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LACUNARY STATISTICAL SUMMABILITY OF SEQUENCES OF SETS

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Abstract. In this paper we define the $WS_\theta-$analog of the Cauchy criterion for convergence and show that it is equivalent to Wijsman lacunary statistical convergence. Also, Wijsman lacunary statistical convergence is compared to other summability methods which are defined in this paper. After giving new definitions for convergence, we prove a result comparing them. In addition, we give the relationship between Wijsman lacunary statistical convergence and Hausdorf lacunary statistical convergence.

1. INTRODUCTION AND BACKGROUND

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [5] and Schoenberg [11]. The concept of lacunary statistical convergence and summability were defined by Fridy and Orhan in [7, 8].

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see, [1],[2],[3],[4],[9],[12],[13],[14]). Nuray and Rhoades [9] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [12] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades.

In this paper, we shall define the concept of Wijsman lacunary statistical Cauchy sequences for sequences of sets and show that this concept is equivalent to the concept of Wijsman lacunary statistically convergence. Also, Wijsman lacunary statistical convergence will be compared to newly defined Wijsman lacunary summability methods. Further, the definition of Wijsman lacunary almost convergence for sequences of sets is introduced and some comparison theorems are given.

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2. DEFINITIONS AND NOTATIONS

Now, we recall the concept of statistical, lacunary statistical, Wijsman, Hausdorff, Wijsman statistiscal, Hausdorff statistical, Wijsman strongly almost, Wijsman almost statistical, Wijsman lacunary statistical convergence, Wijsman lacunary summability, Wijsman strongly lacunary summability and Wijsman Cesàro summability of the sequences of sets (see, [2],[6],[7],[9],[12])

Definition 2.1. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $st - \lim x_k = L$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Definition 2.2. A sequence $x = (x_k)$ is said to be lacunary statistically convergent to the number L if for every $\varepsilon > 0$,

$$\lim_{r} \frac{1}{h_r}|\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0.$$

In this case, we write $S_\theta - \lim x_k = L$ or $x_k \to L(S_\theta)$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Definition 2.3. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k\to\infty} d(x, A_k) = d(x, A),$$

for each $x \in X$. In this case, we write $W - \lim A_k = A$.

As an example, consider the following sequence of circles in the (x, y)-plane:

$$A_k = \{(x, y) : x^2 + y^2 + 2kx = 0\}.$$

As $k \to \infty$ the sequence is Wijsman convergent to the y-axis $A = \{(x, y) : x = 0\}$.

Definition 2.4. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Hausdorff convergent to A if

$$\lim_{k\to\infty} \sup_{x \in X} |d(x, A_k) - d(x, A)| = 0.$$

In this case, we write $H - \lim A_k = A$.

The concepts of Wijsman statistical convergence and Hausdorff statistical convergence were given by Nuray and Rhoades [9] as follows:

Definition 2.5. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if $\{d(x, A_k)\}$ is statistically convergent to $d(x, A)$; i.e., for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}| = 0.$$

In this case, we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$.

Definition 2.6. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Hausdorff statistical convergent to A if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : \sup_{x\in X} |d(x, A_k) - d(x, A)| \ge \varepsilon\}| = 0.$$

In this case, we write $st - \lim_H A_k = A$ or $A_k \to A(HS)$.

Let (X, ρ) be a metric space. For any non-empty closed subsets A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$, for each $x \in X$.

Also, the concepts of Wijsman Cesàro Summability, Wijsman strongly almost convergence and Wijsman almost statistical convergence for sequences of sets were given by Nuray and Rhoades [9] as follows:

Definition 2.7. Let (X, ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman Cesàro summable to A if $\{d(x, A_k)\}$ is Cesàro summable to $d(x, A)$; i.e., for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) = d(x, A).$$

Definition 2.8. Let (X, ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly almost convergent to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)| = 0,$$

uniformly in i.

Definition 2.9. Let (X, ρ) be a metric space. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman almost statistically convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : |d(x, A_{k+i}) - d(x, A)| \ge \varepsilon\}| = 0,$$

uniformly in i.

The concepts of Wijsman lacunary summability, Wijsman strongly lacunary Summability and Wijsman lacunary statistical convergence of sequences of sets were given by Ulusu and Nuray [12] as follows:

Definition 2.10. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary summable to A, if $\{d(x, A_k)\}$ is lacunary summable to $d(x, A)$; i.e., for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} d(x, A_k) = d(x, A).$$

Definition 2.11. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly lacunary Summable to A, if $\{d(x, A_k)\}$ is strongly lacunary summable to $d(x, A)$; i.e., for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} |d(x, A_k) - d(x, A)| = 0.$$

Definition 2.12. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary statistical convergent to A, if $\{d(x, A_k)\}$ is lacunary statistically convergent to $d(x, A)$; i.e., for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r} \frac{1}{h_r} |k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon| = 0.$$

In this case, we write $S_\theta - \lim_W A_k = A$ or $A_k \to A(WS_\theta)$.

Example 2.1. Let $X = \mathbb{R}$ and we define a sequence $\{A_k\}$ as follows:

$$A_k := \begin{cases} \{x \in \mathbb{R} : 2 \leq x \leq k\} & , \quad \text{if } k \geq 2, \quad k_{r-1} < k \leq k_r \\ & \quad\ \ \text{and } k \text{ is a square integer,} \\ \{1\} & , \quad \text{otherwise.} \end{cases}$$

As $k \to \infty$ this sequence is Wijsman lacunary statistical converget to the set $A = \{1\}$.

3. MAIN RESULTS

Definition 3.1. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is said to be a Wijsman lacunary statistical Cauchy sequence if there is a subsequence $\{A_{k'(r)}\}$ of $\{A_k\}$ such that $k'(r) \in I_r$ for each r, $W - \lim_r A_{k'(r)} = A$, and for every $\varepsilon > 0$ and $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |d(x, A_k) - d(x, A_{k'(r)})| \geq \varepsilon\}| = 0. \tag{3.1}$$

Theorem 3.1. *Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. The sequence $\{A_k\}$ is Wijsman lacunary statistical convergent if and only if $\{A_k\}$ is a Wijsman lacunary statistical Cauchy sequence.*

Proof. $(\Rightarrow)$ Let $A_k \to A(WS_\theta)$ and write

$$K^{(j)} := \left\{ k \in \mathbb{N} : |d(x, A_k) - d(x, A)| < \frac{1}{j} \right\},$$

for each $x \in X$ and each $j \in \mathbb{N}$. Hence, for each $j, K^{(j)} \supseteq K^{(j+1)}$ and

$$\lim_{r\to\infty} \frac{|K^{(j)} \cap I_r|}{h_r} = 1.$$

Choose $m(1)$ such that $r \geq m(1)$ implies $\dfrac{|K^{(1)} \cap I_r|}{h_r} > 0$, i.e., $K^{(1)} \cap I_r \neq \emptyset$. Next choose $m(2) > m(1)$ so that $r \geq m(2)$ implies $K^{(2)} \cap I_r \neq \emptyset$. Then, for each r satisfying $m(1) \leq r < m(2)$, choose $k'(r) \in I_r$ such that $k'(r) \in I_r \cap K^{(1)}$, i.e., $|d(x, A_{k'(r)}) - d(x, A)| < 1$. In general, choose $m(p + 1) > m(p)$ such that

$r > m(p+1)$ implies $I_r \cap K^{(p+1)} \neq \emptyset$. Then, for all r satisfying $m(p) \leq r < m(p+1)$, choose $k'(r) \in I_r \cap K^{(p)}$, i.e.,

$$|d(x, A_{k'(r)}) - d(A, x)| < \frac{1}{p}. \tag{3.2}$$

Hence, we get $k'(r) \in I_r$ for every r and (3.2) implies that

$$W - \lim_r d(x, A_{k'(r)}) = d(x, A).$$

Furthermore, for every $\varepsilon > 0$ we have,

$$\begin{aligned}
&\frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A_{k'(r)})| \geq \varepsilon\}| \\
&\qquad\qquad \leq \frac{1}{h_r}|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \frac{\varepsilon}{2}\}| \\
&\qquad\qquad\quad + \frac{1}{h_r}|\{k \in I_r : |d(x, A_{k'(r)}) - d(x, A)| \geq \frac{\varepsilon}{2}\}|.
\end{aligned}$$

Using the assumptions that $A_k \to A(WS_\theta)$ and $W - \lim_r d(x, A_{k'(r)}) = d(x, A)$, we infer (3.1), whence A_k is a Wijsman lacunary statistical Cauchy sequence.

($\Leftarrow$) Conversely, suppose that $\{A_k\}$ is a Wijsman lacunary statistical Cauchy sequence. For every $\varepsilon > 0$, we have

$$\begin{aligned}
|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \quad &\leq \quad \left|\{k \in I_r : |d(x, A_k) - d(x, A_{k'(r)})| \geq \frac{\varepsilon}{2}\}\right| \\
&\quad + \left|\{k \in I_r : |d(x, A_{k'(r)}) - d(x, A)| \geq \frac{\varepsilon}{2}\}\right|
\end{aligned}$$

from which it follows that $A_k \to A(WS_\theta)$. □

Now we give following theorem where Δ denotes the forward difference operator defined by $\Delta d(x, A_i) = d(x, A_i) - d(x, A_{i+1})$.

Theorem 3.2. *Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. If $A_k \to A(WS_\theta)$ and for each $x \in X$*

$$\max\{|\Delta d(x, A_i)| : i \in I_r\} = o(\frac{1}{h_r}) \quad \text{as } r \to \infty,$$

then $W - \lim A_k = A$.

Proof. Assume that $A_k \to A(WS_\theta)$ and by Theorem (3.1), choose a subsequence $\{A_{k'(r)}\}$ of $\{A_k\}$ as in Definition (3.1). Since $k'(r) \in I_r$, for each $x \in X$ we have

$$\begin{aligned}
|d(x, A_k) - d(x, A_{k'(r)})| \quad &\leq \quad \sum_{i=k}^{k'(r)-1} |\Delta d(x, A_i)| \\
&\leq \quad h_r . (\max_{i \in I_r}\{|\Delta d(x, A_i)| : i \in I_r\}) \\
&= \quad o(1)
\end{aligned}$$

and therefore $A_{k'(r)} \to A(WS)$ implies that $A_k \to A(WS)$. □

Theorem 3.3. *Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. If $\{A_k\}$ is a bounded sequence and $A_k \to A(WS_\theta)$, then $\{A_k\}$ is Wijsman Cesàro summable to A.*

Proof. Let (X,ρ) be a metric space, $\theta=\{k_r\}$ be a lacunary sequence and let n be a positive integer with $n\in I_r$; then

$$\frac{1}{n}\sum_{k=1}^{n}\Big(d(x,A_k)-d(x,A)\Big) = \frac{1}{n}\sum_{p=1}^{r-1}\sum_{k\in I_p}\Big(d(x,A_k)-d(x,A)\Big) + \frac{1}{n}\sum_{k=1+k_{r-1}}^{n}\Big(d(x,A_k)-d(x,A)\Big). \tag{3.3}$$

Consider the first term on the right in (3.3),

$$\frac{1}{n}\sum_{p=1}^{r-1}\sum_{k\in I_p}\Big(d(x,A_k)-d(x,A)\Big) \leq \frac{1}{k_{r-1}}\sum_{p=1}^{r-1}\sum_{k\in I_p}\Big|d(x,A_k)-d(x,A)\Big| = \frac{1}{k_{r-1}}\sum_{p=1}^{r-1}h_p.t_p=(Ht)_r, \tag{3.4}$$

say, where

$$t_p=\frac{1}{h_p}\sum_{k\in I_p}|d(x,A_k)-d(x,A)|\,.$$

Since $\{A_k\}$ is bounded and $A_k\to A(WS_\theta)$, it follows from Theorem 1 (ii) of [12] that $t_p\to 0$. Moreover

$$k_{r-1}=\sum_{p=1}^{r-1}h_p\to\infty \qquad \text{as } r\to\infty,$$

because θ is a lacunary sequence, which implies that (3.4) is a regular weighted mean matrix transform of t in [10]; hence,

$$(Ht)_r\to 0. \tag{3.5}$$

Now consider the second term on the right in (3.3). Since $\{A_k\}$ is bounded, there is a constant $M>0$ such that $|d(x,A_k)-d(x,A)|\leq M$, for all k. Therefore, for every $\varepsilon>0$ we have,

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1+k_{r-1}}^{n}\Big(d(x,A_k)-d(x,A)\Big)\right| &\leq \frac{1}{n}\sum_{\substack{k_{r-1}<k\leq n\\ |d(x,A_k)-d(x,A)|\geq\varepsilon}}|d(x,A_k)-d(x,A)| \\ &\quad+\frac{1}{n}\sum_{\substack{k_{r-1}<k\leq n\\ |d(x,A_k)-d(x,A)|<\varepsilon}}|d(x,A_k)-d(x,A)| \\ &\leq \frac{M}{h_r}|\{k\in I_r: |d(x,A_k)-d(x,A)|\geq\varepsilon\}|+\varepsilon.\end{aligned} \tag{3.6}$$

Since $A_k\to A(WS_\theta)$ and ε is an arbitrary, the expression on the left side of (3.6) tends to zero as $r\to\infty$. Hence, (3.3), (3.5) and (3.6) imply that $\{A_k\}$ is Wijsman Cesàro summable to A. □

Definition 3.2. Let (X,ρ) be a metric space and $\theta=\{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k\subseteq X$, we say that $\{A_k\}$ is Wijsman strongly $p-$lacunary summable to A if $\{d(x,A_k)\}$ is strongly $p-$lacunary summable to $d(x,A)$; i.e., for each p positive real number and for each $x\in X$

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{I_r}|d(x,A_k)-d(x,A)|^p=0.$$

Theorem 3.4. *Let (X,ρ) be a metric space, $\theta=\{k_r\}$ be a lacunary sequence and let p positive real number. Then, for any non-empty closed subsets A, $A_k\subseteq X$;*

(i) *$\{A_k\}$ is Wijsman lacunary statistical convergent to A if it is Wijsman strongly $p-$lacunary summable to A.*

(ii) *If $\{A_k\}$ is bounded and Wijsman lacunary statistical convergent to A then it is Wijsman strongly $p-$lacunary summable to A.*

Proof. (i) For any $\{A_k\}$, fix an $\varepsilon>0$. Then

$$\sum_{I_r}|d(x,A_k)-d(x,A)|^p\geq\varepsilon^p|\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\varepsilon\}|,$$

and it follows that if $\{A_k\}$ is Wijsman strongly $p-$lacunary summable to A then $\{A_k\}$ is Wijsman lacunary statistical convergent to A.

(ii) Let $\{A_k\}$ be bounded and Wijsman lacunary statistical convergent to A. Since $\{A_k\}$ is bounded set

$$\sup_k\{d(x,A_k)\}+d(x,A)=M.$$

Since $\{A_k\}$ is Wijsman lacunary statistically convergent to A, for given $\varepsilon>0$ we can select N_ε such that for each $x\in X$

$$\frac{1}{h_r}\left|\left\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}\right|<\frac{\varepsilon}{2M^p},$$

for all $r>N_\varepsilon$ and we let the set

$$L_r=\left\{k\in I_r:|d(x,A_k)-d(x,A)|\geq\left(\frac{\varepsilon}{2}\right)^{\frac{1}{p}}\right\}.$$

Then, for each $x\in X$

$$\begin{aligned}\frac{1}{h_r}\sum_{I_r}|d(x,A_k)-d(x,A)|^p &= \frac{1}{h_r}\Bigg(\sum_{\substack{k\in I_r\\ k\in L_r}}|d(x,A_k)-d(x,A)|^p\\ &\qquad+\sum_{\substack{k\in I_r\\ k\notin L_r}}|d(x,A_k)-d(x,A)|^p\Bigg)\\ &< \frac{1}{h_r}.\frac{h_r.\varepsilon}{2M^p}M^p+\frac{1}{h_r}.\frac{h_r.\varepsilon}{2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Hence, $\{A_k\}$ is Wijsman strongly $p-$lacunary summable to A. $\square$

Definition 3.3. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary almost convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} d(x, A_{k+i}) = d(x, A),$$

uniformly in i.

Definition 3.4. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary strongly almost convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} |d(x, A_{k+i}) - d(x, A)| = 0,$$

uniformly in i.

Example 3.1. Let $X = \mathbb{R}^2$ and we define a sequence $\{A_k\}$ as follows:

$$A_k := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y+1)^2 = \dfrac{1}{k}\right\} & , \quad \text{if} \quad k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ \\ \{(1,0)\} & , \quad \text{otherwise.} \end{cases}$$

As $k \to \infty$ this sequence is Wijsman lacunary strongly almost convergent to the set $A = \{(1,0)\}$.

Definition 3.5. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman lacunary strongly $p-$almost convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} |d(x, A_{k+i}) - d(x, A)|^p = 0,$$

uniformly in i, where p is a positive real number.

Definition 3.6. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary almost statistically convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |d(x, A_{k+i}) - d(x, A)| \geq \varepsilon| = 0,$$

uniformly in i.

Let L_∞, C, $(WAC)_\theta$ and $|WAC|_\theta$, respectively, denote the sets of the all bounded, Wijsman convergent, Wijsman lacunary almost convergent and Wijsman lacunary strongly almost convergent sequences of sets. It is easy to see that

$$C \subset (WAC)_\theta \subset |WAC|_\theta \subset L_\infty.$$

Theorem 3.5. *Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence and p be a positive number. Then, for any non-empty closed subsets A, $A_k \subseteq X$,*

(i) *$\{A_k\}$ is Wijsman lacunary almost statistically convergent to A, if it is Wijsman lacunary strongly $p-$almost converget to A,*

(ii) *If $\{A_k\}$ is bounded and Wijsman lacunary almost statistically convergent to A, then it is Wijsman lacunary strongly $p-$almost convergent to A.*

Proof. The proof is similar to the proof of Theorem (3.4). □

Definition 3.7. Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Hausdorff lacunary statistically convergent to A, if for each $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : \sup_{x\in X} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\} \right| = 0$$

i.e.,

$$\sup_{x\in X} |d(x, A_k) - d(x, A)| < \varepsilon \qquad \text{a.a.k.}$$

in this case, we write $HS_\theta - \lim A_k = A,\ S_\theta - \lim_H A_k = A,\ A_k \to A(HS_\theta)$.

Theorem 3.6. *Let (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary and $\{A_k\}$ be a sequence of non-empty closed subsets of X. If $\{A_k\}$ is Hausdorff lacunary statistical converget, then $\{A_k\}$ is Wijsman lacunary statistical convergent.*

Proof. For any sequence $\{A_k\}$ and for every $\varepsilon > 0$, since

$$|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \leq \left| \left\{ k \in I_r : \sup_{x\in X} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\} \right|$$

we get the result. □

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