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# $\mathcal{I}_2$-CONVERGENCE AND $\mathcal{I}_2$-CAUCHY DOUBLE SEQUENCES*

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**Abstract** In this article, we prove a decomposition theorem for $\mathcal{I}_2$-convergent double sequences and introduce the notions of $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy double sequence, and then study their certain properties. Finally, we introduce the notions of regularly $(\mathcal{I}_2, \mathcal{I})$-convergence and $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence.

**Key words** Ideal; double sequences; $\mathcal{I}$-convergence; $\mathcal{I}$-Cauchy.

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## 1 Introduction

The concept of convergence of a sequence of real numbers was extended to statistical convergence independently by Fast [1] and Schoenberg [2]. A lot of development were made in this area after the works of Šalát [3] and Fridy [4, 5]. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [1, 4–6]. This concept was extended to the double sequences by Mursaleen and Edely [7] and Tripathy independently [8]. Çakan and Altay [9] presented multidimensional analogues of the results presented by Fridy and Orhan [10]. They presented statistically bounded sequences, statistical inferior and statistical superior of double sequences. In addition to these results, they investigated statistical core for double sequences and studied an inequality related to the statistical and $P$-cores of bounded double sequences. Tripathy and Sarma [11] defined the notion of statistically convergent difference double sequence spaces.

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko, Šalát, and Wilczyński [12] as a generalization of statistical convergence, which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Nuray and Ruckle [13] indepedently introduced the same with another name generalized statistical convergence. Kostyrko, Mačaj, Šalát, and Sleziak [14] gave some of basic properties of $\mathcal{I}$-convergence and dealt with extremal $\mathcal{I}$-limit points. Das, Kostyrko, Wilczyński and Malik [15] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Also, Das and

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Malik [16] introduced the concept of $\mathcal{I}$-limit points, $\mathcal{I}$-cluster points and $\mathcal{I}$-limit superior, and $\mathcal{I}$-limit inferior of double sequences.

Nabiev, Pehlivan, and Gürdal [17] proved a decomposition theorem for $\mathcal{I}$-convergent sequences and introduced the notions of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence, and then studied their certain properties. A lot of development were made in this area after the works of [18–36].

In this article, first we investigate some properties of $\mathcal{I}$-convergent of double sequences in a linear metric space. Next, we prove the decomposition theorem of $\mathcal{I}$-convergent of double sequences in a linear metric space and give some results regarding this theorem. Also, we introduce the notions of $\mathcal{I}$-Cauchy double sequence and $\mathcal{I}^*$-Cauchy double sequence, and study their certain properties. Finally, we introduce the notions of regularly $(\mathcal{I}_2, \mathcal{I})$-convergence and regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequence.

## 2 Definitions and Notations

Throughout this article, $\mathbb{N}$ denotes the set of all positive integers, $\chi_A$-the characteristic function of $A \subset \mathbb{N}$, $\mathbb{R}$ the set of all real numbers.

Now, we recall the concept of statistical and ideal convergence of the sequences and basic definitions [1, 7, 12, 15, 27, 37, 38]

**Definition 2.1** A subset $A$ of $\mathbb{N}$ is said to have asymptotic density $d(A)$ if

$$d(A) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_A(k).$$

A sequence $x = (x_n)_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, we have $d(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$.

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$ , there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case, we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be bounded if there exists a positive real number $M$ such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let $K_{mn}$ be the number of $(j, k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit in Pringsheim's sense, then we say that $K$ has double natural density and is denoted by

$$d_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of $X$ is said to be an ideal in $X$ provided:

i) $\emptyset \in \mathcal{I}$,

ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,

iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A nonempty class $\mathcal{F}$ of subsets of $X$ is said to be a filter in $X$ provided:

i) $\emptyset \notin \mathcal{F}$, ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

**Lemma 2.2** ([12]) If $\mathcal{I}$ is a nontrivial ideal in $X$, $X \neq \emptyset$, then the class

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on $X$, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in $X$ is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout this article, we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then, $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

In this section, we consider the $\mathcal{I}_2$ and $\mathcal{I}_2^*$-convergence of double sequences in the more general structure of a metric space $(X, \rho)$. Unless otherwise mentioned, we shall denote the metric space $(X, \rho)$ by $X$ only.

Let $(X, \rho)$ be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in $X$ is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$, we have

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we say that $x$ is $\mathcal{I}_2$-convergent and write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N}\times\mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Let $(X, \rho)$ be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of $X$ is said to be $\mathcal{I}_2^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (that is, $\mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{m,n\to\infty} x_{mn} = L,$$

for $(m, n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L.$$

Let $(X, \rho)$ be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of $X$ is said to be $\mathcal{I}_2$-Cauchy if for every $\varepsilon > 0$, there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$, such that

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \in \mathcal{I}_2.$$

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, \cdots\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, that is, $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Now, we begin with quoting the lemmas due to Das, Kostyrko Wilczyńki and Malik [15] which are needed throughout this article.

**Lemma 2.3** ([15], Theorem 1) Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If $\mathcal{I}^*{}_2 - \lim\limits_{m,n\to\infty} x_{mn} = L$, then $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} x_{mn} = L$.

**Lemma 2.4** ([15], Theorem 3) If $\mathcal{I}_2$ is an admissible ideal of $\mathbb{N}\times\mathbb{N}$ having the property (AP2) and $(X,\rho)$ is an arbitrary metric space, then for an arbitrary double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of elements of $X$, $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} x_{mn} = L$ implies $\mathcal{I}_2^* - \lim\limits_{m,n\to\infty} x_{mn} = L$.

**Lemma 2.5** ([15], Theorem 6) (a) Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If $\lim\limits_{m,n\to\infty} x_{mn} = L$, then $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} x_{mn} = L$.

(b) Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a admissible ideal. If $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} x_{mn} = L$ and $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} y_{mn} = K$, then

(i) $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} (x_{mn} + y_{mn}) = L + K$;

(ii) $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} (x_{mn} y_{mn}) = LK$.

## 3 The Decomposition Theorem for $\mathcal{I}_2$-Convergence of Double Sequences

We extend the decomposition theorem of Nabiev, Pehlivan and Gürdal [17] from ordinary (single) to double sequences as follows.

**Theorem 3.1** Let $(X,\rho)$ be a linear metric space, $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property (AP2) and $x = (x_{mn})_{m,n\in\mathbb{N}}$ be a double sequence in $X$. Then, the following conditions are equivalent:

(i) $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} x_{mn} = L$,

(ii) There exist $y = (y_{mn})_{m,n\in\mathbb{N}}$ and $z = (z_{mn})_{m,n\in\mathbb{N}}$ be two sequences in $X$ such that

$$x = y + z, \quad \lim_{m,n\to\infty} \rho(y_{mn}, L) = 0 \ \text{ and supp } \ z \in \mathcal{I}_2,$$

where supp$z = \{(m,n) \in \mathbb{N}\times\mathbb{N} : z_{mn} \neq \theta\}$ and $\theta$ is the zero element of $X$.

**Proof** (i) $\Rightarrow$ (ii) Let $\mathcal{I}_2 - \lim\limits_{m,n\to\infty} x_{mn} = L$. Then by Lemma 2.4 there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (that is, $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} \rho(x_{mn}, L) = 0.$$

Let us define the double sequence $y = (y_{mn})_{m,n\in\mathbb{N}}$ by

$$y_{mn} = \begin{cases} x_{mn}, & (m,n) \in M \\ L, & (m,n) \in \mathbb{N}\times\mathbb{N}\backslash M. \end{cases} \tag{3.1}$$

It is clear that $y = (y_{mn})$ is in $X$ and

$$\lim_{m,n\to\infty} \rho(y_{mn}, L) = 0.$$

Also, let

$$z_{mn} = x_{mn} - y_{mn}, \quad m,n \in \mathbb{N}. \tag{3.2}$$

As

$$\{(m,n)\in\mathbb{N}\times\mathbb{N} : x_{mn}\neq y_{mn}\}\subset\mathbb{N}\times\mathbb{N}\backslash M\in\mathcal{I}_2,$$

we have

$$\{(m,n)\in\mathbb{N}\times\mathbb{N} : z_{mn}\neq\theta\}\in\mathcal{I}_2.$$

It follows that supp$z\in\mathcal{I}_2$ and by (3.1) and (3.2), we get $x=y+z$.

(ii) $\Rightarrow$ (i) Suppose that there exist two sequences $y=(y_{mn})$ and $z=(z_{mn})$ in $X$ such that

$$x=y+z,\quad \lim_{m,n\to\infty}\rho(y_{mn},L)=0,\ \text{ and supp }\ z=\{(m,n)\in\mathbb{N}\times\mathbb{N} : z_{mn}\neq\theta\}\in\mathcal{I}_2. \tag{3.3}$$

We show that

$$\mathcal{I}_2-\lim_{m,n\to\infty}x_{mn}=L.$$

Let

$$M=\{(m,n)\in\mathbb{N}\times\mathbb{N} : z_{mn}=\theta\}=\mathbb{N}\times\mathbb{N}\backslash\text{supp }\ z. \tag{3.4}$$

As supp$z\in\mathcal{I}_2$, from (3.3) and (3.4), we have $M\in\mathcal{F}(\mathcal{I}_2)$, $x_{mn}=y_{mn}$ for $(m,n)\in M$ and

$$\mathcal{I}_2^*-\lim_{\substack{m,n\to\infty\\(m,n)\in M}}\rho(x_{mn},L)=0.$$

By Lemma 2.3, it follows that

$$\mathcal{I}_2-\lim_{m,n\to\infty}x_{mn}=L.$$

This completes the proof of theorem. □

**Corollary 3.2** Let $(X,\rho)$ be a linear metric space and $\mathcal{I}_2\subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal having the property (AP2). Then,

$$\mathcal{I}_2-\lim_{m,n\to\infty}x_{mn}=L$$

if and only if there exist $y=(y_{mn})$ and $z=(z_{mn})$ in $X$ such that

$$x=y+z,\quad \lim_{m,n\to\infty}\rho(y_{mn},L)=0,\ \text{ and }\ \mathcal{I}_2-\lim_{m,n\to\infty}z_{mn}=\theta.$$

**Proof** Let $\mathcal{I}_2-\lim_{m,n\to\infty}x_{mn}=L$ and $y=(y_{mn})$ is the sequence defined by (3.1). Consider the sequence

$$z_{mn}=x_{mn}-y_{mn}. \tag{3.5}$$

Then, we have

$$\lim_{m,n\to\infty}\rho(y_{mn},L)=0$$

and as $\mathcal{I}_2$ is a strongly admissible ideal, so

$$\mathcal{I}_2-\lim_{m,n\to\infty}y_{mn}=L.$$

By Lemma 2.5 and by (3.5), we have

$$\mathcal{I}_2-\lim_{m,n\to\infty}z_{mn}=\theta.$$

Now, let $x_{mn} = y_{mn} + z_{mn}$, where

$$\lim_{m,n\to\infty} \rho(y_{mn}, L) = 0 \ \ and \ \mathcal{I}_2 - \lim_{m,n\to\infty} z_{mn} = \theta.$$

As $\mathcal{I}_2$ is a strongly admissible ideal, so

$$\mathcal{I}_2 - \lim_{m,n\to\infty} y_{mn} = L$$

and by Lemma 2.5, we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

□

**Remark 3.3** In Theorem 3.1, if (ii) is satisfied, then the strongly admissble ideal $\mathcal{I}_2$ need not have the property $(AP2)$. As

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(z_{mn}, \theta) \geq \varepsilon\} \subset \{(m,n) \in \mathbb{N}\times\mathbb{N} : z_{mn} \neq \theta\} \in \mathcal{I}_2$$

for each $\varepsilon > 0$, then

$$\mathcal{I}_2 - \lim_{m,n\to\infty} z_{mn} = \theta.$$

Thus, we have

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

## 4 $\mathcal{I}_2$-Cauchy Double Sequences

Now, we introduce the notion of $\mathcal{I}_2^*$-Cauchy double sequence.

**Definition 4.1** Let $(X, \rho)$ be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ in $X$ is said to be $\mathcal{I}_2^*$-Cauchy sequence if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (that is, $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that for every $\varepsilon > 0$ and for $(m,n), (s,t) \in M$, $m,n,s,t > k_0 = k_0(\varepsilon)$

$$\rho(x_{mn}, x_{st}) < \varepsilon.$$

In this case, we write

$$\lim_{m,n,s,t\to\infty} \rho(x_{mn}, x_{st}) = 0.$$

**Theorem 4.2** Let $(X, \rho)$ be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a stronly admissible ideal. If $x = (x_{mn})$ in $X$ is an $\mathcal{I}_2^*$-Cauchy sequence, then it is $\mathcal{I}_2$-Cauchy.

**Proof** Suppose that $x = (x_{mn})$ is an $\mathcal{I}_2^*$-Cauchy sequence. Then, there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (that is, $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that

$$\rho(x_{mn}, x_{st}) < \varepsilon,$$

for any $\varepsilon > 0$ and for all $(m,n), (s,t) \in M$, $m,n,s,t \geq k_0 = k_0(\varepsilon)$ and $k_0 \in \mathbb{N}$. Then,

$$\begin{aligned} A(\varepsilon) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \\ &\subset H \cup \big[M \cap \big((\{1,2,3,\cdots,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,3,\cdots,(k_0-1)\})\big)\big]. \end{aligned}$$

As $\mathcal{I}_2$ be a strongly admissible ideal, then,

$$H \cup \big[M \cap \big((\{1,2,3,\cdots,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,3,\cdots,(k_0-1)\})\big)\big] \in \mathcal{I}_2.$$

Therefore, we have

$$A(\varepsilon) \in \mathcal{I}_2.$$

This shows that the double sequence $x = (x_{mn})$ is $\mathcal{I}_2$-Cauchy. □

**Theorem 4.3** Let $(X, \rho)$ be a linear metric space, $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an arbitrary strongly admissible ideal and $(x_{mn})$ in $X$. Then, $\mathcal{I}_2\text{-}\lim_{m,n\to\infty} x_{mn} = L$ implies that $(x_{mn})$ is an $\mathcal{I}_2$-Cauchy sequence.

**Proof** Suppose that $x = (x_{mn})$ is $\mathcal{I}_2$-convergent to $L$. Let $\varepsilon > 0$ be given. Then,

$$A(\frac{\varepsilon}{2}) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, L) \geq \frac{\varepsilon}{2}\} \in \mathcal{I}_2.$$

This implies that the set

$$A^c(\frac{\varepsilon}{2}) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, L) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I}_2)$$

and therefore $A^c(\frac{\varepsilon}{2})$ is non-empty. So, we can choose positive integers $k$ and $l$ such that $(k,l) \notin A(\frac{\varepsilon}{2})$, but then we have

$$\rho(x_{kl}, L) < \frac{\varepsilon}{2}.$$

Let

$$B(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \rho(x_{mn}, x_{kl}) \geq \varepsilon\}.$$

We prove that $B(\varepsilon) \subset A(\frac{\varepsilon}{2})$. Let $(m,n) \in B(\varepsilon)$, then we have

$$\varepsilon \leq \rho(x_{mn}, x_{kl}) \leq \rho(x_{mn}, L) + \rho(x_{kl}, L) < \rho(x_{mn}, L) + \frac{\varepsilon}{2}.$$

This implies that

$$\frac{\varepsilon}{2} < \rho(x_{mn}, L)$$

and therefore $(m,n) \in A(\frac{\varepsilon}{2})$. As $B(\varepsilon) \subset A(\frac{\varepsilon}{2})$ and $A(\frac{\varepsilon}{2}) \in \mathcal{I}_2$, we have

$$B(\varepsilon) \in \mathcal{I}_2.$$

This completes the proof. □

## 5 Regularly $\mathcal{I}_2$-Convergence and Regularly $\mathcal{I}_2$-Cauchy Double Sequences

Now, we denote regularly $(\mathcal{I}_2, \mathcal{I})$-convergence and regularly $(\mathcal{I}_2, \mathcal{I})$-Cauchy double sequences and study their certain properties.

**Definition 5.1** ([29]) Let $\mathcal{I}_2$ be an ideal of $\mathbb{N}\times\mathbb{N}$ and $\mathcal{I}$ be an ideal of $\mathbb{N}$, then a double sequence $x = (x_{mn})$ in $X$ is said to be regularly $(\mathcal{I}_2, \mathcal{I})$-convergent ($r(\mathcal{I}_2, \mathcal{I})$-convergent) if it is $\mathcal{I}_2$-convergent in Pringsheim's sense and for every $\varepsilon > 0$, the followings hold:

$$\{m \in \mathbb{N} : |x_{mn} - L_n| \geq \varepsilon\} \in \mathcal{I}, \tag{5.1}$$

for some $L_n \in X$, for each $n \in \mathbb{N}$ and

$$\{n \in \mathbb{N} : |x_{mn} - K_m| \geq \varepsilon\} \in \mathcal{I}, \tag{5.2}$$

for some $K_m \in X$, for each $m \in \mathbb{N}$.

**Definition 5.2** Let $\mathcal{I}_2$ be an ideal of $\mathbb{N}\times\mathbb{N}$, $\mathcal{I}$ be an ideal of $\mathbb{N}$ and $(X,\rho)$ be a linear metric space, then a double sequence $x=(x_{mn})$ in $X$ is said to be $r(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent if there exist the sets $M\in\mathcal{F}(\mathcal{I}_2)$ (that is, $\mathbb{N}\times\mathbb{N}\setminus M\in\mathcal{I}_2$), $M_1\in\mathcal{F}(\mathcal{I})$, and $M_2\in\mathcal{F}(\mathcal{I})$ (that is, $\mathbb{N}\setminus M_1\in\mathcal{I}$ and $\mathbb{N}\setminus M_2\in\mathcal{I}$) such that the limits

$$\lim_{\substack{m,n\to\infty\\(m,n)\in M}} x_{mn},\ \lim_{\substack{m\to\infty\\ m\in M_1}} x_{mn},(n\in\mathbb{N})\text{ and }\lim_{\substack{n\to\infty\\ n\in M_2}} x_{mn},(m\in\mathbb{N})$$

exist. Note that if $x=(x_{mn})$ in $X$ is regularly convergent to $L\in X$, then the limits $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}x_{mn}$ and $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty}x_{mn}$ exist and are equal to $L\in X$.

**Theorem 5.3** Let $\mathcal{I}_2$ be a strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$, $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, and $(X,\rho)$ be a linear metric space. If a double sequence $x=(x_{mn})$ in $X$ is $r(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent, then $r(\mathcal{I}_2,\mathcal{I})$-convergent.

**Proof** Let $x=(x_{mn})$ be $r(\mathcal{I}_2^*,\mathcal{I}^*)$-convergent. Then, it is $\mathcal{I}_2^*$-convergent and by Lemma 2.3, $\mathcal{I}_2$-convergent. Also, there exist the sets $M_1\in\mathcal{F}(\mathcal{I})$ and $M_2\in\mathcal{F}(\mathcal{I})$, such that

$$(\forall\varepsilon>0)\ \ (\exists m_0\in\mathbb{N})\ \ (\forall m\geq m_0)\ \ (m\in M_1)\ \ \rho(x_{mn},L_n)<\varepsilon,$$

for some $L_n\in X$ and for each $n\in\mathbb{N}$ and

$$(\forall\varepsilon>0)\ \ (\exists n_0\in\mathbb{N})\ \ (\forall n\geq n_0)\ \ (n\in M_2)\ \ \rho(x_{mn},K_m)<\varepsilon,$$

for some $K_m\in X$ and for each $m\in\mathbb{N}$. Hence, we have

$$A_1(\varepsilon)=\{m\in\mathbb{N}:\rho(x_{mn},L_n)\geq\varepsilon\}\subset H_1\cup\{1,2,\cdots,m_0-1\}\text{ for each }n\in\mathbb{N}$$

and

$$A_2(\varepsilon)=\{n\in\mathbb{N}:\rho(x_{mn},K_m)\geq\varepsilon\}\subset H_2\cup\{1,2,\cdots,n_0-1\}\text{ for each }m\in\mathbb{N},$$

for $H_1,H_2\in\mathcal{I}$. As $\mathcal{I}$ is admissible ideal, we have

$$H_1\cup\{1,2,\cdots,m_0-1\},\ \ H_2\cup\{1,2,\cdots,n_0-1\}\in\mathcal{I}$$

and

$$A_1(\varepsilon),\ A_2(\varepsilon)\in\mathcal{I}.$$

This shows that the sequence $x=(x_{mn})$ is $r(\mathcal{I}_2,\mathcal{I})$-convergent. □

**Definition 5.4** Let $\mathcal{I}_2$ be a strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$, $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, and $(X,\rho)$ be a linear metric space, then a double sequence $x=(x_{mn})$ in $X$ is said to be regularly $(\mathcal{I}_2,\mathcal{I})$-Cauchy ($r(\mathcal{I}_2,\mathcal{I})$-Cauchy) if it is $\mathcal{I}_2$-Cauchy in Pringsheim's sense and for every $\varepsilon>0$, there exist $k_n=k_n(\varepsilon)$ and $l_m=l_m(\varepsilon)\in\mathbb{N}$ such that the followings hold:

$$A_1(\varepsilon)=\{m\in\mathbb{N}:\rho(x_{mn},x_{k_nn})\geq\varepsilon\}\in\mathcal{I}\ \ (n\in\mathbb{N})$$

and

$$A_2(\varepsilon)=\{n\in\mathbb{N}:\rho(x_{mn},x_{ml_m})\geq\varepsilon\}\in\mathcal{I}\ \ (m\in\mathbb{N}).$$

A double sequence $x=(x_{mn})$ in $X$ is said to be regularly $(\mathcal{I}_2^*,\mathcal{I}^*)$-Cauchy ($r(\mathcal{I}_2^*,\mathcal{I}^*)$-Cauchy) if there exist the sets $M\in\mathcal{F}(\mathcal{I}_2)$, $M_1\in\mathcal{F}(\mathcal{I})$, and $M_2\in\mathcal{F}(\mathcal{I})$ (that is, $\mathbb{N}\times\mathbb{N}\setminus M\in\mathcal{I}_2$, $\mathbb{N}\setminus M_1\in\mathcal{I}$, and $\mathbb{N}\setminus M_2\in\mathcal{I}$) and for every $\varepsilon>0$, there exist $N=N(\varepsilon)\in\mathbb{N}$, $s=s(\varepsilon)$, $t=t(\varepsilon)$, $k_n=k_n(\varepsilon)$, $l_m=l_m(\varepsilon)\in\mathbb{N}$ such that

$$\rho(x_{mn},x_{st})<\varepsilon\ \ \big(\text{for }(m,n),(s,t)\in M,\ m,n,s,t>N\big),$$

$$\rho(x_{mn}, x_{k_n n}) < \varepsilon \ \big(\text{for each } m \in M_1 \text{ and for each } n \in \mathbb{N}\big), \tag{5.3}$$

and

$$\rho(x_{mn}, x_{ml_m}) < \varepsilon \ \big(\text{for each } n \in M_2 \text{ and for each } m \in \mathbb{N}\big), \tag{5.4}$$

whenever $m, n, k_n, l_m > N$.

**Theorem 5.5** Let $\mathcal{I}_2$ be a strongly admissible ideal of $\mathbb{N}\times\mathbb{N}$, $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, and $(X, \rho)$ be a linear metric space. If the double sequence $x = (x_{mn})$ in $X$ is $r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy, then $r(\mathcal{I}_2, \mathcal{I})$-Cauchy.

**Proof** As double sequence $x = (x_{mn})$ is $r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy, so $\mathcal{I}_2^*$-Cauchy. We know that $\mathcal{I}_2^*$-Cauchy implies $\mathcal{I}_2$-Cauchy by Theorem 4.2. Also, as double sequence $x = (x_{mn})$ is $r(\mathcal{I}_2^*, \mathcal{I}^*)$-Cauchy, so there exist the sets $M_1 \in \mathcal{F}(\mathcal{I})$, $M_2 \in \mathcal{F}(\mathcal{I})$ (that is, $\mathbb{N} \setminus M_1 \in \mathcal{I}$ and $\mathbb{N} \setminus M_2 \in \mathcal{I}$) and for every $\varepsilon > 0$, there exist $k_n = k_n(\varepsilon)$, $l_m = l_m(\varepsilon) \in \mathbb{N}$ such that

$$\rho(x_{mn}, x_{k_n n}) < \varepsilon \ \ \big(\text{for each } m \in M_1 \text{ and for each } n \in \mathbb{N}\big) \tag{5.5}$$

and

$$\rho(x_{mn}, x_{ml_m}) < \varepsilon \ \ \big(\text{for each } n \in M_2 \text{ and for each } m \in \mathbb{N}\big) \tag{5.6}$$

for $N = N(\varepsilon) \in \mathbb{N}$ and $m, n, k_n, l_m \geq N$. Hence, we have

$$A_1(\varepsilon) = \{m \in \mathbb{N} : \rho(x_{mn}, x_{k_n n}) \geq \varepsilon\} \subset H_1 \cup \{1, 2, \cdots, (N-1)\} \ \ (m \in M_1 \text{ and } n \in \mathbb{N})$$

and

$$A_2(\varepsilon) = \{n \in \mathbb{N} : \rho(x_{mn}, x_{ml_m}) \geq \varepsilon\} \subset H_2 \cup \{1, 2, \cdots, (N-1)\} \ \ (n \in M_2 \text{ and } m \in \mathbb{N})$$

for $H_1 = \mathbb{N}\backslash M_1, H_2 = \mathbb{N}\backslash M_2 \in \mathcal{I}$. As $\mathcal{I}$ is admissible ideal,

$$H_1 \cup \{1, 2, \cdots, (N-1)\}, H_2 \cup \{1, 2, \cdots, (N-1)\} \in \mathcal{I}.$$

Therefore, it is clear, $A_1(\varepsilon) \in \mathcal{I}$ and $A_2(\varepsilon) \in \mathcal{I}$. This shows that $x = (x_{mn})$ is $r(\mathcal{I}_2, \mathcal{I})$-Cauchy. □

**Theorem 5.6** Let $\mathcal{I}_2$ be a strongly admissible ideal of $\mathbb{N} \times \mathbb{N}$, $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, and $(X, \rho)$ be a linear metric space. If the double sequence $x = (x_{mn})$ in $X$ is $r(\mathcal{I}_2, \mathcal{I})$-convergent, then $(x_{mn})$ is $r(\mathcal{I}_2, \mathcal{I})$-Cauchy sequence.

**Proof** Let $x = (x_{mn})$ in $X$ be a $r(\mathcal{I}_2, \mathcal{I})$-convergent double sequence. By Theorem 4.3, $x = (x_{mn})$ is $\mathcal{I}_2$-Cauchy sequence. Also, for every $\varepsilon > 0$, we have

$$A_1(\frac{\varepsilon}{2}) = \{m \in \mathbb{N} : \rho(x_{mn}, L_n) \geq \frac{\varepsilon}{2}\} \in \mathcal{I}, \tag{5.7}$$

for some $L_n \in X$, for each $n \in \mathbb{N}$ and

$$A_2(\frac{\varepsilon}{2}) = \{n \in \mathbb{N} : \rho(x_{mn}, K_m) \geq \frac{\varepsilon}{2}\} \in \mathcal{I} \tag{5.8}$$

for some $K_m \in X$, for each $m \in \mathbb{N}$. As $\mathcal{I}$ is admissible ideal, so the sets

$$A_1^c(\frac{\varepsilon}{2}) = \{m \in \mathbb{N} : \rho(x_{mn}, L_n) < \frac{\varepsilon}{2}\} \, (n \in \mathbb{N})$$

and

$$A_2^c(\frac{\varepsilon}{2}) = \{n \in \mathbb{N} : \rho(x_{mn}, K_m) < \frac{\varepsilon}{2}\} \, (m \in \mathbb{N})$$

are nonempty and belong to $\mathcal{F}(\mathcal{I})$. For $k_n \notin A_1(\frac{\varepsilon}{2})$ ($n \in \mathbb{N}$ and $k_n > 0$), we have

$$\rho(x_{k_n n}, L_n) < \frac{\varepsilon}{2}\,(n \in \mathbb{N}).$$

Now, we define the set

$$B_1(\varepsilon) = \{m \in \mathbb{N} : \rho(x_{mn}, x_{k_n n}) \geq \varepsilon\} \ \ (n \in \mathbb{N}),$$

where $k_n = k_n(\varepsilon)$ for $\varepsilon > 0$. We must prove that

$$B_1(\varepsilon) \subset A_1(\frac{\varepsilon}{2}).$$

Let $m \in B_1(\varepsilon)$. Then, for $k_n \notin A_1(\frac{\varepsilon}{2})$ ($n \in \mathbb{N}$ and $k_n > 0$), we have

$$\varepsilon \leq \rho(x_{mn}, x_{k_n n}) \leq \rho(x_{mn}, L_n) + \rho(x_{k_n n}, L_n) < \rho(x_{mn}, L_n) + \frac{\varepsilon}{2}.$$

This shows that

$$\frac{\varepsilon}{2} < \rho(x_{mn}, L_n)$$

and therefore $m \in A_1(\frac{\varepsilon}{2})$. Hence, we have

$$B_1(\varepsilon) \subset A_1(\frac{\varepsilon}{2}).$$

Similarly, for $l_m \notin A_2(\frac{\varepsilon}{2})$ ($m \in \mathbb{N}$ and $l_m > 0$), we have

$$\rho(x_{ml_m}, K_m) < \frac{\varepsilon}{2} \quad (m \in \mathbb{N})$$

Therefore, it can be seen that

$$B_2(\varepsilon) \subset A_2(\frac{\varepsilon}{2}),$$

where

$$B_2(\varepsilon) = \{m \in \mathbb{N} : \rho(x_{mn}, x_{ml_m}) \geq \varepsilon\} \quad (m \in \mathbb{N}).$$

This shows that $(x_{mn})$ is $r(\mathcal{I}_2, \mathcal{I})$-Cauchy sequence. $\square$

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