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Multipliers for Bounded Statistical Convergence of Double Sequences

Erdinç Dündar

fethi gemuhluoglu anadolu ögretmen lisesi Malatya, Turkey
erdincdundar79@gmail.com.tr, erdinc.dundar@inonu.edu.tr

Yurdal Sever

Kocatepe Universitesi, Afyon, Turkey
yurdalsever@hotmail.com, ysever@aku.edu.tr

Abstract. Multipliers and factorizations for bounded statistically convergent sequences were studied in μ-density by Connor et al. [J. Connor, K. Demirci, C. Orhan, Multipliers and factorizations for bounded statistically convergent sequences, Analysis 22 (2002), 321-333]. In this paper we get analogous results of multipliers for bounded statistically convergent double sequences.

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1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [22]. A lot of development have been made in this area after the works of Śalát [21] and Fridy [13, 14]. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [11, 13, 14, 19]. This concept was extended to the double sequences by Mursaleen and Edely [17]. akan and Altay [6] presented multidimensional analogues of the results of Fridy and Orhan [15]. Das and Bhunia [7] extended the idea of statistical convergence of a double sequence to μ-statistical convergence and convergence in μ-density using a two valued measure μ.

The study of the multipliers of one sequence space into another is a well-established area of research and has been the object of several investigations over the last fifty years. Demirci and Orhan [9] studied the bounded multiplier space of all bounded A-statistically convergent sequences, and using the

"βN program" they gave an analogue of a result of Fridy and Miller [12] for bounded multipliers. Connor, Demirci and Orhan [2] studied multipliers and factorizations for bounded statistically convergent sequences and a related result in μ-density. Yardımcı [23] studied multipliers for bounded $\mathcal{I}$-convergent sequences. Also, Dündar and Altay [10] investigated analogous results of multipliers for bounded $\mathcal{I}_2$-convergent double sequences.

In this paper we study multipliers for bounded statistical convergence of double sequences in μ_2-density.

2. Definitions and Notations

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers, χ_A-the characteristic function of $A \subset \mathbb{N}$, $\mathbb{R}$ the set of all real numbers. We often regard χ_A as sequence (x_{mn}), where $x_{mn} = \chi_A(m,n)$, $A \subset \mathbb{N} \times \mathbb{N}$; note in particular, that e can be regarded as the double sequence of all 1's.

Now, we recall the concepts of convergence, statistical and ideal convergence of the sequences (See [2, 7, 10, 11, 17, 20]).

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be (Pringsheim) convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$|x_{mn} - L| < \varepsilon,$$

whenever $m, n > N_\varepsilon$. In this case we write

$$P - \lim_{m,n\to\infty} x_{mn} = L \quad or \quad \lim_{m,n\to\infty} x_{mn} = L.$$

By ℓ^2_∞, $c^2(b)$ and $c^2_0(b)$ we denote the space of all bounded, bounded convergent and bounded null double sequences, respectively.

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j,k) \in K$ such that $j \le m$, $k \le n$. If the sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$d_2(K_{mn}) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \ge \varepsilon\}$. In this case we write

$$st_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

Throughout the paper μ_2 will denote a complete $\{0,1\}$ valued finite additive measure defined on an algebra Γ of subsets of $\mathbb{N}\times\mathbb{N}$ that contains all subsets of $\mathbb{N}\times\mathbb{N}$ that are contained in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$ and $\mu_2(A) = 0$ if A is contained in the union of a finite number of rows and columns of $\mathbb{N} \times \mathbb{N}$.

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be μ_2-statistically convergent to $L \in \mathbb{R}$ if and only if for any $\varepsilon > 0$,

$$\mu_2\big(\{(m,n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}\big) = 0.$$

In this case we write

$$st_{\mu_2} - \lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in μ_2-density, if there exists an $A \in \Gamma$ with $\mu_2(A) = 1$ such that $x = (x_{mn})_{(m,n)\in A}$ is convergent to L.

If C_{μ_2} and $C^*_{\mu_2}$ denote respectively the sets of all double sequences which are μ_2-statistically convergent and convergent in μ_2-density then as in [5] (see also [8]) it is easy to prove that $C^*_{\mu_2}$ is a dense subset of C_{μ_2} which again is closed in ℓ^2_∞ (the set of all bounded double sequences of real numbers endowed with the sup metric). Further, following the methods of [5] one can easily verify that there exists a measure μ_2 such that it is always possible to construct a double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ which is μ_2-statistically convergent but does not converge to any point in μ_2-density.

The measure μ_2 is said to satisfy the condition $(APO2)$ if for every sequence $\{A_i\}_{i\in\mathbb{N}}$ of mutually disjoint μ_2-null sets (i.e. $\mu_2(A) = 0$ for all $i \in \mathbb{N}$) there exists a countable family of sets $\{B_i\}_{i\in\mathbb{N}}$ such that $A_i \Delta B_i$ is included in the union of a finite number rows and columns of $\mathbb{N} \times \mathbb{N}$ for every $i \in \mathbb{N}$ and $\mu_2(B) = 0$, where $B = \bigcup_{i\in\mathbb{N}} B_i$ (hence $\mu_2(B_i) = 0$ for every $i \in \mathbb{N}$).

Now, we give definition of multiplier for double sequences.

Let E and F be two double sequence spaces. A multiplier from E into F is a sequence $u = (u_{mn})_{m,n\in\mathbb{N}}$ such that

$$ux = (u_{mn}x_{mn}) \in F$$

whenever $x = (x_{mn})_{m,n\in\mathbb{N}} \in E$. The linear space of all such multipliers will be denoted by $m(E,F)$. Bounded multipliers will be denoted by $M(E,F)$. Hence we write

$$M(E,F) = \ell^2_\infty \cap m(E,F).$$

If $E = F$, then we write $m(E)$ and $M(E)$ instead of $m(E,F)$ and $M(E,F)$, respectively.

Now we begin with quoting the lemmas due to Dündar, Altay [10] and Das, Bhunia [7] which are needed throughout the paper.

Lemma 2.1. [10, Theorem 3.2] *If E and F are subspaces of ℓ^2_∞ that contain $c^2_0(b)$, then $c^2_0(b) \subset m(E,F) \subset \ell^2_\infty$.*

Lemma 2.2. [10, Lemma 3.4] $m\big(c^2_0(b)\big) = \ell^2_\infty$.

Lemma 2.3. [7, Theorem 1] *$C_{\mu_2} = C^*_{\mu_2}$ if μ_2 satisfies the condition $(APO2)$.*

Lemma 2.4. [7, Theorem 2] *If $C_{\mu_2} = C^*_{\mu_2}$ for a measure μ_2, then μ_2 has the condition $(APO2)$.*

3. Main Results

In this section, we deal with the multipliers on or into $st_{\mu_2}(b)$ and $st^0_{\mu_2}(b)$. By st_{μ_2}, $st^0_{\mu_2}$, $st_{\mu_2}(b)$ and $st^0_{\mu_2}(b)$ we denote the sets of all μ_2-statistically convergent double sequences, μ_2-statistically null double sequences, bounded μ_2-statistically convergent double sequences and bounded μ_2-statistically null double sequences, respectively.

Theorem 3.1. *Let μ_2 be an arbitrary density. Then,*

$$m\big(st^0_{\mu_2}(b)\big) = M\big(st^0_{\mu_2}(b)\big) = \ell^2_\infty.$$

Proof. We show that

$$m\big(st^0_{\mu_2}(b)\big) = \ell^2_\infty.$$

By Lemma 2.1, the inclusion

$$m\big(st^0_{\mu_2}(b)\big) \subset \ell^2_\infty$$

holds.

Now, we prove that

$$\ell^2_\infty \subset m\big(st^0_{\mu_2}(b)\big).$$

Let $u \in \ell^2_\infty$ and $z \in st^0_{\mu_2}(b)$. Then for $\varepsilon > 0$, we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |u_{mn}z_{mn}| \geq \varepsilon\} \subseteq \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : |z_{mn}| \geq \frac{\varepsilon}{\|u\|_\infty + 1}\right\}.$$

Since

$$\mu_2\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : |z_{mn}| \geq \frac{\varepsilon}{\|u\|_\infty + 1}\right\} = 0,$$

so we can write

$$\mu_2\Big\{(m,n) \in \mathbb{N}\times\mathbb{N} : |u_{mn}z_{mn}| \geq \varepsilon\Big\} = 0.$$

Also, since $u, z \in \ell^2_\infty$ so uz is bounded and hence

$$\ell^2_\infty \subset m\big(st^0_{\mu_2}(b)\big).$$

This completes the proof of theorem. □

Combining Lemma 2.1 and Lemma 2.2, we have the following theorem.

Theorem 3.2. *Let μ_2 be an arbitrary density. Then,*

$$m(c^2_0(b), st^0_{\mu_2}(b)) = \ell^2_\infty.$$

Theorem 3.3. *Let μ_2 be an arbitrary density. Then,*

$$c^2_0(b) \subset m\big(st_{\mu_2}(b), c^2(b)\big) \subseteq c^2(b).$$

Proof. For $u \in c_0^2(b)$ and $x \in st_{\mu_2}(b) \subset \ell_\infty^2$, by Lemma 2.2 since

$$ux \in c_0^2(b) \subset c^2(b),$$

so we have

$$c_0^2(b) \subset m\big(st_{\mu_2}(b), c^2(b)\big).$$

Let $u \in m\big(st_{\mu_2}(b), c^2(b)\big)$. Since $e \in st_{\mu_2}(b)$, so

$$ue = u \in c^2(b)$$

and we have

$$m\big(st_{\mu_2}(b), c^2(b)\big) \subseteq c^2(b).$$

Thus, the proof of theorem is completed. □

Theorem 3.4. *Let μ_2 be an arbitrary density. If $c^2(b)$ is a proper subset of $st_{\mu_2}(b)$, then*

$$m\big(st_{\mu_2}(b), c^2(b)\big) = c_0^2(b)$$

Proof. By Theorem 3.3, we know that

$$c_0^2(b) \subset m\big(st_{\mu_2}(b), c^2(b)\big).$$

We show that

$$u \notin m\big(st_{\mu_2}(b), c^2(b)\big)$$

for $u \in c^2(b) \backslash c_0^2(b)$. Then, there exists a number l such that

$$\lim_{m,n\to\infty} u_{mn} = l \neq 0.$$

Let $z \in st_{\mu_2}(b) \setminus c^2(b)$, and, without loss of generality, suppose z is μ_2-statistically convergent to 1. Then, there is an $\varepsilon > 0$ such that

$$A = \{(m,n) : |z_{mn} - 1| \geq \varepsilon\}.$$

Note that $\mu_2(A) = 0$.

Define $x = (x_{mn})$ by

$$x_{mn} = \chi_{A^c}(m,n)$$

and observe that x is bounded and convergent in μ_2-density to 1, hence

$$x \in st_{\mu_2}(b).$$

Also note ux converges to $\ell \neq 0$ along A^c and to 0 along A, hence $ux \notin c^2(b)$ and thus

$$u \notin m(st_{\mu_2}(b), c^2(b)).$$

Hence, we have

$$m(st_{\mu_2}(b), c^2(b)) \subset c_0^2(b).$$

□

Theorem 3.5. *Let μ_2 be a density with condition $(APO2)$. Then*

$$m(st^0_{\mu_2}(b), c_0^2(b)) = \{u \in \ell^2_\infty : u\chi_E \in c_0^2(b) \textit{ for all } E \textit{ such that } \mu_2(E) = 0\}.$$

Proof. Let $K = \{u \in \ell^2_\infty : u\chi_E \in c^2_0(b)$ for all E such that $\mu_2(E) = 0\}$. First note that if $\mu_2(E) = 0$, then

$$\chi_E \in st^0_{\mu_2}(b)$$

and hence, if $u \in m(st^0_{\mu_2}(b), c^2_0(b))$ then

$$u\chi_E \in c^2_0(b)$$

or u goes to 0 along E. Thus, we have

$$m(st^0_{\mu_2}(b), c^2_0(b)) \subseteq K.$$

Now, let $u \in K$ and $x \in st^0_{\mu_2}(b)$. Then, with condition $(APO2)$ by Lemma 2.3 there is an $A \subseteq \mathbb{N} \times \mathbb{N}$ such that

$$x\chi_{A^c} \in c^2_0(b) \ \ and \ \ \mu_2(A) = 0.$$

With condition $(APO2)$ by Lemma 2.3, as

$$ux = ux\chi_{A^c} + ux\chi_A$$

and both terms of the right hand side are null sequences, $ux \in c^2_0(b)$. Thus we have

$$K \subseteq m(st^0_{\mu_2}(b), c^2_0(b)).$$

This completes the proof of theorem. □

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