Research Article

On Asymptotically Lacunary Statistical Equivalent Set Sequences

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This paper presents three definitions which are natural combination of the definitions of asymptotic equivalence, statistical convergence, lacunary statistical convergence, and Wijsman convergence. In addition, we also present asymptotically equivalent (Wijsman sense) analogs of theorems in Patterson and Savaş (2006).

1. Introduction

In 1993, Marouf presented definitions for asymptotically equivalent and asymptotic regular matrices. In 2003, Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In 2006, Patterson and Savaş extended the definitions presented in [1] to lacunary sequences. In addition to these definitions, natural inclusion theorems were presented. The concept of Wijsman statistical convergence is implementation of the concept of statistical convergence to sequences of sets presented by Nuray and Rhoades in 2012. Similar to this concept, the concept of Wijsman lacunary statistical convergence was presented by Ulusu and Nuray in 2012. This paper extends the definitions presented in [2] to Wijsman statistical convergent sequences and Wijsman lacunary statistical convergent sequences. In addition to these definitions, natural inclusion theorems will also be presented.

2. Definitions and Notations

Definition 1 (see Marouf [3]). Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_{k} \frac{x_k}{y_k} = 1 \qquad (1)$$

(denoted by $x \sim y$).

Definition 2 (see Fridy [4]). The sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n} \frac{1}{n} \left| \{ k \le n : |x_k - L| \ge \varepsilon \} \right| = 0. \qquad (2)$$

In this case we write $st - \lim x_k = L$.

The next definition is natural combination of Definitions 1 and 2.

Definition 3 (see Patterson [1]). Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_{n} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{x_k}{y_k} - L \right| \ge \varepsilon \right\} \right| = 0 \qquad (3)$$

(denoted by $x \overset{S_L}{\sim} y$) and simply asymptotically statistically equivalent if $L = 1$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio k_r/k_{r-1} will be abbreviated by q_r.

Definition 4 (see Patterson and Savaş [2]). Let θ be a lacunary sequence; the two nonnegative sequences $x = (x_k)$ and

$y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0 \tag{4}$$

(denoted by $x \overset{S_\theta^L}{\sim} y$) and simply asymptotically lacunary statistically equivalent if $L = 1$.

Definition 5 (see Patterson & Savaş [2]). Let θ be a lacunary sequence; two nonnegative number sequences $x = (x_k)$ and $y = (y_k)$ are strongly asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| = 0 \tag{5}$$

(denoted by $x \overset{N_\theta^L}{\sim} y$) and strongly simply asymptotically lacunary equivalent if $L = 1$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any nonempty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, A). \tag{6}$$

Definition 6 (see Baronti & Papini [5]). Let (X, ρ) be a metric space. For any nonempty closed subset $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A) \tag{7}$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

Definition 7 (see Nuray & Rhoades [6]). Let (X, ρ) a metric space. For any nonempty closed subset $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistically convergent to A if $\{d(x, A_k)\}$ is statistically convergent to $d(x, A)$; that is, for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| = 0. \tag{8}$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

Also the concept of bounded sequence for sequences of sets was given by Nuray and Rhoades.

Definition 8 (see Nuray & Rhoades [6]). Let (X, ρ) be a metric space. For any nonempty closed subset A_k of X, we say that the sequence $\{A_k\}$ is bounded if

$$\sup_k d(x, A_k) < \infty \tag{9}$$

for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

Definition 9 (see Ulusu & Nuray [7]). Let (X, ρ) be a metric space and let $\theta = \{k_r\}$ be a lacunary sequence. For any nonempty closed subset $A, A_k \subseteq X$, we say that the sequence

$\{A_k\}$ is Wijsman lacunarily statistically convergent to A if $\{d(x, A_k)\}$ is lacunarily statistically convergent to $d(x, A)$; that is, for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_r \frac{1}{h_r} \left| \{ k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon \} \right| = 0. \tag{10}$$

In this case we write $S_\theta - \lim_W = A$ or $A_k \to A(WS_\theta)$.

Following these results we introduce three new notions that are asymptotically statistical equivalent (Wijsman sense) of multiple L, asymptotically lacunary statistical equivalent (Wijsman sense) of multiple L, and strongly asymptotically lacunary equivalent (Wijsman sense) of multiple L.

Definition 10. Let (X, ρ) be a metric space. For any non-empty closed subset $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent (Wijsman sense) if for each $x \in X$,

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1 \tag{11}$$

(denoted by $A_k \sim B_k$).

As an example, consider the following sequences of circles in the (x, y)-plane:

$$\begin{aligned} A_k &= \left\{ (x, y) : x^2 + y^2 + 2kx = 0 \right\}, \\ B_k &= \left\{ (x, y) : x^2 + y^2 - 2kx = 0 \right\}. \end{aligned} \tag{12}$$

Since

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1, \tag{13}$$

the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent (Wijsman sense); that is, $A_k \sim B_k$.

Definition 11. Let (X, ρ) be a metric space. For any non-empty closed subset $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically statistically equivalent (Wijsman sense) of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0 \tag{14}$$

(denoted by $\{A_k\} \overset{WS_L}{\sim} \{B_k\}$) and simply asymptotical statistical equivalent (Wijsman sense) if $L = 1$.

As an example, consider the following sequences of circles in the (x, y)-plane:

$$A_k = \begin{cases} \{(x, y) : x^2 + y^2 + 2ky = 0\}, \\ \qquad \text{if } k \text{ is a square integer,} \\ \{(1, 1)\}, \\ \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

$$B_k = \begin{cases} \{(x, y) : x^2 + y^2 - 2ky = 0\}, \\ \qquad \text{if } k \text{ is a square integer,} \\ \{(1, 1)\}, \\ \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{15}$$

Since

$$\lim_n \frac{1}{n} \left| \left\{ k \le n : \left| \frac{d(x, A_k)}{d(x, B_k)} - 1 \right| \ge \varepsilon \right\} \right| = 0, \tag{16}$$

the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically statistically equivalent (Wijsman sense); that is, $\{A_k\} \overset{WS_1}{\sim} \{B_k\}$.

Definition 12. Let (X, ρ) be a metric space and let θ be a lacunary sequence. For any non-empty closed subset A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically lacunary equivalent (Wijsman sense) of multiple L if for each $x \in X$,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \frac{d(x, A_k)}{d(x, B_k)} = L \tag{17}$$

(denoted by $\{A_k\} \overset{WN_\theta^L}{\sim} \{B_k\}$) and simply asymptotically lacunarily equivalent (Wijsman sense) if $L = 1$.

Definition 13. Let (X, ρ) be a metric space and let θ be a lacunary sequence. For any non-empty closed subset A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically lacunary equivalent (Wijsman sense) of multiple L if for each $x \in X$,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| = 0 \tag{18}$$

(denoted by $\{A_k\} \overset{[WN]_\theta^L}{\sim} \{B_k\}$) and simply strongly asymptotically lacunarily equivalent (Wijsman sense) if $L = 1$.

As an example, consider the following sequences:

$$A_k := \begin{cases} \left\{ (x, y) \in \mathbb{R}^2 : \frac{(x - \sqrt{k})^2}{k} + \frac{y^2}{2k} = 1 \right\}, \\ \qquad \text{if } k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r} \right] \\ \{(1, 1)\}, \\ \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

$$B_k := \begin{cases} \left\{ (x, y) \in \mathbb{R}^2 : \frac{(x + \sqrt{k})^2}{k} + \frac{y^2}{2k} = 1 \right\}, \\ \qquad \text{if } k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r} \right] \\ \{(1, 1)\}, \\ \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{19}$$

Since

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - 1 \right| = 0, \tag{20}$$

the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically lacunarily equivalent (Wijsman sense); that is, $\{A_k\} \overset{WN_\theta^1}{\sim} \{B_k\}$.

Definition 14. Let (X, ρ) be a metric space and let θ be a lacunary sequence. For any non-empty closed subset $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically lacunarily statistical equivalent (Wijsman sense) of multiple L if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right| = 0 \tag{21}$$

(denoted by $\{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\}$) and simply asymptotically lacunarily statistically equivalent (Wijsman sense) if $L = 1$.

As an example, consider the following sequences:

$$A_k := \begin{cases} \left\{ (x, y) \in \mathbb{R}^2 : x^2 + (y - 1)^2 = \frac{1}{k} \right\}, \\ \qquad \text{if } k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r} \right], \\ \qquad\qquad k \text{ is a square integer,} \\ \{(0, 0)\}, \\ \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

$$B_k := \begin{cases} \left\{ (x, y) \in \mathbb{R}^2 : x^2 + (y + 1)^2 = \frac{1}{k} \right\}, \\ \qquad \text{if } k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r} \right], \\ \qquad\qquad k \text{ is a square integer,} \\ \{(0, 0)\}, \\ \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{22}$$

Since

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - 1 \right| \ge \varepsilon \right\} \right| = 0, \tag{23}$$

the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically lacunarily statistically equivalent (Wijsman sense); that is, $\{A_k\} \overset{WS_\theta^1}{\sim} \{B_k\}$.

3. Main Results

Theorem 15. *Let (X, ρ) be a metric space, let $\theta = \{k_r\}$ be a lacunary sequence, and let A_k, B_k be non-empty closed subsets of X:*

(i) (a) $\{A_k\} \overset{[WN]_\theta^L}{\sim} \{B_k\} \Rightarrow \{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\}$

    (b) *$[WN]_\theta^L$ is a proper subset of WS_θ^L,*

(ii) $\{A_k\} \in L_\infty$ and $\{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\} \Rightarrow \{A_k\} \overset{[WN]_\theta^L}{\sim} \{B_k\}$,

(iii) $WS_\theta^L \cap L_\infty = [WN]_\theta^L \cap L_\infty$,

where L_∞ denotes the set of bounded sequences of sets.

Proof. (i)-(a). Let $\varepsilon > 0$ and $\{A_k\} \overset{[WN]_\theta^L}{\sim} \{B_k\}$. Then we can write

$$\sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right|$$

$$\geq \sum_{\substack{k \in I_r \\ |d(x,A_k)/d(x,B_k)-L| \geq \varepsilon}} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \qquad (24)$$

$$\geq \varepsilon \cdot \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|$$

which yields the result.

(ii)-(b). Suppose that $[WN]_\theta^L \subset WS_\theta^L$. Let $\{A_k\}$ and $\{B_k\}$ be the following sequences:

$$A_k = \begin{cases} \{k\}, & \text{if } k_{r-1} < k \leq k_{r-1} + \left[\sqrt{h_r}\right] \\ & \qquad\qquad r = 1, 2, \ldots \\ \{0\}, & \text{otherwise}, \end{cases} \qquad (25)$$

$$B_k = \{0\} \quad \forall k.$$

Note that $\{A_k\}$ is not bounded. We have, for every $\varepsilon > 0$ and for each $x \in X$,

$$\frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - 1 \right| \geq \varepsilon \right\} \right| = \frac{\left[\sqrt{h_r}\right]}{h_r} \longrightarrow 0 \qquad (26)$$

$$\text{as } r \longrightarrow \infty.$$

That is, $\{A_k\} \overset{WS_\theta^1}{\sim} \{B_k\}$. On the other hand,

$$\frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \nrightarrow 0 \quad \text{as } r \longrightarrow \infty. \qquad (27)$$

Hence $\{A_k\} \overset{[WN]_\theta^L}{\nsim} \{B_k\}$.

(ii) Suppose that $\{A_k\} \in L_\infty$ and $\{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\}$. Then we can assume that

$$\left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \leq M \qquad (28)$$

for each $x \in X$ and all k.

Given $\varepsilon > 0$, we get

$$\frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right|$$

$$= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |d(x,A_k)/d(x,B_k)-L| \geq \varepsilon}} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \qquad (29)$$

$$+ \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |d(x,A_k)/d(x,B_k)-L| < \varepsilon}} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right|$$

$$\leq \frac{M}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| + \varepsilon.$$

Therefore $\{A_k\} \overset{[WN]_\theta^L}{\sim} \{B_k\}$.

(iii) This is an immediate consequences of (i)and (ii). □

Theorem 16. *Let (X, ρ) be a metric space and let A_k, B_k be non-empty closed subsets of X. If $\theta = \{k_r\}$ is a lacunary sequence with $\liminf_r q_r > 1$, then*

$$\{A_k\} \overset{WS_L}{\sim} \{B_k\} \Longrightarrow \{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\}. \qquad (30)$$

Proof. Suppose first that $\liminf_r q_r > 1$, then there exists a $\lambda > 0$ such that $q_r \geq 1 + \lambda$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\lambda}{1 + \lambda}. \qquad (31)$$

If $\{A_k\} \overset{WS_L}{\sim} \{B_k\}$, then for every $\varepsilon > 0$, for each $x \in X$, and for sufficiently large r, we have

$$\frac{1}{k_r} \left| \left\{ k \leq k_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right|$$

$$\geq \frac{1}{k_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \qquad (32)$$

$$\geq \frac{\lambda}{1 + \lambda} \cdot \left(\frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \right).$$

This completes the proof. □

Theorem 17. *Let (X, ρ) be a metric space and let A_k, B_k be non-empty closed subsets of X. If $\theta = \{k_r\}$ is a lacunary sequence with $\limsup_r q_r < \infty$, then*

$$\{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\} \Longrightarrow \{A_k\} \overset{WS_L}{\sim} \{B_k\}. \qquad (33)$$

Proof. Let $\limsup_r q_r < \infty$. Then there is an $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $\{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\}$ and $\varepsilon > 0$. There exists $R > 0$ such that for every $j \geq R$

$$A_j = \frac{1}{h_j} \left| \left\{ k \in I_j : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| < \varepsilon. \qquad (34)$$

We can also find $H > 0$ such that $A_j < H$ for all $j = 1, 2, \ldots$. Now let t be any integer satisfying $k_{r-1} < t \le k_r$, where $r > R$.

Then we can write

$$
\frac{1}{t} \left| \left\{ k \le t : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right|
$$

$$
\le \frac{1}{k_{r-1}} \left| \left\{ k \le k_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right|
$$

$$
= \frac{1}{k_{r-1}} \left\{ \left| \left\{ k \in I_1 : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right| \right\}
$$

$$
+ \frac{1}{k_{r-1}} \left\{ \left| \left\{ k \in I_2 : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right| \right\}
$$

$$
+ \cdots + \frac{1}{k_{r-1}}
$$

$$
\times \left\{ \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right| \right\}
$$

$$
= \frac{k_1}{k_{r-1} k_1} \left| \left\{ k \in I_1 : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right|
$$

$$
+ \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}
$$

$$
\times \left| \left\{ k \in I_2 : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right|
$$

$$
+ \cdots + \frac{k_R - k_{R-1}}{k_{r-1}(k_R - k_{R-1})}
$$

$$
\times \left| \left\{ k \in I_R : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right|
$$

$$
+ \cdots + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})}
$$

$$
\times \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \ge \varepsilon \right\} \right|
$$

$$
= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}} A_R
$$

$$
+ \frac{k_{R+1} - k_R}{k_{r-1}} A_{R+1} + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}} A_r
$$

$$
\le \left\{ \sup_{j \ge 1} A_j \right\} \frac{k_R}{k_{r-1}} + \left\{ \sup_{j \ge B} A_j \right\} \frac{k_r - k_R}{k_{r-1}}
$$

$$
\le H \frac{k_B}{k_{r-1}} + \varepsilon M.
$$

$$(35)$$

This completes the proof. □

Combining Theorems 16 and 17 we have the following.

Theorem 18. *Let (X, ρ) be a metric space and let A_k, B_k be non-empty closed subsets of X. If $\theta = \{k_r\}$ is a lacunary sequence with $1 < \liminf_r q_r \le \limsup_r q_r < \infty$, then*

$$
\{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\} = \{A_k\} \overset{WS_L}{\sim} \{B_k\}. \tag{36}
$$

Proof. This is an immediate consequence of Theorems 16 and 17. □

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