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WIJSMAN $\mathcal{I}_2$-INVARIANT CONVERGENCE OF DOUBLE SEQUENCES OF SETS

ŞÜKRÜ TORTOP, ERDİNÇ DÜNDAR

Abstract. In this paper, we study the concepts of Wijsman invariant convergence, Wijsman invariant statistical convergence, Wijsman $\mathcal{I}_2$-invariant convergence ($\mathcal{I}^{\sigma}_{W_2}$), Wijsman $\mathcal{I}^*_2$-invariant convergence (${\mathcal{I}^*}^{\sigma}_{W_2}$), Wijsman p-strongly invariant convergence ($[W_2V_\sigma]_p$) of double sequence of sets and investigate the relationships among Wijsman invariant convergence, $[W_2V_\sigma]_p$, $\mathcal{I}^{\sigma}_{W_2}$ and ${\mathcal{I}^*}^{\sigma}_{W_2}$. Also, we introduce the concepts of $\mathcal{I}^{\sigma}_{W_2}$-Cauchy double sequence and ${\mathcal{I}^*}^{\sigma}_{W_2}$-Cauchy double sequence of sets.

1. Introduction and Background

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ denotes the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [10] and Schoenberg [35]. This concept was extended to the double sequences by Mursaleen and Edely [20]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko, Šalát and Wilczyński [16] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$. Das et al. [**?**] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. A lot of development have been made in this area after the various studies of researchers [**?**, **?**, **?**, 5, 6, 8, 22].

Nuray and Rhoades [23] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [42] defined the Wijsman lacunary statistical convergence of set sequences and considered its relation with Wijsman statistical convergence which was defined by Nuray and Rhoades. Kişi and Nuray [15] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence. Also, the concept of convergence of sequences has been extended to convergence, statistical convergence and ideal convergence of sequences of sets by several authors [36–38, 43–46].

Several authors including Raimi [30], Schaefer [34], Mursaleen [21], Savaş [31], Pancaroğlu and Nuray [27], and others have studied invariant convergent sequences [19, 25]. Savaş and Nuray [33] introduced the concepts of σ-statistical convergence

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and lacunary σ-statistical convergence and gave some inclusion relations. Recently, the concept of strong σ-convergence was generalized by Savaş [31]. Nuray et al. [26] defined the concepts of σ-uniform density of subsets A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Ulusu and Nuray [41] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers. Pancaroğlu et al. [29] studied Wijsman $\mathcal{I}$-invariant convergence of sequences of sets. Also, Çakan et al. [4] investigated σ-convergence and σ-core of double sequences.

2. Definitions and Notations

Now, we recall the basic definitions and concepts (see [1–4, 16, 18, 23, 24, 26–28, 30, 32, 34, 41]).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N} \notin \mathcal{I}$. Nontrivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

(i) $\emptyset \notin \mathcal{F}$, (ii) For each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, (iii) For each $A \in \mathcal{F}$ and each $B \supseteq A$ we have $B \in \mathcal{F}$.

If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$ is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

Throughout the paper we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (X, ρ) be a metric space. A sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$, $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2$. In this case, we say that x is $\mathcal{I}_2$-convergent and we write $\mathcal{I}_2 - \lim_{m,n \to \infty} x_{mn} = L$.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. If σ is translation mappings that is, $\sigma(n) = n + 1$, then σ-mean is often called a Banach limit.

A bounded sequence $x = (x_k)$ is said to be strongly σ-convergent to L if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |x_{\sigma^k(m)} - L| = 0 \text{ uniformly in } \quad m$$

and in this case, we write $x_k \to L[V_\sigma]$. By $[V_\sigma]$, we denote the set of all strongly σ-convergent sequences.

In the case $\sigma(n) = n + 1$, the space $[V_\sigma]$ is the set of strongly almost convergent sequences $[\hat{c}]$.

The concept of strong σ-convergence was generalized by Savaş [31] as below:

$$[V_\sigma]_p = \Big\{ x = (x_k) : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} |x_{\sigma^k(n)} - L|^p = 0 \ \text{ uniformly in } \ n \Big\},$$

where $0 < p < \infty$. If $p = 1$, then $[V_\sigma]_p = [V_\sigma]$. It is known that $[V_\sigma]_p \subset \ell_\infty$.

A sequence $x = (x_k)$ is σ-statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m} \Big| \big\{ k \le m : |x_{\sigma^k(n)} - L| \ge \varepsilon \big\} \Big| = 0,$$

uniformly in n . In this case we write $S_\sigma - \lim x = L$ or $x_k \to L(S_\sigma)$.

A bounded double sequence $x = (x_{kj})$ of real numbers is said to be σ-convergent to a limit l if

$$\lim_{mn} \frac{1}{mn} \sum_{k=0}^{m} \sum_{j=0}^{n} x_{\sigma^k(s), \sigma^j(t)} = l$$

uniformly in s, t. In this case, we write $\sigma_2 - \lim x = l$.

Let $A \subseteq \mathbb{N}$ and

$$s_n := \min_m \big| A \cap \big\{ \sigma(m), \sigma^2(m), ..., \sigma^n(m) \big\} \big|$$

and

$$S_n := \max_m \big| A \cap \big\{ \sigma(m), \sigma^2(m), ..., \sigma^n(m) \big\} \big|.$$

If the limits $\underline{V}(A) := \lim_{n\to\infty} \frac{s_n}{n}$ and $\overline{V}(A) := \lim_{n\to\infty} \frac{S_n}{n}$ exist, then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A) = \overline{V}(A)$, then $V(A) = \underline{V}(A) = \overline{V}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A \subseteq \mathbb{N}$ with $V(A) = 0$.

A sequence (x_k) is said to be $\mathcal{I}_\sigma$-convergent to the number L if for every $\varepsilon > 0$, $A_\varepsilon = \big\{ k : |x_k - L| \ge \varepsilon \big\} \in \mathcal{I}_\sigma$ that is, $V(A_\varepsilon) = 0$. In this case, we write $\mathcal{I}_\sigma - \lim x_k = L$.

Let (X, ρ) be a separable metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by $d(x, A) = \inf_{a\in A} \rho(x, a)$.

Throughout the paper, let (X, ρ) be a separable metric space and A, A_{kj} be any non-empty closed subsets of X.

A double sequence of sets $\{A_{kj}\}$ is $\mathcal{I}_{W_2}$-convergent to A if for each $x \in X$ and for every $\varepsilon > 0$, $\{(k, j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \ge \varepsilon\} \in \mathcal{I}_2$. In this case, we write $\mathcal{I}_{W_2} - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$.

A double sequence of sets $\{A_{kj}\}$ is $\mathcal{I}^*_{W_2}$-convergent to A if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2$) such that for each $x \in X$

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} d(x, A_{kj}) = d(x, A).$$

In this case, we write $\mathcal{I}^*_{W_2} - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$.

A double sequence of sets $\{A_{kj}\}$ is $\mathcal{I}_2$-Cauchy sequence if for each $x \in X$ and for every $\varepsilon > 0$, there exists (p,q) in $\mathbb{N} \times \mathbb{N}$ such that

$$\{(k,j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A_{pq})| \geq \varepsilon\} \in \mathcal{I}_2.$$

A double sequence of sets $\{A_{kj}\}$ is $\mathcal{I}^*_{W_2}$-Cauchy if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2)$ $(\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2)$ such that for each $x \in X$, $\lim\limits_{k,j,p,q\to\infty} |d(x, A_{kj}) - d(x, A_{pq})| = 0$, for $(k,j), (p,q) \in M_2$.

A double sequence $\{A_{kj}\}$ is said to be bounded if $\sup\limits_{k,j} d(x, A_{kj}) < \infty$, for each $x \in X$. The set of all bounded double sequences of sets will be denoted by L^2_∞.

A sequence $\{A_k\}$ is said to be Wijsman invariant convergent to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_{\sigma^k(m)}) = d(x, A), \text{ uniformly in m.}$$

A sequence $\{A_k\}$ is said to be Wijsman strongly invariant convergent to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{\sigma^k(m)}) - d(x, A)| = 0, \text{ uniformly in m.}$$

A sequence $\{A_k\}$ is said to be Wijsman invariant statistical convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} |\{0 \leq k \leq n : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon\}| = 0, \text{ uniformly in m.}$$

A sequence $\{A_k\}$ is said to be Wijsman p-strongly invariant convergent to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{\sigma^k(m)}) - d(x, A)|^p = 0, \text{ uniformly in m,}$$

where $0 < p < \infty$.

A sequence $\{A_k\}$ is said to be Wijsman $\mathcal{I}$-invariant convergent or $\mathcal{I}^W_\sigma$-convergent to A if for every $\varepsilon > 0$, $A_\varepsilon = \{k : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_\sigma$ that is, $V(A_\varepsilon) = 0$. In this case, we write $A_k \to A(\mathcal{I}^W_\sigma)$ and the set of all Wijsman $\mathcal{I}$-invariant convergent sequences of sets will be denoted $\mathcal{I}^W_\sigma$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that $E_j \Delta F_j \in \mathcal{I}^0_2$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_2$ (hence, $F_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

3. Main Results

In this section, we study the concepts of Wijsman invariant convergence, Wijsman invariant statistical convergence, Wijsman $\mathcal{I}_2$-invariant convergence, Wijsman $\mathcal{I}^*_2$-invariant convergence, Wijsman p-strongly invariant convergence double sequence of sets and investigate the relationships among Wijsman invariant convergence, $[W_2V_\sigma]_p$, $\mathcal{I}^\sigma_{W_2}$ and $\mathcal{I}^{*\sigma}_{W_2}$. Also, we introduce the concepts of $\mathcal{I}^\sigma_{W_2}$-Cauchy double sequence and $\mathcal{I}^{*\sigma}_{W_2}$-Cauchy double sequence of sets.

Definition 3.1. *Let $A \subseteq \mathbb{N} \times \mathbb{N}$ and*

$$s_{mn} := \min_{k,j} \left| A \cap \left\{ \left(\sigma(k), \sigma(j)\right), \left(\sigma^2(k), \sigma^2(j)\right), ..., \left(\sigma^m(k), \sigma^n(j)\right) \right\} \right|$$

and

$$S_{mn} := \max_{k,j} \left| A \cap \left\{ \left(\sigma(k), \sigma(j)\right), \left(\sigma^2(k), \sigma^2(j)\right), ..., \left(\sigma^m(k), \sigma^n(j)\right) \right\} \right|.$$

If the following limits exists

$$\underline{V_2}(A) := \lim_{m,n\to\infty} \frac{s_{mn}}{mn} \quad and \quad \overline{V_2}(A) := \lim_{m,n\to\infty} \frac{S_{mn}}{mn}$$

then they are called a lower and an upper σ-uniform density of the set A, respectively. If $\underline{V_2}(A) = \overline{V_2}(A)$, then $V_2(A) = \underline{V_2}(A) = \overline{V_2}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_2^\sigma$ the class of all $A \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(A) = 0$.

Definition 3.2. *A double sequence $\{A_{kj}\}$ is said to be Wijsman invariant convergent to A if for each $x \in X$,*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{\sigma^k(s),\sigma^j(t)}) = d(x, A), \text{ uniformly in } s,t.$$

Definition 3.3. *A double sequence $\{A_{kj}\}$ is said to be Wijsman $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_{W_2}^\sigma$-convergent to A, if for every $\varepsilon > 0$,*

$$A(\varepsilon, x) = \{(k,j) : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_2^\sigma$$

that is, $V_2(A(\varepsilon, x)) = 0$. In this case, we write $A_{kj} \to A(\mathcal{I}_{W_2}^\sigma)$ and the set of all Wijsman $\mathcal{I}_2$-invariant convergent double sequences of sets will be denoted by $\mathcal{I}_{W_2}^\sigma$.

Definition 3.4. *Let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2^*$-invariant convergent or $\mathcal{I}_{W_2}^{*\sigma}$-convergent to A if and only if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ ($\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2^\sigma$) such that for each $x \in X$,*

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} d(x, A_{kj}) = d(x, A).$$

In this case, $\mathcal{I}_{W_2}^{*\sigma} - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$

Theorem 3.1. *Let $\mathcal{I}_2^\sigma \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If a sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{*\sigma}$-convergent to A, then this sequence is $\mathcal{I}_{W_2}^\sigma$-convergent to A.*

Proof. Since $\mathcal{I}_{W_2}^{*\sigma} - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A)$, there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^\sigma)$ ($\mathbb{N} \times \mathbb{N} \backslash M_2 = H \in \mathcal{I}_2^\sigma$) such that for each $x \in X$,

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} d(x, A_{kj}) = d(x, A).$$

Let $\varepsilon > 0$. Then, there exists $k_0 \in \mathbb{N}$ such that for each $x \in X$,

$$|d(x, A_{kj}) - d(x, A)| < \varepsilon,$$

for all $(k,j) \in M_2$ and $k, j \geq k_0$. Hence, for every $\varepsilon > 0$ and each $x \in X$, we have

$$\begin{aligned} T(\varepsilon, x) &= \{(k,j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \\ &\subset H \cup \Big(M_2 \cap \big((\{1,2,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,...,(k_0-1)\})\big)\Big). \end{aligned}$$

Since $\mathcal{I}_2^{\sigma} \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal,

$$H \cup \Big(M_2 \cap \big((\{1,2,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,...,(k_0-1)\}) \big) \Big) \in \mathcal{I}_2^{\sigma},$$

so we have $T(\varepsilon, x) \in \mathcal{I}_2^{\sigma}$ that is $V_2(T(\varepsilon,x)) = 0$. Hence,

$$\mathcal{I}_{W_2}^{\sigma} - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A).$$

□

Theorem 3.2. *Let $\mathcal{I}_2^{\sigma} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property $(AP2)$. If $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{\sigma}$-convergent to A, then $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{*\sigma}$-convergent to A.*

Proof. Suppose that $\mathcal{I}_2^{\sigma}$ satisfies the property $(AP2)$. Let $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{\sigma}$-convergent to A. Then,

$$T(\varepsilon, x) = T_{\varepsilon} = \{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_2^{\sigma}$$

for every $\varepsilon > 0$ and for each $x \in X$. Put

$$T_1 = T(1, x) = \{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq 1\}$$

and

$$T_v = T(v, x) = \left\{ (k,j) \in \mathbb{N}\times\mathbb{N} : \frac{1}{v} \leq |d(x, A_{kj}) - d(x, A)| < \frac{1}{v-1} \right\},$$

for $v \geq 2$ and $v \in \mathbb{N}$. Obviously $T_i \cap T_j = \emptyset$ for $i \neq j$ and $T_i \in \mathcal{I}_2^{\sigma}$ for each $i \in \mathbb{N}$. By the property $(AP2)$ there exits a sequence of sets $\{E_v\}_{v\in\mathbb{N}}$ such that $T_i \Delta E_i$ is included in finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each i and $E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{I}_2^{\sigma}$.

We shall prove that for $M_2 = \mathbb{N}\times\mathbb{N} \backslash E$ we have

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} d(x, A_{kj}) = d(x, A).$$

Let $\eta > 0$ be given. Choose $v \in \mathbb{N}$ such that $\frac{1}{v} < \eta$. Then,

$$\{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \eta\} \subset \bigcup_{i=1}^{v} T_i.$$

Since $T_i \Delta E_i$, $i = 1, 2, ...$ are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{v} T_i \right) \cap \big\{(k,j) : k \geq n_0 \wedge j \geq n_0\big\} = \left(\bigcup_{i=1}^{v} E_i \right) \cap \big\{(k,j) : k \geq n_0 \wedge j \geq n_0\big\}.$$

If $k, j > n_0$ and $(k,j) \notin E$, then

$$(k,j) \notin \bigcup_{i=1}^{v} E_i \text{ and } (k,j) \notin \bigcup_{i=1}^{v} T_i.$$

This implies that

$$|d(x, A_{kj}) - d(x, A)| < \frac{1}{v} < \eta.$$

Hence, we have

$$\lim_{\substack{k,j\to\infty \\ (k,j)\in M_2}} d(x, A_{kj}) = d(x, A).$$

□

Now, we define the concepts of Wijsman $\mathcal{I}_2^{\sigma}$–invariant Cauchy and Wijsman $\mathcal{I}_2^{*\sigma}$-invariant Cauchy double sequence of sets.

Definition 3.5. *A double sequence $\{A_{kj}\}$ is said to be Wijsman $\mathcal{I}_2$-invariant Cauchy sequence or $\mathcal{I}_{W_2}^{\sigma}$-Cauchy sequence, if for every $\varepsilon > 0$ and for each $x \in X$, there exist numbers $r = r(\varepsilon, x), s = s(\varepsilon, x) \in \mathbb{N}$ such that*

$$A(\varepsilon, x) = \big\{(k, j) : |d(x, A_{kj}) - d(x, A_{rs})| \geq \varepsilon\big\} \in \mathcal{I}_2^{\sigma},$$

that is, $V_2\big(A(\varepsilon, x)\big) = 0$.

Definition 3.6. *A double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{*\sigma}$-Cauchy if there exists a set $M_2 \in \mathcal{F}(\mathcal{I}_2^{\sigma})$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M_2 = H \in \mathcal{I}_2^{\sigma}$) such that for every $x \in X$ and $(k, j), (p, q) \in M_2$*

$$\lim_{k,j,p,q\to\infty} |d(x, A_{kj}) - d(x, A_{pq})| = 0.$$

We give following theorems which show relationships between $\mathcal{I}_{W_2}^{\sigma}$-convergence, $\mathcal{I}_{W_2}^{\sigma}$-Cauchy sequence and $\mathcal{I}_{W_2}^{*\sigma}$-Cauchy sequence. The proof of them are similar to the proof of Theorems in [8, 24], so we omit them.

Theorem 3.3. *If a double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{\sigma}$-convergent, then $\{A_{kj}\}$ is an $\mathcal{I}_{W_2}^{\sigma}$-Cauchy double sequence of sets.*

Theorem 3.4. *If a double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{*\sigma}$-Cauchy double sequence, then $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{\sigma}$-Cauchy double sequence of sets.*

Theorem 3.5. *Let $\mathcal{I}_2^{\sigma}$ has property $(AP2)$. Then, the concepts $\mathcal{I}_{W_2}^{\sigma}$-Cauchy sequence and $\mathcal{I}_{W_2}^{*\sigma}$-Cauchy sequence of sets coincide.*

Definition 3.7. *A double sequence $\{A_{kj}\}$ is said to be Wijsman strongly invariant convergent to A, if for each $x \in X$,*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \big|d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, A)\big| = 0, \text{ uniformly in } s,t.$$

Definition 3.8. *A double sequence $\{A_{kj}\}$ is said to be Wijsman p-strongly invariant convergent to A, if for each $x \in X$,*

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} \big|d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, A)\big|^p = 0, \text{ uniformly in } s,t.$$

where $0 < p < \infty$. In this case, we write $A_{kj} \to A([W_2V_\sigma]_p)$. Also, the set of all Wijsman p-strongly invariant convergent sequences of sets will be denoted by $[W_2V_\sigma]_p$.

Theorem 3.6. *Let $\{A_{kj}\}$ be bounded sequence. If $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{\sigma}$-convergent to A, then $\{A_{kj}\}$ is Wijsman invariant convergent to A.*

Proof. Let $m, n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. For each $x \in X$, we estimate

$$u(s, t, m, n, x) = \left|\frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{\sigma^k(s),\sigma^j(t)}) - d(x, A)\right|.$$

Then, for each $x \in X$ we have

$$u(s, t, m, n, x) \leq u^1(s, t, m, n, x) + u^2(s, t, m, n, x)$$

where

$$u^1(s,t,m,n,x) = \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|\geq\varepsilon}}^{m,n} |d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)|$$

and

$$u^2(s,t,m,n,x) = \frac{1}{mn} \sum_{\substack{k,j=1,1 \\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|<\varepsilon}}^{m,n} |d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)|.$$

Therefore, we have

$$u^2(s,t,m,n,x) < \varepsilon,$$

for each $x \in X$ and for every $s,t = 1,2,\dots$. The boundedness of $\{A_{kj}\}$ implies that there exists $L > 0$ such that

$$|d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)| \leq L, \quad (k,s,j,t = 1,2,\dots),$$

then this implies that

$$\begin{aligned} u^1(s,t,m,n,x) &\leq \frac{L}{mn}|\{1 \leq k \leq m, 1 \leq j \leq n : |d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)| \geq \varepsilon\}| \\ &\leq L\frac{\max_{s,t}|\{1 \leq k \leq m, 1 \leq j \leq n : |d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)| \geq \varepsilon\}|}{mn} \\ &= L\frac{S_{mn}}{mn}. \end{aligned}$$

Hence $\{A_{kj}\}$ is Wijsman invariant convergent to A. □

Theorem 3.7. *Let $\mathcal{I}_2^{\sigma} \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and $0 < p < \infty$.*

- **(i):** *If $A_{kj} \to A([W_2V_\sigma]_p)$, then $A_{kj} \to A(\mathcal{I}_{W_2}^{\sigma})$.*
- **(ii):** *If $\{A_{kj}\} \in L_\infty^2$ and $A_{kj} \to A(\mathcal{I}_{W_2}^{\sigma})$, then $A_{kj} \to A([W_2V_\sigma]_p)$.*
- **(iii):** *If $\{A_{kj}\} \in L_\infty^2$, then $\{A_{kj}\}$ is $\mathcal{I}_{W_2}^{\sigma}$-convergent to A if and only if $A_{kj} \to A([W_2V_\sigma]_p)$.*

Proof. (i) : Assume that $A_{kj} \to A([W_2V_\sigma]_p)$, for every $\varepsilon > 0$ and for each $x \in X$. Then, we can write

$$\begin{aligned} &\sum_{k,j=1,1}^{m,n} \left|d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)\right|^p \\ &\qquad \geq \sum_{\substack{k,j=1,1 \\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|\geq\varepsilon}}^{m,n} |d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)|^p \\ &\qquad \geq \varepsilon^p|\{k \leq m, j \leq n : |d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)| \geq \varepsilon\}| \\ &\qquad \geq \varepsilon^p \min_{s,t}|\{k \leq m, j \leq n : |d(x,A_{\sigma^k(s),\sigma^j(t)}) - d(x,A)| \geq \varepsilon\}| \end{aligned}$$

and

$$\frac{1}{mn}\sum_{k,j=1,1}^{m,n}\left|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)\right|^p$$
$$\geq \varepsilon^p\frac{\min_{s,t}|\{k\leq m, j\leq n: |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|\geq\varepsilon\}|}{mn}$$
$$=\varepsilon^p\frac{s_{mn}}{mn}$$

for every $s,t=1,2,\ldots$. This implies

$$\lim_{m,n\to\infty}\frac{s_{mn}}{mn}=0$$

and so $\{A_{kj}\}$ is $(\mathcal{I}_{W_2}^{\sigma})$-convergent to A.

(ii) : Suppose that $\{A_{kj}\}\in L_\infty^2$ and $A_{kj}\to A(\mathcal{I}_{W_2}^{\sigma})$. Let $0<p<\infty$ and $\varepsilon>0$. By assumption we have $V_2(A(\varepsilon,x))=0$. Since $\{A_{kj}\}$ is bounded, $\{A_{kj}\}$ implies that there exists $L>0$ such that for each $x\in X$,

$$|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|\leq L$$

for all k,s,j and t. Then, we have

$$\frac{1}{mn}\sum_{k,j=1,1}^{m,n}\left|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)\right|^p$$
$$=\frac{1}{mn}\sum_{\substack{k,j=1,1\\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|\geq\varepsilon}}^{m,n}|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|^p$$
$$+\frac{1}{mn}\sum_{\substack{k,j=1,1\\ |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|<\varepsilon}}^{m,n}|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|^p$$
$$\leq L\frac{\max_{s,t}|\{k\leq m, j\leq n: |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|\geq\varepsilon\}|}{mn}+\varepsilon^p$$
$$\leq L\frac{S_{mn}}{mn}+\varepsilon^p,$$

for each $x\in X$. Hence, for each $x\in X$ we obtain

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{k,j=1,1}^{m,n}\left|d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)\right|^p=0,\ \text{uniformly in s,t.}$$

(iii) : This is immediate consequence of (i) and (ii). □

Now, we define Wijsman invariant statistical convergence of double sequences of sets. We shall state a theorem that gives a relation between W_2S_σ and $\mathcal{I}_{W_2}^{\sigma}$ without proof.

Definition 3.9. *A double sequence $\{A_{kj}\}$ is said to be Wijsman invariant statistical convergent or W_2S_σ-convergent to A, if for every $\varepsilon>0$ and for each $x\in X$,*

$$\lim_{m,n\to\infty}\frac{1}{mn}|\{k\leq m, j\leq n: |d(x,A_{\sigma^k(s),\sigma^j(t)})-d(x,A)|\geq\varepsilon\}|=0,\ \textit{uniformly in s,t.}$$

Theorem 3.8. *A sequence $\{A_{kj}\}$ is W_2S_σ-convergent to A if and only if it is $\mathcal{I}_{W_2}^{\sigma}$-convergent to A.*

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Şükrü Tortop
Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey
E-mail address: stortop@aku.edu.tr

Erdinç Dündar
Department of Mathematics, Afyon Kocatepe University, Afyonkarahisar, Turkey
E-mail address: edundar@aku.edu.tr, erdincdundar79@gmail.com