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STRONGLY $\mathcal{I}_2$-LACUNARY CONVERGENCE AND $\mathcal{I}_2$-LACUNARY CAUCHY DOUBLE SEQUENCES OF SETS

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Abstract

In this paper, we study the concepts of the Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence, Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy double sequences and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequences of sets and investigate the properties and relationships between them.

Keywords and phrases : Lacunary sequence, $\mathcal{I}_2$-convergence, $\mathcal{I}_2$-Cauchy, Double sequence of sets, Wijsman convergence.

AMS Subject Classification : 40A05, 40A35.

1 INTRODUCTION

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [10] and Schoenberg [23].

This concept was extended to the double sequences by Mursaleen and Edely [16]. Çakan and Altay [5] presented multidimensional analogues of the results presented by Fridy and Orhan [11].

Nuray and Ruckle [20] independently introduced the same with another name generalized statistical convergence. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko, Šalát and Wilczyński [14] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of development have been made in this area after the works of [8, 15, 17].

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, [2, 3, 4, 19, 28, 29]). Nuray and Rhoades [19] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [26] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wiijsman statistical convergence, which was defined by Nuray and Rhoades.

Kişi and Nuray [12] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence. Sever et al. [24] studied the concepts of Wijsman strongly lacunary convergence, Wijsman strongly $\mathcal{I}$-lacunary convergence, Wijsman strongly $\mathcal{I}^*$-lacunary convergence and Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequences of sets. Dündar et al. [7] examinated the ideas of Wijsman strongly lacunary Cauchy, Wijsman strongly $\mathcal{I}$-lacunary Cauchy and Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequences of sets. Nuray et al. [21] studied Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigate the relationships between them. Nuray et al. [22] studied the concepts of Wijsman $\mathcal{I}, \mathcal{I}^*$-convergence and Wijsman $\mathcal{I}, \mathcal{I}^*$-Cauchy double sequences of sets.

In this paper, we study the concepts of Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence, Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequences and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequences of sets and investigate the properties and relationships between them.

2 DEFINITIONS AND NOTATIONS

Now, we recall the basic definitions and concepts (See [1, 2, 3, 4, 6, 7, 8, 9, 14, 18, 19, 22, 24, 26, 27, 28, 29]).

Throughout the paper, we let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X.

For any point $x \in X$, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

We say that the sequence $\{A_k\}$ is Wijsman convergent to A if $\lim_{k\to\infty} d(x, A_k)$ $= d(x, A)$, for each $x \in X$. In this case we write $W - \lim A_k = A$.

We say that the sequence $\{A_k\}$ is Wijsman Cauchy sequence, if for $\varepsilon > 0$ and for each $x \in X$, there is a positive integer k_0 such that for all $m, n > k_0$, $|d(x, A_m) - d(x, A_n)| < \varepsilon$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by $q_{r.}$

Let $\theta = \{k_r\}$ be a lacunary sequence. We say that the sequence $\{A_k\}$ is

Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \to A([WN]_\theta)$ or $A_k \stackrel{[WN]_\theta}{\to} A$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $F \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

(i) $\emptyset \notin F$, (ii) For each $A, B \in F$ we have $A \cap B \in F$, (iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$.

$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$, then the set $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$ is a filter in $\mathbb{N}$, called the filter associated with $\mathcal{I}$.

Let θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. We say that the sequence $\{A_k\}$ is said to be Wijsman strongly $\mathcal{I}$-lacunary convergent to A or $N_\theta\left[\mathcal{I}_W\right]$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$, the set

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\}$$

belongs to $\mathcal{I}$. In this case, we write $A_k \to A\left(N_\theta\left[\mathcal{I}_W\right]\right)$.

Let (X, ρ) be a separable metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. We say that the sequence $\{A_k\}$ is Wijsman strongly $\mathcal{I}^*$-lacunary convergent to A if and only if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in F(\mathcal{I})$ for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_{m_k}) - d(x, A)| = 0.$$

In this case, we write $A_k \to A\left(N_\theta\left[\mathcal{I}_W^*\right]\right)$.

Let θ be lacunary sequence. The sequence $\{A_k\}$ is Wijsman strongly lacunary Cauchy if for every $\varepsilon > 0$ and for each $x \in X$, there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k,p \in I_r} |d(x, A_k) - d(x, A_p)| < \varepsilon,$$

for every $k, p \geq k_0$.

Let θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. The sequence $\{A_k\}$ is Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A_{k_0})| \geq \varepsilon \right\} \in \mathcal{I}.$$

Let (X, ρ) be a separable metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal. The sequence $\{A_k\}$ is Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence if for every $\varepsilon > 0$ and for each $x \in X$, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ and there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k,p \in I_r} |d(x, A_{m_k}) - d(x, A_{m_p})| < \varepsilon$$

for every $k, p \geq k_0$.

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A) \;\; or \;\; \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A)$$

for each $x \in X$. In this case, we write $W_2 - \lim A_{kj} = A$.

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \;\; and \;\; j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad r, u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u, \;\; h_{ru} = h_r \bar{h}_u, \;\; I_{ru} = \{(k,j) : k_{r-1} < k \leq k_r \;\; and \;\; j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \;\; and \;\; q_u = \frac{j_u}{j_{u-1}}.$$

Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)| = 0.$$

In this case, we write $A_{kj} \stackrel{[W_2 N_\theta]}{\longrightarrow} A$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i,j) \notin A)\}$. Then, $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$, for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Throughout the paper, we let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal, (X, ρ) be a separable metric space and A, A_k be any non-empty closed subsets of X.

We say that a double sequence of sets $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-convergent to A, if for each $x \in X$ and for every $\varepsilon > 0$, $\{(k,j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we write $\mathcal{I}_{W_2} - \lim\limits_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$.

We say that the double sequence of sets $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2^*$-convergent to A, if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2$) such that for each $x \in X$

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} d(x, A_{kj}) = d(x, A).$$

In this case, we write $\mathcal{I}_{W_2}^* - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$.

Lemma 2.1.(8], Theorem 3.3). Let $\{P_i\}_{i=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}\times\mathbb{N}$ such that $P_i \in F(\mathcal{I}_2)$ for each i, where $\mathcal{F}(\mathcal{I}_2)$ is a filter associate with a strongly admissible ideal $\mathcal{I}_2$ with the property $(AP2)$. Then, there exists a set $P \subset \mathbb{N}\times\mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and the set $P\backslash P_i$ is finite for all i.

3 MAIN RESULTS

Throughout the paper we take (X, ρ) be a separable metric space, $\theta = \{k_{rj}\}$ be a double lacunary sequence, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and A, A_{kj} be non-empty closed subsets of X.

Definition 3.1. *The sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta\left[\mathcal{I}_{W_2}\right]$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$, the set*

$$A(\varepsilon, x) = \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\left(N_\theta\left[\mathcal{I}_{W_2}\right]\right)$.

Theorem 3.2. *If $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A, then it is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A.*

Proof. Let $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A. For every $\varepsilon > 0$ and for each $x \in X$ there exists $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that

$$\frac{1}{h_r\bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \varepsilon,$$

for all $k, j \geq k_0$. Then, we have

$$T(x,\varepsilon) = \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\} \subset \{1,2,\cdots,k_0-1\}.$$

Since $\mathcal{I}_2$ is a strongly admissible ideal we have $\{1,2,\cdots,k_0-1\} \in \mathcal{I}_2$ and so $T(x,\varepsilon) \in \mathcal{I}_2$. This completes the proof. □

Definition 3.3. *The sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2^*$-lacunary convergent to A if and only if there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,*

$$\lim_{r,u\to\infty} \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} d(x,A_{kj}) = d(x,A).$$

In this case, we write $A_{kj} \to A\left(N_\theta\left(\mathcal{I}_{W_2}^*\right)\right)$.

Definition 3.4. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A if and only if there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,*

$$\lim_{r,u\to\infty} \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| = 0.$$

In this case, we write $A_{kj} \to A\left(N_\theta\left[\mathcal{I}_{W_2}^*\right]\right)$.

Theorem 3.5. *If the sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A.*

Proof. Suppose that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A. Then, there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ for each $x \in X$,

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| < \varepsilon,$$

for every $\varepsilon > 0$ and for all $k, j \geq k_0 = k_0(\varepsilon, x) \in \mathbb{N}$. Hence, for every $\varepsilon > 0$ and for each $x \in X$ we have

$$\begin{aligned} T(\varepsilon, x) &= \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \\ &\subset H \cup \Big(M' \cap \big((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}) \big) \Big), \end{aligned}$$

for $\mathbb{N} \times \mathbb{N} \backslash M' = H \in \mathcal{I}_2$. Since $\mathcal{I}_2$ is an admissible ideal we have

$$H \cup \Big(M' \cap \big((\{1, 2, ..., (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, ..., (k_0 - 1)\}) \big) \Big) \in \mathcal{I}_2$$

and so $T(\varepsilon, x) \in \mathcal{I}_2$. Hence, this completes the proof. □

Theorem 3.6. *Let $\mathcal{I}_2 \subseteq 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal with property $(AP2)$. If $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary convergent to A.*

Proof. Suppose that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A. Then, for every $\varepsilon > 0$ and for each $x \in X$

$$T(\varepsilon, x) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

Put

$$T_1 = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq 1 \right\}$$

and

$$T_p = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{p} \leq \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \frac{1}{p-1} \right\},$$

for $p \geq 2$ and $p \in \mathbb{N}$. It is clear that $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exits a sequence of sets $\{V_p\}_{p \in \mathbb{N}}$ such that

$T_j \Delta V_j$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each j and $V = \bigcup_{j=1}^{\infty} V_j \in \mathcal{I}_2$. We prove that for each $x \in X$

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| = 0,$$

for $M = \mathbb{N} \times \mathbb{N} \backslash V \in F(\mathcal{I}_2)$. Let $\delta > 0$ be given. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Then, for each $x \in X$.

$$\left\{(r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \delta\right\} \subset \bigcup_{j=1}^{q-1} T_j.$$

Since $T_j \Delta V_j$ is a finite set for $j \in \{1, 2, \cdots, q - 1\}$, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} T_j\right) \cap \{(k, j) \in \mathbb{N} \times \mathbb{N} : k \geq n_0 \wedge j \geq n_0\}$$

$$= \left(\bigcup_{j=1}^{q-1} V_j\right) \cap \{(k, j) \in \mathbb{N} \times \mathbb{N} : k \geq n_0 \wedge j \geq n_0\}.$$

If $k, j \geq n_0$ and $(k, j) \notin V$, then

$$(k, j) \notin \bigcup_{j=1}^{q-1} V_j \text{ and so } (k, j) \notin \bigcup_{j=1}^{q-1} T_j.$$

Thus, for each $x \in X$ we have

$$\frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| = 0.$$

Hence, for each $x \in X$ we have $A_{kj} \to A\left(N_\theta\left[\mathcal{I}^*_{W_2}\right]\right)$. This completes the proof. □

Definition 3.7. *The sequence $\{A_{kj}\}$ is Wijsman strongly lacunary Cauchy if for every $\varepsilon > 0$ and for each $x \in X$, there exists $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that*

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j),(s,t)\in I_{ru}}|d(x,A_{kj})-d(x,A_{st})|<\varepsilon,$$

for every $k, j, s, t \geq k_0$.

Definition 3.8. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence if for each $\varepsilon > 0$ and $x \in X$, there exists numbers $s = s(\varepsilon, x), t = t(\varepsilon, x) \in \mathbb{N}$ such that*

$$A(\varepsilon,x)=\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A_{st})|\geq\varepsilon\right\}\in\mathcal{I}_2.$$

Theorem 3.9. *If $\{A_{kj}\}$ is Wijsman strongly lacunary Cauchy sequence, then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Proof. The proof is routine verification so we omit it. □

Theorem 3.10. *If $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent then $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence.*

Proof. Let $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A. Then, for every $\varepsilon > 0$ and for each $x \in X$, we have

$$T\left(\frac{\varepsilon}{2},x\right)=\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|\geq\frac{\varepsilon}{2}\right\}\in\mathcal{I}_2.$$

Since $\mathcal{I}_2$ is a strongly admissible ideal, the set

$$T^c\left(\frac{\varepsilon}{2},x\right)=\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|<\frac{\varepsilon}{2}\right\}$$

is non-empty and belongs to $F(\mathcal{I}_2)$. So, we can choose positive integers r, u such that $(r,u) \notin T(\frac{\varepsilon}{2}, x)$, we have

$$\frac{1}{h_r\overline{h}_u}\sum_{(k_0,j_0)\in I_{ru}}|d(x,A_{k_0j_0})-d(x,A)|<\frac{\varepsilon}{2}.$$

Now, we define the set

$$B(\varepsilon,x)=\left\{(r,u)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_r\overline{h}_u}\sum_{(k,j),(k_0,j_0)\in I_{ru}}|d(x,A_{kj})-d(x,A_{k_0j_0})|\geq\varepsilon\right\}.$$

We show that $B(\varepsilon,x)\subset T(\frac{\varepsilon}{2},x)$. Let $(r,u)\in B(\varepsilon,x)$ then, we have

$$\begin{aligned}\varepsilon &\leq \frac{1}{h_r\overline{h}_u}\sum_{(k,j),(k_0,j_0)\in I_{ru}}|d(x,A_{kj})-d(x,A_{k_0j_0})|\\ &\leq \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|+\frac{1}{h_r\overline{h}_u}\sum_{(k_0,j_0)\in I_{ru}}|d(x,A_{k_0j_0})-d(x,A)|\\ &< \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|+\frac{\varepsilon}{2}.\end{aligned}$$

This implies that

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A)|>\frac{\varepsilon}{2}$$

and therefore $(r,u)\in T(\frac{\varepsilon}{2},x)$. Hence, we have $B(\varepsilon,x)\subset T(\frac{\varepsilon}{2},x)$. This shows that $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence. □

Definition 3.11. *The sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence if for every $\varepsilon>0$ and for each $x\in X$, there exists a set $M=\{(k,j)\in\mathbb{N}\times\mathbb{N}\}$ such that $M'=\{(r,u)\in\mathbb{N}\times\mathbb{N}:(k,j)\in I_{ru}\}\in\mathcal{F}(\mathcal{I}_2)$ and a number $N=N(\varepsilon,x)\in\mathbb{N}$ such that*

$$\frac{1}{h_r\overline{h}_u}\sum_{(k,j),(s,t)\in I_{ru}}|d(x,A_{kj})-d(x,A_{st})|<\varepsilon$$

for every $k,j,s,t\geq N$.

Theorem 3.12. *If the double sequence $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence then $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Proof. Suppose that $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence. Then, for every $\varepsilon > 0$ and for each $x \in X$, there exists a set $M = \{(k,j) \in \mathbb{N}\times\mathbb{N}\}$ such that $M' = \{(r,u) \in \mathbb{N}\times\mathbb{N} : (k,j) \in I_{ru}\} \in \mathcal{F}(\mathcal{I}_2)$ and a number $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that

$$\frac{1}{h_r\bar{h}_u}\sum_{(k,j),(s,t)\in I_{ru}} |d(x,A_{kj}) - d(x,A_{st})| < \varepsilon$$

for every $k,j,s,t \geq k_0$.

Let $H = \mathbb{N}\times\mathbb{N}\backslash M'$. It is obvious that $H \in \mathcal{I}_2$ and

$$\begin{aligned} T(\varepsilon,x) \quad &= \left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \tfrac{1}{h_r\bar{h}_u}\sum_{(k,j),(s,t)\in I_{ru}} |d(x,A_{kj}) - d(x,A_{st})| \geq \varepsilon\right\} \\ &\subset H \cup \Big(M' \cap \big((\{1,2,...,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,...,(k_0-1)\})\big)\Big). \end{aligned}$$

As $\mathcal{I}_2$ be a strongly admissible ideal then,

$$H \cup \Big(M' \cap \big((\{1,2,...,(k_0-1)\}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,...,(k_0-1)\})\big)\Big) \in \mathcal{I}_2.$$

Therefore, we have $T(\varepsilon,x) \in \mathcal{I}_2$, that is, $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets. □

Combining Theorem 3.5 and Theorem 3.10, we have following Theorem:

Theorem 3.13. *If the double sequence $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2^*$-lacunary convergence then $\{A_{kj}\}$ is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets.*

Theorem 3.14. *If $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is an admissible ideal with the property $(AP2)$ then the concepts Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy double sequence and Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy double sequence of sets coincide in X.*

Proof. If a sequence is Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence, then it is Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence of sets by Theorem 3.12, where $\mathcal{I}_2$ need not have the property $(AP2)$.

Now, it is sufficient to prove that a sequence $\{A_{kj}\}$ in X is a Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence under assumption that it is a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence. Let $\{A_{kj}\}$ in X be a Wijsman strongly $\mathcal{I}_2$-lacunary Cauchy sequence. Then, for every $\varepsilon > 0$ and for each $x \in X$, there exists numbers $s = s(\varepsilon, x), t = t(\varepsilon, x) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

Let

$$P_i = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \tfrac{1}{h_r \overline{h}_u} \textstyle\sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{s_i t_i})| < \tfrac{1}{i} \right\}; \quad (i = 1, 2, \ldots),$$

where $s_i = s(1 \backslash i), t_i = t(1 \backslash i)$. It is clear that $P_i \in \mathcal{F}(\mathcal{I}_2)$, $(i = 1, 2, \cdots)$. Since $\mathcal{I}_2$ has the property $(AP2)$, then by Lemma 2.1 there exists a set $P \subset \mathbb{N} \times \mathbb{N}$ such that $P \in \mathcal{F}(\mathcal{I}_2)$ and $P \backslash P_i$ is finite for all i. Now, we show that

$$\lim_{k,n,s,t \to \infty} \frac{1}{h_r \overline{h}_u} \sum_{(k,j),(s,t) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{st})| = 0,$$

for each $x \in X$ and for $(k,j), (s,t) \in P$. To prove this, let $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $m > 2/\varepsilon$. If $(k,j), (s,t) \in P$ then $P \backslash P_m$ is a finite set, so there exists $v = v(m)$ such that $(k,j), (s,t) \in P_m$ for all $k, j, s, t > v(m)$. Therefore, for each x in X,

$$\frac{1}{h_r \overline{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A_{s_m t_m})| < \frac{1}{m}$$

and

$$\frac{1}{h_r \overline{h}_u} \sum_{(s,t) \in I_{ru}} |d(x, A_{st}) - d(x, A_{s_m t_m})| < \frac{1}{m},$$

for all $k, j, s, t > v(m)$. Hence, it follows that

$$\begin{aligned}\frac{1}{h_r\bar{h}_u}\sum_{(k,j),(s,t)\in I_{ru}}|d(x,A_{kj})-d(x,A_{st})| &\leq \frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}}|d(x,A_{kj})-d(x,A_{s_mt_m})| \\ &+\frac{1}{h_r\bar{h}_u}\sum_{(s,t)\in I_{ru}}|d(x,A_{st})-d(x,A_{s_mt_m})| \\ &< \frac{1}{m}+\frac{1}{m}=\frac{2}{m} \\ &< \varepsilon,\end{aligned}$$

for all $k, j, s, t > v(m)$ and for each x in X. Thus, for any $\varepsilon > 0$ there exists $v = v(\varepsilon)$ such that for $k, j, s, t > v(\varepsilon)$ and $(k, j), (s, t) \in P \in \mathcal{F}(\mathcal{I}_2)$

$$\frac{1}{h_r\bar{h}_u}\sum_{(k,j),(s,t)\in I_{ru}}|d(x,A_{kj})-d(x,A_{st})| < \varepsilon,$$

for each x in X. This shows that the sequence $\{A_{kj}\}$ in X is Wijsman strongly $\mathcal{I}_2^*$-lacunary Cauchy sequence of sets.

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