Statistical Lacunary Invariant Summability of Double Sequences

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Abstract: In this study, we give definitions of lacunary σ-summability, strongly p-lacunary σ-summability and statistical lacunary σ-convergence for double sequences. We also examine the existence of some relations among the definitions of statistical lacunary σ-convergence, lacunary invariant statistical convergence and strongly p-lacunary σ-summability.

1. Introduction and Background

The concept of statistical convergence was first introduced by Fast [2] and since then it has been studied by Šalát [15], Fridy [3] and many others, too.
A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \Big| \big\{ k \leq n : |x_k - L| \geq \varepsilon \big\} \Big| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.
Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\phi(e) = 1$, where $e = (1, 1, 1, ...)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$.
In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit.
Many authors have studied on the concepts of invariant mean and invariant convergence (see, [5, 6, 8, 10, 14, 16, 20]).
By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$ (see, [9]).
The space of lacunary strong σ-convergent sequences L_θ was defined by Savaş [17] as below:

$$L_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} |x_{\sigma^k(m)} - L| = 0, \text{ uniformly in } m \right\}.$$

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Savaş and Nuray [18] introduced the concept of lacunary σ-statistically convergent sequence as follows:
Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be $S_{\sigma\theta}$-convergent to L if for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |x_{\sigma^k(m)} - L| \geq \varepsilon \right\} \right| = 0,$$

uniformly in m. It is denoted by $x_k \to L(S_{\sigma\theta})$.
The concept of lacunary invariant summability and the space $[V_{\theta\sigma}]_q$ were defined by Pancaroğlu and Nuray [11] as below:
Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary invariant summable to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} x_{\sigma^k(m)} = L,$$

uniformly in m.
Let $0 < q < \infty$. A sequence $x = (x_k)$ is said to be strongly lacunary q-invariant convergent to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_{\sigma^k(m)} - L|^q = 0,$$

uniformly in m. It is denoted by $x_k \to L\big([V_{\theta\sigma}]_q\big)$
The concepts of convergence for double sequences have been studied by many authors (see, [1, 4, 12, 13, 21]).
A double sequence $x = (x_{kj})$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j \geq N_\varepsilon$.
A double sequence $x = (x_{kj})$ is said to be bounded if there exists an $M > 0$ such that $|x_{kj}| < M$ for all k and j, i.e., if $\sup_{k,j} |x_{kj}| < \infty$.
The set of all bounded double sequences will be denoted by ℓ_∞^2.
Mursaleen and Edely [7] introduced the concept of statistically convergence for double sequences as follows:
A double sequence $x = (x_{kj})$ is said to be statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{m,n\to\infty} \frac{1}{mn} \left| \left\{ (k,j),\ k \leq m \text{ and } j \leq n : \ |x_{kj} - L| \geq \varepsilon \right\} \right| = 0.$$

The double sequence $\theta_2 = \{k_r, j_u\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \ \text{ and } \ j_0 = 0,\ \bar{h}_u = j_u - j_{u-1} \to \infty \ \text{ as } \ r,u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u,\ \ h_{ru} = h_r \bar{h}_u,\ \ I_{ru} = \left\{ (k,j) : k_{r-1} < k \leq k_r \ \ and \ \ j_{u-1} < j \leq j_u \right\}.$$

Using the double lacunary sequence concept, the concept of lacunary σ-statistically convergence for double sequences and similar concepts were defined by Savaş and Patterson [19] as below:
Let $\theta_2 = \{k_r, j_u\}$ be a double lacunary sequence. A double sequence $x = (x_{kj})$ is said to be lacunary invariant statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left| \left\{ (k,j) \in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon \right\} \right| = 0,$$

uniformly in m and n. It is denoted by $x_{kj} \to L\big(S_2^{\sigma\theta}\big)$.
A double sequence $x = (x_{kj})$ is said to be strongly lacunary invariant convergent to L if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L| = 0,$$

uniformly in m and n. It is denoted by $x_{kj} \to L\big([V_2^{\sigma\theta}]\big)$.

2. Statistical Lacunary Invariant Summability of Double Sequences

In this study, we give definitions of lacunary σ-summability, strongly p-lacunary σ-summability and statistical lacunary σ-convergence for double sequences. We also examine the existence of some relations among the definitions of statistical lacunary σ-convergence, lacunary invariant statistical convergence and strongly p-lacunary σ-summability.

Definition 2.1 Let $\theta_2 = \{k_r, j_u\}$ be a double lacunary sequence. A double sequence $x = (x_{kj})$ is said to be statistical lacunary σ-convergent to L if for every $\varepsilon > 0$

$$\lim_{v,w\to\infty} \frac{1}{vw} \left| \left\{ (k,j),\ k \le v \text{ and } j \le w : \left| \frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} x_{\sigma^k(m),\sigma^j(n)} - L \right| \ge \varepsilon \right\} \right| = 0,$$

uniformly in m and n . In this case, we write $x_{kj} \to L(S_2^{\theta\sigma})$.

In other words, a double sequence $x = (x_{kj})$ is statistical lacunary σ-convergent to L if and only if the sequence

$$\left(\frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} x_{\sigma^k(m),\sigma^j(n)} \right)$$

is statistical convergent to L.

Theorem 2.2 *Assume that $x = (x_{kj}) \in \ell_\infty^2$. If x is lacunary invariant statistical convergent to L, then this sequence is statistical lacunary σ-convergent to L.*

Proof. Let $x = (x_{kj})$ be a bounded double sequence and lacunary invariant statistical convergent to L. Let take a set $A(\varepsilon)$ as follows:

$$A(\varepsilon) = \left\{ k_{r-1} \le k \le k_r,\ j_{u-1} \le j \le j_u : |x_{\sigma^k(m),\sigma^j(n)} - L| \ge \varepsilon \right\},$$

for each $m \ge 1$ and $n \ge 1$. Then we have

$$\begin{aligned} \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{\sigma^k(m),\sigma^j(n)} - L \right| &= \left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} \left(x_{\sigma^k(m),\sigma^j(n)} - L \right) \right| \\ &\le \left| \frac{1}{h_{ru}} \sum_{(k,j)\in A(\varepsilon)} \left(x_{\sigma^k(m),\sigma^j(n)} - L \right) \right| \\ &\le \left(\sup_{k,j} |x_{\sigma^k(m),\sigma^j(n)} - L| \right) \frac{1}{h_{ru}} |A(\varepsilon)| \to 0 \end{aligned}$$

as $r, u \to \infty$, which implies

$$\left| \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} x_{\sigma^k(m),\sigma^j(n)} - L \right| \to 0,$$

for all m and n. That is, $x = (x_{kj})$ is statistical lacunary σ-convergent to L. □

Definition 2.3 Let $\theta_2 = \{k_r, j_u\}$ be a double lacunary sequence. A double sequence $x = (x_{kj})$ is said to be lacunary σ-summable to L if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} x_{\sigma^k(m),\sigma^j(n)} = L,$$

uniformly in m and n. In this case, we write $x_{kj} \to L(V_2^{\sigma\theta})$.

Definition 2.4 Let $\theta_2 = \{k_r, j_u\}$ be a double lacunary sequence and $0 < p < \infty$. A double sequence $x = (x_{kj})$ is said to be strongly p-lacunary σ-summable to L if

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{k,j\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p = 0,$$

uniformly in m and n. In this case, we write $x_{kj} \to L([V_2^{\sigma\theta}]_p)$.

Theorem 2.5 *If a double sequence $x = (x_{kj})$ is strongly p-lacunary σ-summable to L, then this sequence is lacunary invariant statistical convergent to L.*

Proof. Let $x = (x_{kj})$ is strongly p-lacunary σ-summable to L. Then, for each $m \geq 1$ and $n \geq 1$

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p = \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p + \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ |x_{\sigma^k(m),\sigma^j(n)} - L| < \varepsilon}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p,$$

therefore we have

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p \geq \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p \geq \frac{1}{h_{ru}} \varepsilon^p \cdot |A(\varepsilon)|.$$

So if limit is taken as $r,u \to \infty$, we have

$$\varepsilon^p \cdot \lim_{r,u\to\infty} \frac{1}{h_{ru}} \left| \left\{ (k,j) \in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon \right\} \right| \leq \lim_{r,u\to\infty} \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p \to 0.$$

That is, $x = (x_{kj})$ is lacunary invariant statistical convergent to L. □

Theorem 2.6 *Assume that $x = (x_{kj}) \in \ell_\infty^2$. If x is lacunary invariant statistical convergent to L, then this sequence is strongly p-lacunary σ-summable to L.*

Proof. Suppose that $x = (x_{kj})$ is a bounded double sequence and lacunary invariant statistical convergent to L. Since x is bounded, there exists $M > 0$ such that

$$|x_{\sigma^k(m),\sigma^j(n)} - L| \leq M$$

uniformly in m and n. Now that $x = (x_{kj})$ is lacunary invariant statistical convergent to L, for every $\varepsilon > 0$ we have

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left| \left\{ (k,j) \in I_{ru} : |x_{\sigma^k(m),\sigma^j(n)} - L| \geq \varepsilon \right\} \right| = 0$$

uniformly in m and n. Also, we can write

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p = \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ (k,j)\in A(\varepsilon)}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p + \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ (k,j)\notin A(\varepsilon)}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p = t^{(1)}(r,u) + t^{(2)}(r,u)$$

where

$$t^{(1)}(r,u) = \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ (k,j)\in A(\varepsilon)}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p$$

and

$$t^{(2)}(r,u) = \frac{1}{h_{ru}} \sum_{\substack{(k,j)\in I_{ru} \\ (k,j)\notin A(\varepsilon)}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p.$$

Now if $(k,j) \notin A(\varepsilon)$, then $t^{(2)}(r,u) < \varepsilon^p$. If $(k,j) \in A(\varepsilon)$, then

$$t^{(1)}(r,u) \leq \left(\sup_{k,j} |x_{\sigma^k(m),\sigma^j(n)} - L| \right) \frac{|A(\varepsilon)|}{h_{ru}} \leq M \frac{|A(\varepsilon)|}{h_{ru}} \to 0.$$

Thus

$$\frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |x_{\sigma^k(m),\sigma^j(n)} - L|^p \to 0,$$

uniformly in m and n. □

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