G(x, x_0)$ be the Green's function for the Sturm-Liouville problem

$$\mathcal{L}u = (pu')' - qu = -\lambda u(x),$$

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \beta_1 u(b) + \beta_2 u'(b) = 0. \tag{4.8.1}$$

We note that $G(x, x_0)$ is continuous and symmetric on $[a, b] \times [a, b]$. Define

$$(Ku)(x) = \int_a^b G(x, x_0)u(x_0)\mathrm{d}x_0.$$

There is exists a finite number of nonzero eigenvalues $\{\mu_i\}$ with corresponding normalized eigenfunctions $\{\chi_i(x)\}$ such that $|\mu_{i+1}| \leq |\mu_i|$ and $(\chi_i, \chi_j) = \delta_{ij}$.

Remark 4.8.7: If the kernel is $G(x, x_0)$, then there is a sequence of eigenfunctions $\{\chi_i(x)\}$ with corresponding eigenvalues $\{\mu_i\}$ with $\mu_i \to 0$ for the operator K.

Theorem 4.8.8: *Let $f \in C^2[a, b]$ and satisfy the boundary conditions (4.8.2). Then*

$$f = \sum_{i=1}^{\infty}(f, \chi_i)\chi_i,$$

where the convergence is uniform on $[a, b]$.

Proof: Let $f \in C^2[a, b]$ and satisfy (4.8.1). Then $u = \mathcal{L}f \in C[a, b]$. We know from the properties of Green's function that $f = -Ku$. From Theorem 4.8.5

$$f = -Ku = -\sum_{i=1}^{\infty}(Ku, \chi_i)\chi_i = \sum_{i=1}^{\infty}(f, \chi_i)\chi_i.$$

□

Theorem 4.8.9: $\{\chi_i(x)\}$ *is a complete orthonormal set in* $[a, b]$.

Proof: Let $f \in S$ and let $\varepsilon > 0$ be given. Then there exists a function $g \in C^2[a, b]$ such that $\|f - g\| < \varepsilon$. By using the triangle inequality, we have

$$\left\| f - \sum_{i=1}^{m} (f, \chi_i)\chi_i \right\| \leq \|f - g\| + \left\| g - \sum_{i=1}^{m} (g, \chi_i)\chi_i \right\| + \left\| \sum_{i=1}^{m} (g - f, \chi_i)\chi_i \right\|.$$

By Bessel's inequality,

$$\left\| \sum_{i=1}^{m} (g - f, \chi_i)\chi_i \right\|^2 = \sum_{i=1}^{m} |(g - f, \chi_i)|^2 \leq ||g - f||^2 < \varepsilon^2,$$

and there exists an integer $M = M(\varepsilon)$ such that for $m > M$,

$$\left\| g - \sum_{i=1}^{m} (g, \chi_i)\chi_i \right\| < \varepsilon.$$

Hence,

$$\left\| f - \sum_{i=1}^{m} (f, \chi_i)\chi_i \right\| < 3\varepsilon \quad \text{for } m > M.$$

$\square$

Remark 4.8.10: The eigenfunctions $\{u_n(x)\}$ of $\mathcal{L}u = -\lambda u$ with boundary conditions (BC3) form a complete orthogonal set in $[a, b]$ if $\lambda = 0$ is not an eigenvalue.

Example 4.8.11: Show that $\{e^{\frac{1}{2}x} \sin(\frac{nx}{2})\}$ is a complete orthogonal set on $[0, 2\pi]$.

Solution: Consider the Sturm-Liouville problem

$$u'' - u' + \lambda u = 0, \ 0 < x < 2\pi,$$
$$u(0) = 0 = u(2\pi).$$

To solve this equation we look at the characteristic equation

$$r^2 - r + \lambda = 0.$$

It has roots

$$r = \frac{1 \pm \sqrt{1 - 4\lambda}}{2}.$$

We consider the following three situations separately:

Case 1: $1-4\lambda > 0$. In this case we have two real roots $r = \frac{1+\sqrt{1-4\lambda}}{2}$ and $r = \frac{1-\sqrt{1-4\lambda}}{2}$. Thus, the solution looks like

$$u = c_1 e^{\frac{1+\sqrt{1-4\lambda}}{2}x} + c_2 e^{\frac{1-\sqrt{1-4\lambda}}{2}x}.$$

Applying to the boundary conditions we get

$$u(0) = c_1 + c_2 = 0 \implies c_1 = -c_2,$$
$$u(2\pi) = c_1(e^{(1+\sqrt{1-4\lambda})\pi} - e^{(1-\sqrt{1-4\lambda})\pi}) = 0.$$

Since the term in the parenthesis is non-zero we see that $c_1 = 0$ and thus $c_2 = 0$. So our only solution in this case is the trivial one $u = 0$.

Case 2: $1 - 4\lambda < 0$. In this case we get complex roots $r = \frac{1}{2} \pm i\sqrt{4\lambda - 1}$. So the solution is

$$u = c_1 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{4\lambda - 1}}{2}x\right) + c_2 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{4\lambda - 1}}{2}x\right).$$

Plugging in the boundary conditions gives

$$u(0) = c_2 = 0,$$
$$u(2\pi) = c_1 e^{\pi} \sin(\pi\sqrt{4\lambda - 1}).$$

In order not to have a trivial solution we assume $c_1 \neq 0$ then this equation implies

$$\sin(\pi\sqrt{4\lambda - 1}) = 0 \implies \pi\sqrt{4\lambda - 1} = n\pi,\ n = 1, 2, \cdots .$$

So that, the λ's that satisfy this equation are

$$\lambda_n = \frac{n^2 + 1}{4},\ n = 1, 2, \cdots .$$

And the eigenfunction corresponding to λ_n is

$$e^{\frac{1}{2}x} \sin\left(\frac{nx}{2}\right).$$

Case 3: $1 - 4\lambda = 0$. In this case the two roots are $\frac{1}{2}$. Hence, we have

$$u = c_1 e^{\frac{1}{2}x} + c_2 e^{\frac{1}{2}x}.$$

By applying the boundary conditions we get $c_1 = 0 = c_2$. Thus λ is not an eigenvalue and the only solution for this case is $u \equiv 0$.

As a result, when we consider the last remark which is given above we see that $\{e^{\frac{1}{2}x} \sin(\frac{nx}{2})\}$ is a complete orthogonal on $[0, 2\pi]$. □

A few notes about eigenvalues and eigenfunctions

Following the e.g. (Lo 2000, Mc Owen 1996) consider the Sturm-Liouville problem

$$\mathcal{L}u = (pu')' - qu = -\lambda\rho u, \ a < x < b, \tag{4.8.2}$$

$$u(a) = 0 = u(b) \tag{4.8.3}$$

with the additional condition $q(x) \geq 0$ on $[a, b]$.

Theorem 4.8.12: *All the eigenvalues of* (4.8.2)-(4.8.3) *are positive.*

Proof: Let λ be an eigenvalue and let $u(x)$ be the corresponding eigenfunction. Multiplying (4.8.2) by $u(x)$ and integrating by parts from a to b, we have

$$0 = \int_a^b u\left[(pu')' - qu + \lambda\rho u\right] \mathrm{d}x = upu'|_a^b + \int_a^b \left[-pu'^2 - qu^2 + \lambda\rho u^2\right] \mathrm{d}x.$$

By using the boundary conditions, we get

$$\lambda = \frac{\int_a^b \left[pu'^2 + qu^2\right] \mathrm{d}x}{\int_a^b \rho u^2 \mathrm{d}x} \geq 0$$

since $p > 0$, $q \geq 0$ and $\rho > 0$ on $[a, b]$. $\square$

In case of $\lambda = 0$, then u is a constant. From (4.8.3), $u \equiv 0$, a contradiction. So that, $\lambda > 0$.

The Green's function for the problem

$$\mathcal{L}u = 0, \quad u(a) = 0 = u(b)$$

exists. Because $\lambda = 0$ is not an eigenvalue. The corresponding nonhomogenous problem

$$\mathcal{L}w = -f, \quad w(a) = 0 = w(b)$$

has a unique solution given by

$$w(x) = \int_a^b G(x, x_0) f(x_0) \mathrm{d}x_0.$$

If we set $f(x) = \lambda\rho(x)u(x)$, then the eigenfunction $u(x)$ of (4.8.2)-(4.8.3) with the corresponding eigenvalue λ satisfies

$$u(x) = \lambda \int_a^b G(x, x_0)\rho(x_0)u(x_0)\mathrm{d}x_0.$$

Let $\psi(x) = \sqrt{\rho(x)}u(x)$ and $\mu = \frac{1}{\lambda}$. Then, we obtain

$$K\psi = \int_a^b \sqrt{\rho(x)}G(x, x_0)\sqrt{\rho(x_0)}\psi(x_0)\mathrm{d}x_0 = \mu\psi(x). \tag{4.8.4}$$

Hence, there exists a sequence of eigenvalues $\{\mu_k\}$ of (4.8.4) with normalized eigenfunctions $\{\chi_k(x)\}$ such that $|\mu_k| \geq |\mu_{k+1}|$ and $|\mu_k| \to 0$. Since $\lambda_k = \frac{1}{\mu_k}$ are eigenvalues of (4.8.2)-(4.8.3) and λ_k is positive and simple.

Theorem 4.8.13: *The nth-eigenvalue for the Sturm-Liouville problem* (4.8.2)-(4.8.3) *is the minimum value of the functional*

$$I(u) = \frac{\int_a^b [pu'^2 + qu^2]\,\mathrm{d}x}{\int_a^b \rho u^2 \mathrm{d}x} \tag{4.8.5}$$

for the class of continuous, piecewise smooth functions satisfying (4.8.3) *and* $(u, u_i) = 0$, $i = 1, \ldots, (n-1)$, *where* $u_i(x)$ *are the first* $(n-1)$ *normalized eigenfunctions. The nth eigenfunction* $u_n(x)$ *is the corresponding normalized minimizing function.*

Theorem 4.8.14 (Courant's Theorem)**:** *Let* $\phi_1(x), \ldots, \phi_{n-1}(x)$ *be arbitrary continuous functions on* $[a, b]$. *Let* $\Lambda(\phi_1, \ldots, \phi_{n-1})$ *be the minimum of the functional* $I(u)$ *in* (4.8.5) *where class* H *of admissible functions* $u(x)$ *is the space of all continuous, piecewise smooth functions* $u(x)$ *in* $[a, b]$ *with* $u(a) = 0 = u(b)$ *and* $(u, \phi_i) = 0$ *for* $i = 1, 2, \ldots, n-1$. *Then the nth eigenvalue* λ_n *of* (4.8.2)-(4.8.3) *is*

$$\lambda_n = \max \Lambda(\phi_1, \ldots \phi_{n-1})$$

for all possible choices of the functions $\phi_1, \ldots, \phi_{n-1}$.

5 BOOLEAN DIFFERENTIAL EQUATIONS

In this section, we will discuss Boolean Differential Equations which is one of the important part of Model Theory in Mathematical Logic. After giving some definitions and theorems, we will be able to solve the examples about the subject. At the end of this section, we will be started to investigate the relation between Sturm-Liouville Problems and Boolean Differential Equations. This thesis will be starting point for further studies. Now, let us start with the essential definitions.

A Boolean Differential Equation is an equation that includes derivative operations and differential operators of an unknown Boolean function. All differentials of Boolean variables, differential operations and derivative operations of Boolean functions are Boolean functions with special properties.

5.1 Logic (Boolean) Function

We begin by defining the concept of a boolean function.

Definition 5.1.1: A Boolean function of n variables is a function f of B^n into B, where B is the set $\{0, 1\}$, n is a positive integer, and B^n denotes the n-fold cartesian product of the set B with itself. A point $x^* = (x_1, x_2, \ldots, x_n) \in B^n$ is a true point (resp. false point) of the Boolean function f if $f(x^*) = 1$ (resp. $f(x^*) = 0$). (Int.Ref.5)

The most elementary way to define a Boolean function f is to provide its truth table, i.e. to give a complete list of all the points in B^n together with the value of the function at each point.
Any decision that can be answered yes/no or true/false can be mathematically represented as a combination of logic functions. George Boole invented and published this form of mathematics (Boolean Algebra) in 1847. The 3 basic logic functions, which can be used to solve any Boolean equation, are (Steinbah 1974):

$$NOT$$
$$AND$$
$$OR$$

Other common logic functions, that are combinations of the basic 3, are:

$$NAND$$
$$NOR$$
$$XOR$$

5.2 Boolean Equation

A Boolean equation equals two given Boolean functions and its solution is a set of Boolean vectors, i.e., if $f(x)$ and $g(x)$ are logic functions then $f(x) = g(x)$ is a logic equation, and its solution is a set of vectors b where $f(b) = g(b) = 0$ or $f(b) = g(b) = 1$. In practical applications both functions are given by logic expressions that consist of variables connected by logic operations and structured by parantheses. (Int.Ref.6)

Now, let us give a standart system of logical equations is expressed as

$$\begin{cases} f_1(p_1, p_2, \ldots, p_n) = c_1, \\ f_2(p_1, p_2, \ldots, p_n) = c_2, \\ \vdots \\ f_m(p_1, p_2, \ldots, p_n) = c_m, \end{cases} \tag{5.2.1}$$

where $f_i, i = 1, \ldots, m$ are logical functions, $p_i, i = 1, \ldots, m$ are logical constants. A set of logical constants $d_i, i = 1, 2, \ldots, n$, such that $p_i = di, i = 1, 2, \ldots, n$, satisfy (5.2.1) is said to be solution of (5.2.1).

As an example, consider the following system:

$$\begin{cases} p \wedge q = c_1 \\ q \vee r = c_2 \\ r \leftrightarrow (\overline{p}) = c_3 \end{cases}$$

1. Assume the logical constants are

$$c_1 = 1, c_2 = 1, c_3 = 1$$

A straightforward verifications shows that

$$p = 1$$

$$q = 1$$

$$r = 0$$

2. Assume the logical constans are

$$c_1 = 1, c_2 = 0, c_3 = 1$$

It can be checked that there is no solution.

3. Assume the logical constants are

$$c_1 = 0, c_2 = 1, c_3 = 0$$

There are then two solutions:

$$\begin{cases} p_1 = 1 \\ q_1 = 0 \\ r_1 = 1 \end{cases}$$

and

$$\begin{cases} p_1 = 0 \\ q_1 = 1 \\ r_1 = 0 \end{cases}$$

It shows that the solutions of system of logical equations are quite different from those of linear algebraic equations where the type of solution dpends only on the coefficients of the system.

5.3 Boolean Derivative

Now following the e.g. (Int.Ref.7), we describe a single variable by an italic letter like x_i and a set of variables by a bold letter like $\mathbf{x_0}$. The first group of derivative operations explores the change of the function value with regard to the change of a single variable x_i. Hence, the subsets, evaluated by simple derivative operations, include two function values which are reached by changing x_i.
Before giving the first definition, let us introduce the meaning of " $\oplus$ " with using " $\vee$ " and " $\wedge$ " which are the basic notations of logic equations. We will use this notations in equations and definitions.

The exclusive disjunction: $p \oplus q$ can be expressed in terms of the logical conjunction ($\vee$), the disjunction ($\wedge$), and the negation ($\overline{p}$) or ($\overline{q}$) as follows:

$$p \oplus q = (p \wedge q) \vee \overline{(p \vee q)}$$

$$p \oplus q = (p \wedge \overline{q}) \vee (\overline{p} \wedge q)$$

Additionally, we have the relation as follows:

$$r = p \wedge q \leftrightarrow r = p \cdot q(mod2)$$

$$r = p \oplus q \leftrightarrow r = p + q(mod2)$$

In summary, we can write the notations like:

$$\begin{aligned} p \oplus q &= (p \wedge \overline{q}) \vee (\overline{p} \wedge q) = p\overline{q} + \overline{p}q \\ &= (p \vee q] \wedge (\overline{p} \vee \overline{q}) = (p+q)(\overline{p}+\overline{q}) \\ &= (p \vee q) \wedge \overline{(p \wedge q)} = (p+q)\overline{(pq)} \end{aligned}$$

Definition 5.3.1: Let $f(x) = f(x_i, x_1)$ be a logic function of n variables. Then

$$\frac{\partial f(x)}{\partial x_i} = f(x_i = 0, x_1) \oplus f(x_i = 1, x_1) \tag{5.3.1}$$

is the (simple) derivative (where the binary operator defined by $p \oplus q = 0$ if $p = q$ and $p \oplus q = 1$, if $p \neq q$),

$$\min_{x_i} f(x) = f(x_i = 0, x_1) \wedge f(x_i = 1, x_1)$$

the (simple) minimum and

$$\max_{x_i} f(x) = f(x_i = 0, x_1) \vee f(x_i = 1, x_1)$$

the (simple) maximum of the logic function $f(x)$ with regard to the variable x_i. (Int.Ref.7)

The second group of derivative operations explores the change of a function with regard to the set of variables x_0. Hence, the subsets, evaluated by vectorial derivative operations, include again two function values which are reached by changing all variables of the set x_0 at the same point of time.

Definition 5.3.2: Let $f(x) = f(x_0, x_1)$ be a logic function of n variables. Then

$$\frac{\partial f(x_0, x_1)}{\partial x_0} = f(x_0, x_1) \oplus f(\overline{x}_0, x_1) \tag{5.3.2}$$

is the vectorial derivative,

$$\min_{x_0} f(x_0, x_1) = f(x_0, x_1) \wedge f(\overline{x}_0, x_1)$$

the vectorial minimum, and

$$\max_{x_0} f(x) = f(x_0, x_1) \vee f(\overline{x}_0, x_1)$$

the vectorial maximum of the logic function $f(x_0, x_1)$ with regard to the variables of x_0. (Int.Ref.7)

The result of each derivative operation of a Boolean function is again a Boolean function. Hence, derivative operations can be executed iteratively for the result of a previous derivative operation.

Definition 5.3.3: Let $f(x) = f(x_0, x_1)$ be a logic function of n variables, and let $x_0 = (x_1, x_2, \ldots, x_m)$. Then

$$\frac{\partial^m f(x_0, x_1)}{\partial x_1 \partial x_2 \ldots \partial x_m} = \frac{\partial}{\partial x_m}(\ldots(\frac{\partial}{\partial x_2}(\frac{\partial f(x_0, x_1)}{\partial x_1}))\ldots)$$

is the m-fold derivative,

$$\min_{x_0}{}^m f(x_0, x_1) = \min_{x_m}(\ldots(\min_{x_2}(\min_{x_1} f(x_0, x_1)))\ldots)$$

the m-fold minimum,

$$\max_{x_0}{}^m f(x_0, x_1) = \max_{x_m}(\ldots(\max_{x_2}(\max_{x_1} f(x_0, x_1)))\ldots)$$

the m-fold maximum and

$$\Delta_{x_0} f(x_0, x_1) = \min_{x_0}{}^m f(x_0, x_1) \oplus \max_{x_0}{}^m f(x_0, x_1)$$

the Δ- operation of the function $f(x_0, x_1)$ with regard to the set of variables x_0.

The Boolean derivative $\frac{\partial f(x)}{\partial x_i}$ of the Booleean function $f(x) = f(x_1, x_2, \ldots, x_n)$ with respect to variables $x_1, x_2, \ldots, x_n$ is defined in the form

$$\frac{\partial f(x)}{\partial x_i} = f(x_1, x_2, \ldots, x_n) \oplus f(x_1, x_2, \ldots, \overline{x}_i, \ldots, x_n) \tag{5.3.3}$$

This notation represents the change in the value of the variable x_i is inverted to $\overline{x}_i$. Relationship (5.3) is used to compute the logical derivative with respect to a variable when the Boolean function $f(x)$ is defined in symbolic form.

Example 5.3.4: For the function of three variables

$$f(x) = x_1x_2 \vee x_3,$$

compute the Boolean derivatives with respect to the variables x_1, x_2 and x_3 ($\vee$ is the disjunction symbol). Using (5.3), we obtain the symbolic expression of the variable with respect to x_1:

$$\frac{\partial f(x)}{\partial x_1} = (x_1x_2 \vee x_3) \oplus (\overline{x}_1x_2 \vee x_3) = x_2\overline{x}_3$$

Similarly with respect to the other variables

$$\frac{\partial f(x)}{\partial x_2} = x_1\overline{x}_3$$

and

$$\frac{\partial f(x)}{\partial x_3} = \overline{x_1x_2}.$$

5.4 Boolean Differential Equations

The topic of Boolean equations has been a hot topic of research for almost two centuries and its importance can hardly be overestimated. Boolean-equation solving permeates many areas of modern science such as logical design, biology, grammars, chemistry, law, medicine, spectroscopy, and graph theory. Many important problems in operations research can be reduced to the problem of solving a system of Boolean equations. A notable example is the problem of an n-person coalition game with a domination relation between different strategies. The solutions of Boolean equations serve also as an important tool in the treatment of pseudo-Boolean equations and inequalities, and their associated problems in integer linear programming. (Int.Ref.6)

Let us take the Boolean function

$$f(x) = f(x_1, x_2, x_3) = x_1 \vee x_2x_3. \tag{5.4.1}$$

Using the Definition 5.3.1 we get the vectorial derivative with regard to (x_1, x_3) as follows:

$$
\begin{aligned}
\frac{\partial f(x_1,x_2,x_3)}{\partial(x_1,x_3)} &= f(x_1,x_2,x_3) \oplus f(\overline{x}_1,x_2,\overline{x}_3) \\
&= (x_1 \vee x_2x_3) \oplus (\overline{x}_1 \vee x_2\overline{x}_3) \\
&= (x_1 \vee \overline{x}_1x_2x_3) \oplus (\overline{x}_1 \vee x_1x_2\overline{x}_3) \\
&= (x_1 \oplus \overline{x}_1x_2x_3) \oplus (\overline{x}_1 \oplus x_1x_2\overline{x}_3) \\
&= 1 \oplus \overline{x}_1x_2x_3 \oplus x_1x_2\overline{x}_3) \\
&= 1 \oplus x_2(\overline{x}_1x_3 \oplus x_1\overline{x}_3) \\
&= 1 \oplus x_2(x_1 \oplus x_3) \\
&= \overline{x}_2 \vee \overline{(x_1 \oplus x_3)} \\
&= \overline{x}_2 \vee (\overline{x_1} \oplus x_3). \qquad (*)
\end{aligned}
$$

Hence, the result of this vectorial derivative is the Boolean function

$$g(x_1,x_2,x_3) = \overline{x}_2 \vee (\overline{x}_1 \oplus x_3) \tag{5.4.2}$$

and we have the Boolean differential equation:

$$\frac{\partial f(x_1,x_2,x_3)}{\partial(x_1,x_3)} = g(x_1,x_2,x_3). \tag{5.4.3}$$

The function (5.4.2) is uniquely defined by the given function (5.4.1), the definition of the vectorial derivative (5.4.3) and the direction of change described by the taken subset of variables (x_1, x_2).

All calculation steps of (*) can also be executed in the reverse direction. Hence, the function $f(x_1,x_2,x_3)$ (5.4.1) is a solution of the Boolean differential equation (5.4.3) where the function $g(x_1,x_2,x_3)$ is defined by (5.4.2).

Now the question arises whether the function $f(x_1,x_2,x_3)$ is uniquely defined by the function $g(x_1,x_2,x_3)$ and Boolean differential equation (5.4.3) The answer to this question is NO. There are 15 other Boolean functions $f_i(x_1,x_2,x_3)$ which solve the BDE (5.4.3) for the function $g(x_1,x_2,x_3)$). All 16 solution functions of the BDE (5.4.3) for the the function $g(x_1,x_2,x_3)$ are (Int.Ref.6):

$$
\begin{aligned}
f_0(x_1,x_2,x_3) &= x_1(\overline{x}_1 \vee x_3),\\
f_1(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus \overline{x}_2(\overline{x}_1 \oplus x_3)\\
f_2(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus \overline{x}_2(x_1 \oplus x_3)\\
f_3(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus \overline{x}_2\\
f_4(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus x_2(\overline{x}_1 \oplus x_3)\\
f_5(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus (\overline{x}_1 \oplus x_3)\\
f_6(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus (x_1 \oplus x_2 \oplus x_3)\\
f_7(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus (\overline{x}_2 \vee (\overline{x}_1 \oplus x_3))\\
f_8(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus x_2(x_1 \oplus x_3)\\
&= (x_1\overline{x}_2\overline{x}_3 \oplus x_1\overline{x}_2x_3 \oplus x_1x_2x_3) \oplus (x_1x_2\overline{x}_3)\\
&= x_1 \vee x_2x_3,\\
f_9(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus (\overline{x}_1 \oplus x_2 \oplus x_3)\\
f_{10}(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3 \oplus (x_1 \oplus x_3)\\
f_{11}(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus (\overline{x}_2 \vee (x_1 \oplus x_3))\\
f_{12}(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus x_2\\
f_{13}(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus (x_2 \vee (\overline{x}_1 \oplus x_3))\\
f_{14}(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus (x_2 \vee (x_1 \oplus x_3))\\
f_{15}(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \oplus 1
\end{aligned}
$$

It can be verified directly (by inserting these 16 functions into the BDE) (5.4.3) that the calculated vectorial derivatives are equal to the function $g(x_1, x_2, x_3)$.
The enumeration of these 16 solution function shows that all solution function have a common basic structure

$$f_i(x_1, x_2, x_3) = g_0(x_1, x_2, x_3) \oplus h_i(x_1, x_2, x_3) \tag{5.4.4}$$

with

$$
\begin{aligned}
g_0(x_1,x_2,x_3) &= x_1(\overline{x}_2 \vee x_3) \\
&= x_1\overline{x}_2 \vee x_1x_3 \\
&= x_1\overline{x}_2 \vee x_1x_3 \vee 0 \\
&= x_1\overline{x}_2 \vee x_1x_1x_3 \vee x_1\bar{x_1}\bar{x_3} \\
&= x_1 \wedge (\overline{x}_2 \vee x_1x_3 \vee \bar{x_1}\bar{x_3}) \\
&= x_1 \wedge \overline{x}_2 \vee (\overline{x}_1 \oplus x_3) \\
&= x_1 \wedge g(x_1,x_2,x_3). && (**)
\end{aligned}
$$

The variable x_1 is selected from the set (x_1, x_3) used to define the direction of the vectorial derivative. The other variable x_3 can be chosen in (**) to specify the function $g_0(x_1, x_2, x_3) = x_1(\overline{x_2} \vee x_3)$. This function $g_0(x_1, x_2, x_3)$ can be used to create together with the same 16 functions $h_i(x_1, x_2, x_3)$ exactly the same set of 16 solution function of the BDE (5.4.3) for the function $g_0(x_1, x_2, x_3)$ (5.4.2).
The transformation for the solution function $f_8(x_1, x_2, x_3)$ confirms that the used initial function $f_1(x_1, x_2, x_3) = f_8(x_1, x_2, x_3)$ (5.4.1) is an element of the solution set. (Rudeanu 1974)
All functions $h_i(x_1, x_2, x_3)$ hold the BDE

$$
\frac{\partial h_i(x_1,x_2,x_3)}{\partial(x_1,x_3)} = 0.
$$

This property can be checked easily. Either the functions $h_i(x_1, x_2, x_3)$ does not depend on (x_1, x_3), than we have e.g.

$$
\begin{aligned}
\frac{\partial h_3(x_1,x_2,x_3)}{\partial(x_1,x_3)} &= \frac{\partial(\overline{x}_2)}{\partial(x_1,x_3)} \\
&= \overline{x}_2 \oplus x_2 = 0,
\end{aligned}
$$

or the variables (x_1, x_3) appear in the function $h_i(x_1, x_2, x_3)$ connected by an $\oplus$-operation, than we have e.g. $h_i(x_1, x_2, x_3)$

$$\frac{\partial h_1(x_1,x_2,x_3)}{\partial(x_1,x_3)} = \frac{\partial(\overline{x}_2(\overline{x}_1 \oplus x_3))}{\partial(x_1,x_3)}$$
$$= \overline{x}_2(\overline{x}_1 \oplus x_3) \oplus \overline{x}_2(x_1 \oplus \overline{x}_3)$$
$$= \overline{x}_2\overline{(x_1 \oplus x_3)} \oplus \overline{x}_2\overline{(x_1 \oplus x_3)} = 0.$$

There is one remaining question concerning the BDE of a single derivative operation like (5.4.3): Are there solution functions $f(x_1, x_2, x_3)$ for each function $g(x_1, x_2, x_3)$? The answer to this question is NO too. It can be verified, for instance, by checking all 256 functions $f(x_1, x_2, x_3)$ that no function $f(x_1, x_2, x_3)$ exists as solution of the BDE (5.4.3) where the function $g(x_1, x_2, x_3) = x_1, x_2, x_3$. The reason for that come from the definition of the vectorial derivative.
Due to

$$g(x_0, x_1) = \frac{\partial f(x_0, x_1)}{\partial x_0} = f(x_0, x_1) \oplus f(\overline{x}_0, x_1)$$

we get

$$g(\overline{x}_0, x_1) = \frac{\partial f(\overline{x}_0, x_1)}{\partial x_0} = f(\overline{x}_0, x_1) \oplus f(x_0, x_1)$$

and consequently

$$g(x_0, x_1) = g(\overline{x}_0, x_1)$$

which can be expressed by

$$g(x_0, x_1) = g(\overline{x}_0, x_1)$$
$$g(x_0, x_1) \oplus g(\overline{x}_0, x_1) = g(\overline{x}_0, x_1) = g(\overline{x}_0, x_1)$$
$$g(x_0, x_1) \oplus g(\overline{x}_0, x_1) = 0$$
$$\frac{\partial g(x_0, x_1)}{\partial x_0} = 0. \tag{5.4.5}$$

Hence, the function $g(x_1, x_2, x_3)$ in the BDE (5.4.3) must hold the condition (5.4.5) in order to find solution functions $f(x_1, x_2, x_3)$. For that reason (5.4.5) is called integrability condition for the vectorial derivatives of Boolean functions.

We learn form from this example:

1. A Boolean differential equation (5.4.3) includes the unknown function $f(x_1, x_2, x_3)$.
2. There are solutions of a BDE like (5.4.3) only if the righthand function $g(x_1, x_2, x_3)$ holds a special integrability condition.
3. As shown in (5.4.4), the general solution of an inhomogeneous BDE is built using a single special solution of the inhomogeneous BDE and the set of all solutions of the associated homogeneous BDE. The associated homogeneous BDE is developed by replacing the righthand side of an inhomogeneous BDE by 0.
4. Generally, the solution of a Boolean differential equation is a set of Boolean functions. This is a significant difference to Boolean equations. The solution of a Boolean equation is a set of Boolean vectors.

Example 5.4.1: Let the following Boolean differential equation be given: $\frac{\partial f}{\partial x_1} = 1$, where f is two-variable Boolean function. To solve the equations by the method of unspecified coefficients, it needs to be represented by the canonical exclusive OR form:

$$f = f^{(0)}\overline{x}_1\overline{x}_2 \oplus f^{(1)}\overline{x}_1 x_2 \oplus f^{(3)} x_1 x_2$$

with unspecified coefficients $f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)}$. Then we have to compute the Boolean difference with respect to variable x_1 and make it equal to 1:

$$\begin{aligned}\frac{\partial f}{\partial x_1} &= f^{(0)}\overline{x}_2 \oplus f^{(1)} x_2 \oplus f^{(2)}\overline{x}_2 \oplus f^{(3)} x_2 \\ &= (f^{(0)} + f^{(2)})\overline{x}_2 + (f^{(1)} \oplus f^{(3)}) x_2 \\ &= 1\end{aligned}$$

Next, one can substitute all possible values of x_2 and write the system of Boolean equations as follows:

$$f^{(0)} \oplus f^{(2)} = 1$$
$$f^{(1)} \oplus f^{(3)} = 1.$$

Next, combine these two equations by one

$$(f^{(0)} \oplus f^{(2)}) \cdot (f^{(1)} \oplus f^{(3)}) = 1.$$

So, the initial Boolean differential equation is reduced to one logic equation with 4 unknown variables $f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)}$. The solution of the equation is the set of functions $\{x_1, x_1 \oplus x_2, \overline{x}_1 \oplus x_2, \overline{x}_1\}$.

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