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ROUGH CONVERGENCE OF DOUBLE SEQUENCES

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Abstract. In this paper, we introduce the notion of rough convergence and the set of rough limit points of a double sequence and obtained two rough convergence criteria associated with this set. Later, we proved that this set is closed and convex. Finally, we examined the relations between the set of cluster points and the set of rough limit points of a double sequence.

1. Introduction and preliminaries

The well-know Pringsheim [18] convergence of double sequences is defined as the convergence of nets, where the set of indexes $\mathbb{N}\times\mathbb{N}$ is ordered in the natural way. The main drawback of this convergence is that a convergent double sequence fails in general to be bounded. The notion of regular convergence introduced by Hardy [10] lacks this disadvantage. In addition to the Pringsheim convergence the regular convergence requires the convergence of rows and columns of a double sequence. These two most important kinds of convergence and some related notions were considered in the classical works of Robison [20], Kojima [12] and Hamilton [9] in connection with maps defined by 4-dimensional matrices.

Nowadays double sequence has become one of the most active area of research in the field of summability. Altay and Başar [3] defined some new spaces of double sequences and also examined some properties of those sequence spaces. A lot of developments have been made in this area after the works of [4, 5, 6, 14].

The idea of rough convergence was first introduced by Phu [15] in finite-dimensional normed spaces. He showed that the set $\mathtt{LIM}^r x$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\mathtt{LIM}^r x$ on the roughness degree r. Phu [16] defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r -continuous at every point $x \in X$ under the assumption $\dim Y < \infty$ and $r > 0$ where X and Y are normed spaces. He [17] extended the results given in [15] to infinite-dimensional normed spaces.

Aytar [1] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also,

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Aytar [2] studied that the r-limit set of the sequence is equal to the intersection of these sets and that r-core of the sequence is equal to the union of these sets. Dündar and Çakan [7] investigated of rough ideal convergence and defined the set of rough ideal limit points of a sequence.

In this paper, we introduce the notion of rough convergence and the set of rough limit points of a double sequence and obtained two rough convergence criteria associated with this set. Later, we proved that this set is closed and convex. Finally, we examined the relations between the set of cluster points and the set of rough limit points of a double sequence.

We note that our results and proof techniques presented in this paper are analogues of those in Phu's [15] paper and Aytar's [1] paper. Namely, the actual origin of most of these results and proof techniques is them papers.

2. Definitions and notations

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively.

Now, we recall the concepts of convergence of the double sequences, rough convergence of the sequences (see [1, 3, 7, 15, 19]).

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense (shortly, p-convergent to $L \in \mathbb{R}$), if for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

Throughout the paper, let r be a nonnegative real number.

Let $x = (x_n)$ be a sequence in some normed linear space $(X, \|.\|)$. $x = (x_n)$ is said to be rough convergent (r-convergent) to x_*, denoted by $x_n \xrightarrow{r} x_*$ if

$$\forall\varepsilon > 0 \ \exists n_\varepsilon \in \mathbb{N} : \ \ n \geq n_\varepsilon \Rightarrow \|x_n - x_*\| < r + \varepsilon, \tag{2.1}$$

or equivalently, if

$$\limsup \|x_n - x_*\| \leq r. \tag{2.2}$$

The set

$$\text{LIM}^r x_n := \{x_* \in \mathbb{R}^n : x_n \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_n)$.

A sequence $x = (x_n)$ is said to be r-convergent if $\text{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_n)$. For $r = 0$, we get the ordinary convergence.

Throughout the paper, $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a double sequence $x = (x_{mn})$ such that $(x_{mn}) \in \mathbb{R}^n$, $m, n \in \mathbb{N}$.

3. Main Results

Definition 3.1. *The double sequence $x = (x_{mn})$ is said to be rough convergent (r-convergent) in Prinsgheim sense to x_*, denoted by $x_{mn} \xrightarrow{r} x_*$ provided that*

$$\forall \varepsilon > 0 \ \exists k_\varepsilon \in \mathbb{N} : \ \ m, n \geq k_\varepsilon \Rightarrow \|x_{mn} - x_*\| < r + \varepsilon, \tag{3.1}$$

or equivalently, if

$$\limsup \|x_{mn} - x_*\| \leq r. \tag{3.2}$$

Here r is called the roughness degree. If we take $r = 0$, then we obtain the ordinary convergence of a double sequence. Assume that a double sequence $y = (y_{mn})$ is convergent and cannot be measured or calculated exactly; one has to do with an approximated sequence $x = (x_{mn})$ satisfying $\|x_{mn} - y_{mn}\| \leq r$ for all m, n where $r > 0$ is an upper bound of approximation error. Then, the sequence $x = (x_{mn})$ is no more convergent in the Prindsheim's sense, but

$$\|x_{mn} - x_*\| \leq \|x_{mn} - y_{mn}\| + \|y_{mn} - x_*\| \leq r + \|y_{mn} - x_*\| \tag{3.3}$$

implies that it is r-convergent in the sense of (3.1).

If (3.1) holds, x_* is an rough limit point of $x = (x_{mn})$, which is usually no more unique (for $r > 0$). So we have to consider the so-called rough limit set (or shortly: r-limit) of $x = (x_{mn})$ defined by

$$\text{LIM}^r x_{mn} := \{x_* \in \mathbb{R}^n : x_{mn} \xrightarrow{r} x_*\}. \tag{3.4}$$

A double sequence $x = (x_{mn})$ is said to be r-convergent if $\text{LIM}^r x_{mn} \neq \emptyset$. In this case, r is called a convergence degree of $x = (x_{mn})$.

As noted above, we cannot say that the r-limit of a double sequence is unique for the roughness degree $r > 0$. The following result is related to the this fact.

Theorem 3.2. *For a double sequence $x = (x_{mn})$, we have $\text{diam}(\text{LIM}^r x_{mn}) \leq 2r$. In general, $\text{diam}(\text{LIM}^r x_{mn})$ has no smaller bound.*

Proof. Assume that

$$\text{diam}(\text{LIM}^r x_{mn}) = \sup\{\|y - z\| : y, z \in \text{LIM}^r x_{mn}\} > 2r.$$

Then, there exist $y, z \in \text{LIM}^r x_{mn}$ such that $\|y - z\| > 2r$. Take $\varepsilon \in (0, \frac{\|y-z\|}{2} - r)$. Because $y, z \in \text{LIM}^r x_{mn}$, it follows from (3.1) and (3.4) that there is a $k = k_\varepsilon \in \mathbb{N}$ such that

$$\|x_{mn} - y\| < r + \varepsilon \quad \text{and} \quad \|x_{mn} - z\| < r + \varepsilon, \text{ for } m, n \geq k.$$

This implies

$$\|y - z\| \leq \|x_{mn} - y\| + \|x_{mn} - z\| < 2(r + \varepsilon) < 2(r + \frac{\|y - z\|}{2} - r) = \|y - z\|,$$

This is a contradiction. Hence, $\text{diam}(\text{LIM}^r x_{mn}) \leq 2r$.

Now, consider a double sequence $x = (x_{mn})$ such that

$$\lim_{m,n\to\infty} x_{mn} = x_*.$$

Let $\varepsilon > 0$. Then, it follows from

$$\|x_{mn} - y\| \leq \|x_{mn} - x_*\| + \|x_* - y\| \leq \|x_{mn} - x_*\| + r$$

and so

$$\|x_{mn} - y\| \leq r + \varepsilon,$$

for $y \in \overline{B_r}(x_*) := \{y \in \mathbb{R}^n : \|y - x_*\| \leq r\}$. By (3.1) and (3.4) we have $\mathrm{LIM}^r x_{mn} = \overline{B_r}(x_*)$. Because $\mathrm{diam}(\overline{B_r}(x_*)) = 2r$, this shows that in general, the upper bound $2r$ of the diameter of the set $\mathrm{LIM}^r x_{mn}$ cannot be decreased anymore. □

Now, we give the topological and geometrical properties of the r-limit set of a double sequence.

Theorem 3.3. *The r-limit set of an arbitrary double sequence $x = (x_{mn})$ is closed*

Proof. If $\mathrm{LIM}^r x_{mn} = \emptyset$, then there is nothing to prove. Assume that $\mathrm{LIM}^r x_{mn} \neq \emptyset$. Then, we can choose a sequence $(y_{mn}) \subseteq \mathrm{LIM}^r x_{mn}$ such that $y_{mn} \to y_*$ for $m, n \to \infty$. We will show that $y_* \in \mathrm{LIM}^r x_{mn}$.

Let $\varepsilon > 0$ be given. Because $y_{mn} \to y_*$, there exists $k = k_\varepsilon \in \mathbb{N}$ such that

$$\|y_{mn} - y_*\| < \varepsilon, \quad \text{for all } m, n \geq k.$$

Now choose an $m_0, n_0 \in \mathbb{N}$ such that $m_0, n_0 \geq k$. Then we can write

$$\|y_{m_0 n_0} - y_*\| < \varepsilon.$$

On the other hand, because $(y_{mn}) \subseteq \mathrm{LIM}^r x_{mn}$, we have $y_{m_0 n_0} \in \mathrm{LIM}^r x$, namely,

$$\|x_{mn} - y_{m_0 n_0}\| < r + \varepsilon.$$

Consequently,

$$\|x_{mn} - y_*\| \leq \|x_{mn} - y_{m_0 n_0}\| + \|y_{m_0 n_0} - y_*\| < r + 2\varepsilon, \ \ if \ \ m, n, m_0, n_0 \geq k.$$

This implies that $y_* \in \mathrm{LIM}^r x_{mn}$. □

Theorem 3.4. *The r-limit set of a double sequence $x = (x_{mn})$ is convex.*

Proof. Assume that $y_0, y_1 \in \mathrm{LIM}^r x_{mn}$ for the double sequence $x = (x_{mn})$. For every $\varepsilon > 0$ there exists a $k = k_\varepsilon \in \mathbb{N}$ such that

$$\|x_{mn} - y_0\| < r + \varepsilon \text{ and } \|x_{mn} - y_1\| < r + \varepsilon,$$

whenever $m, n \geq k$. Thus, we have

$$\begin{aligned} \|x_{mn} - [(1-\lambda)y_0 + \lambda y_1]\| &= \|(1-\lambda)(x_{mn} - y_0) + \lambda(x_{mn} - y_1)\| \\ &< r + \varepsilon, \end{aligned}$$

for $m, n \geq k$ and for $\lambda \in [0, 1]$.This implies that

$$(1-\lambda)y_0 + \lambda y_1 \in \mathrm{LIM}^r x_{mn},$$

for $\lambda \in [0, 1]$, so $\mathrm{LIM}^r x_{mn}$ is convex. □

Theorem 3.5. *Let $r > 0$. Then a double sequence $x = (x_{mn})$ is r-convergent to x_* if and only if there exists a double sequence $y = (y_{mn})$ such that*

$$y_{mn} \to x_* \text{ and } \|x_{mn} - y_{mn}\| \leq r, \text{ for each } m, n \in \mathbb{N}. \tag{3.5}$$

Proof. Assume that $x_{mn} \xrightarrow{r} x_*$. Then, by (3.2) we have

$$\limsup \|x_{mn} - x_*\| \leq r. \tag{3.6}$$

Now, define

$$y_{mn} := \begin{cases} x_* & , \ if \ \ \|x_{mn} - x_*\| \leq r \\ x_{mn} + r\frac{x_* - x_{mn}}{\|x_{mn} - x_*\|} & , \ otherwise \end{cases}.$$

Then, we can write

$$\|y_{mn} - x_*\| = \begin{cases} 0 & , \ if \ \ \|x_{mn} - x_*\| \leq r \\ \|x_{mn} - x_*\| - r & , \ otherwise \end{cases}$$

and by definition of y_{mn}, we have

$$\|x_{mn} - y_{mn}\| \leq r, \tag{3.7}$$

for all $m, n \in \mathbb{N}$. By (3.6) and by definition of y_{mn}, we get

$$\limsup \|y_{mn} - x_*\| = 0.$$

This implies that $y_{mn} \to x_*$.

Assume that (3.5) holds. Because $y_{mn} \to x_*$, then for all $\varepsilon > 0$ there exists $k = k_\varepsilon \in \mathbb{N}$ such that

$$\|y_{mn} - x_*\| < \varepsilon, \ \ for \ \ m, n \geq k.$$

Since $\|x_{mn} - y_{mn}\| \leq r$, this immediately

$$\|x_{mn} - x_*\| \leq \|x_{mn} - y_{mn}\| + \|y_{mn} - x_*\| < r + \varepsilon, \ \ for \ \ m, n \geq k.$$

This implies that $x_{mn} \xrightarrow{r} x_*$. □

We finally complete this work by giving the relation between the set of cluster points and the set of r-limit points of a sequence.

Theorem 3.6. *(i) If c is a cluster point of the double sequence $x = (x_{mn})$, then*

$$\text{LIM}^r x_{mn} \subseteq \overline{B}_r(c). \tag{3.8}$$

(ii) Let $\mathcal{C}_x$ be the set of cluster points of $x = (x_{mn})$. Then

$$\text{LIM}^r x_{mn} = \bigcap_{c \in \mathcal{C}_x} \overline{B}_r(c) = \{x_* \in \mathbb{R}^n : \mathcal{C}_x \subseteq \overline{B}_r(x_*)\}. \tag{3.9}$$

Proof. (i) For an arbitrary cluster point c of $x = (x_{mn})$, we have

$$\|x_* - c\| \leq r, \ \text{for all } x_* \in \text{LIM}^r x_{mn},$$

otherwise there are infinite $x = (x_{mn})$ satisfying

$$\|x_{mn} - x_*\| \geq r + \varepsilon, \ \ \text{with} \ \ \varepsilon := \frac{\|x_* - c\| - r}{2} > 0,$$

because c is an cluster point of (x_{mn}), which contradicts with the fact that $x_* \in \text{LIM}^r x_{mn}$. Hence, $\text{LIM}^r x_{mn} \subseteq \overline{B}_r(c)$ must be true.

(ii)From (3.8), we have

$$\text{LIM}^r x_{mn} \subseteq \bigcap_{c \in \mathcal{C}_x} \overline{B}_r(c). \tag{3.10}$$

Now, let

$$y \in \bigcap_{c \in \mathcal{C}_x} \overline{B}_r(c).$$

Then, we have

$$\|y - c\| \leq r,$$

for all $c \in \mathcal{C}_x$, which is equivalent to $\mathcal{C}_x \subseteq \overline{B}_r(y)$, i.e.,

$$\bigcap_{c \in \mathcal{C}_x} \overline{B}_r(c) \subseteq \{x_* \in \mathbb{R}^n : \mathcal{C}_x \subseteq \overline{B}_r(x_*)\}. \tag{3.11}$$

Now, let $y \notin \mathrm{LIM}^r x_{mn}$. Then, there is an $\varepsilon > 0$ such that there exists infinite x_{mn} satisfying $\|x_{mn} - y\| \geq r + \varepsilon$, which implies the existence of a cluster point c of the double sequence $x = (x_{mn})$ with $\|y - c\| \geq r + \varepsilon$, i.e.,

$$\mathcal{C}_x \not\subseteq \overline{B}_r(y) \text{ and } y \notin \{x_* \in \mathbb{R}^n : \mathcal{C}_x \subseteq \overline{B}_r(x_*)\}.$$

Hence, $y \in \mathrm{LIM}^r x_{mn}$ follows from $y \in \{x_* \in \mathbb{R}^n : \mathcal{C}_x \subseteq \overline{B}_r(x_*)\}$, i.e.,

$$\{x_* \in \mathbb{R}^n : \mathcal{C}_x \subseteq \overline{B}_r(x_*)\} \subseteq \mathrm{LIM}^r x_{mn}. \tag{3.12}$$

Therefore the inclusions (3.10)-(3.12) ensure that (3.9) i.e.,

$$\mathrm{LIM}^r x_{mn} = \bigcap_{c \in \mathcal{C}_x} \overline{B}_r(c) = \{x_* \in \mathbb{R}^n : \mathcal{C}_x \subseteq \overline{B}_r(x_*)\}.$$

□

Theorem 3.7. *Let $x = (x_{mn})$ be a bounded double sequence. If $r \geq \mathrm{diam}(\mathcal{C}_x)$, then we have $\mathcal{C}_x \subseteq \mathrm{LIM}^r x_{mn}$.*

Proof. Let $c \notin \mathrm{LIM}^r x_{mn}$. Then, there exists an $\varepsilon > 0$ such that

$$\|x_{mn} - c\| \geq r + \varepsilon. \tag{3.13}$$

Since $x = (x_{mn})$ is bounded and from the inequality (3.13), there exists an cluster point c_1 such that $\|c - c_1\| > r + \varepsilon_1$ where $\varepsilon_1 := \frac{\varepsilon}{2}$. So we get

$$\mathrm{diam}(\mathcal{C}_x) > r + \varepsilon_1,$$

which proves the theorem.

The converse of this theorem is also holds, i.e., if $\mathcal{C}_x \subseteq LIM^r x_{mn}$ then we have $r \geq diam(\mathcal{C}_x)$. □

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