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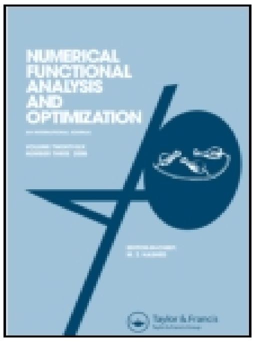

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On Rough $\mathcal{I}_2$-Convergence of Double Sequences

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ABSTRACT
In this study, we introduce the notion of rough $\mathcal{I}_2$-convergence and the set of rough $\mathcal{I}_2$-limit points of a double sequence and obtained two rough $\mathcal{I}_2$-convergence criteria associated with this set. Later, we proved that this set is closed and convex. Finally, we examined the relationships between the set of cluster points and the set of rough $\mathcal{I}_2$-limit points of a double sequence.

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1. Background and introduction

Throughout the article $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [11] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of a subset of the set of natural numbers. Nuray and Ruckle [14] introduced the same with another name, which generalized statistical convergence. Kostyrko et al. [12] studied the idea of $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points, and Demirci [6] studied the concepts of $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior. Das et al. [4] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Also, Das and Malik [5] introduced the concept of $\mathcal{I}$-limit points, $\mathcal{I}$-cluster points and $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of double sequences. A lot of progress has been made in this area after the works of [1, 8–10, 13, 18–21].

The idea of rough convergence was first introduced by Phu [15] in finite-dimensional normed spaces. In [15], he showed that the set $\mathrm{LIM}^r x$ is bounded, closed, and convex and introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\mathrm{LIM}^r x$ on the roughness degree r. In another article [16] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r -continuous at every point $x \in X$ under the assumption $dim Y < \infty$ and $r > 0$ where X and

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Y are normed spaces. In [17], he extended the results given in [15] to infinite-dimensional normed spaces. Aytar [2] found of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [3] found that the r-limit set of the sequence is equal to the intersection of these sets and that r-core of the sequence is equal to the union of these sets. Recently, Dündar and Çakan [7, 8] introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and studied the notion of rough convergence and the set of rough limit points of a double sequence.

In this article, we introduce the notion of rough $\mathcal{I}_2$-convergence and the set of rough $\mathcal{I}_2$-limit points of a double sequence and obtained two rough $\mathcal{I}_2$-convergence criteria associated with this set. Later, we proved that this set is closed and convex. Finally, we examined the relations between the set of cluster points and the set of rough $\mathcal{I}_2$-limit points of a double sequence.

We note that our results and proof techniques presented in this article are $\mathcal{I}$ analogues of those in Phu's [15] article. Namely, the actual origin of most of these results and proof techniques is his article. The following our theorems and results are the $\mathcal{I}$-extension of theorems and results in [15].

Definition 1.1 ([4]). A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

Definition 1.2 ([4]). A double sequence $x = (x_{mn})$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

Definition 1.3 ([11]). Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

i) $\emptyset \in \mathcal{I}$,
ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Definition 1.4 ([11]). Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

i) $\emptyset \notin \mathcal{F}$,
ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$.

Lemma 1.5 ([11]). *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

Definition 1.6 ([11]). Let (X, ρ) be a linear metric space and $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal. a sequence $(x_i)_{i\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-convergent to $\xi \in X$ ($\mathcal{I} - \lim_{i\to\infty} x_i = \xi$) if and only if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{i \in \mathbb{N} : \rho(x_i, \xi) \geq \varepsilon\}$ belongs to $\mathcal{I}$. The element ξ is called the $\mathcal{I}$-limit of the sequence $x = (x_i)_{i\in\mathbb{N}}$.

Note that if $\mathcal{I}$ is an admissible ideal, then usual convergence in X implies $\mathcal{I}$-convergence in X.

Definition 1.7 ([6]). For a sequence $x = (x_i)$ of real numbers, the notions of ideal limit superior and ideal limit inferior are defined as follows:

$$\mathcal{I} - \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset \\ -\infty, & \text{if } B_x = \emptyset \end{cases}$$

and

$$\mathcal{I} - \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset \\ +\infty, & \text{if } A_x = \emptyset \end{cases},$$

where $A_x = \{a \in \mathbb{R} : \{i \in \mathbb{N} : x_i < a\} \notin \mathcal{I}\}$ and $B_x = \{b \in \mathbb{R} : \{i \in \mathbb{N} : x_i > b\} \notin \mathcal{I}\}$.

Definition 1.8 ([4]). A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in \mathbb{N}$.

It is evident that a strongly admissible ideal is admissible also.

Throughout the article, we take $\mathcal{I}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Definition 1.9 ([4]). Let (X, ρ) be a metric space A double sequence $x = (x_{mn})$ in X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we say that x is $\mathcal{I}_2$-convergent and we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Definition 1.10 ([5]). Let $x = (x_{jk})$ be a double sequence of real numbers and

$$A_y = \{a \in \mathbb{R} : \{(j, k) \in \mathbb{N} \times \mathbb{N} : x_{jk} < a\} \notin \mathcal{I}_2\}$$

and

$$B_y = \{b \in \mathbb{R} : \{(j,k) \in \mathbb{N} \times \mathbb{N} : x_{jk} > b\} \notin \mathcal{I}_2\}.$$

Then $\mathcal{I}_2$-limit superior and $\mathcal{I}_2$-limit inferior of x are defined as follows:

$$\mathcal{I}_2 - \limsup x = \begin{cases} \sup B_x, & \textit{if } B_x \neq \emptyset \\ -\infty, & \textit{if } B_x = \emptyset \end{cases}$$

and

$$\mathcal{I}_2 - \liminf x = \begin{cases} \inf A_x, & \textit{if } A_x \neq \emptyset \\ +\infty, & \textit{if } A_x = \emptyset. \end{cases}$$

Within the article let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_i) \subset \mathbb{R}^n$.

Definition 1.11 ([15]). The sequence $x = (x_i)$ is said to be r-convergent to x_*, denoted by $x_i \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \ \exists i_\varepsilon \in \mathbb{N} : \ i \geq i_\varepsilon \Rightarrow \|x_i - x_*\| < r + \varepsilon.$$

The set

$$\mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_i \xrightarrow{r} x_*\}$$

is called the r-limit set of the sequence $x = (x_i)$. A sequence $x = (x_i)$ is said to be r-convergent if $\mathrm{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_i)$. For $r = 0$, we get the ordinary convergence. There are several reasons for this interest (see [15]).

Definition 1.12 ([7]). A sequence $x = (x_i)$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}^n$, written as $\mathcal{I}$-$\lim x = L$, provided that the set $\{i \in \mathbb{N} : \|x_i - L\| \geq \varepsilon\} \in \mathcal{I}$, for every $\varepsilon > 0$. In this case, L is called the $\mathcal{I}$-limit of the sequence x.

Definition 1.13 ([7]). $c \in \mathbb{R}^n$ is called a $\mathcal{I}$-cluster point of a sequence $x = (x_i)$ provided that

$$\{i \in \mathbb{N} : \|x_i - c\| < \varepsilon\} \notin \mathcal{I}$$

for every $\varepsilon > 0$. We denote the set of all $\mathcal{I}$-cluster points of the sequence x by $\mathcal{I}(\Gamma_x)$.

A sequence $x = (x_i)$ is said to be $\mathcal{I}$-bounded if there exists a positive real number M such that $\{i \in \mathbb{N} : \|x_i\| \geq M\} \in \mathcal{I}$.

Definition 1.14 ([7]). A sequence $x = (x_i)$ is said to be rough $\mathcal{I}$-convergent (r-$\mathcal{I}$-convergent) to x_* with the roughness degree r, denoted by $x_i \xrightarrow{r-\mathcal{I}} x_*$

provided that

$$\{i \in \mathbb{N} : \|x_i - x_*\| \geq r + \varepsilon\}$$

belongs to $\mathcal{I}$ for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I} - \limsup \|x_i - x_*\| \leq r \tag{1.1}$$

is satisfied. In addition, we can write $x_i \xrightarrow{r-\mathcal{I}} x_*$ if the inequality $\|x_i - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all i.

Throughout the article, we consider a sequence $x = (x_{mn})$ such that $(x_{mn}) \in \mathbb{R}^n$, $m, n \in \mathbb{N}$.

Definition 1.15 ([8]). The double sequence $x = (x_{mn})$ is said to be rough convergent (r-convergent) to x_* with the roughness degree r, denoted by $x_{mn} \xrightarrow{r} x_*$ provided that

$$\forall \varepsilon > 0 \ \exists k_\varepsilon \in \mathbb{N} : \quad m, n \geq k_\varepsilon \Rightarrow \|x_{mn} - x_*\| < r + \varepsilon, \tag{1.2}$$

or equivalently, if

$$\limsup \ \|x_{mn} - x_*\| \leq r. \tag{1.3}$$

2. Main results

Definition 2.1. For some given real number $r \geq 0$, a sequence $x = (x_{mn})$ is said to be r-$\mathcal{I}_2$-convergent to x_* with the roughness degree r, denoted by $x_{mn} \xrightarrow{r-\mathcal{I}_2} x_*$, provided that

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| \geq r + \varepsilon\} \in \mathcal{I}_2, \tag{2.1}$$

for every $\varepsilon > 0$; or equivalently, if the condition

$$\mathcal{I}_2 - \limsup \|x_{mn} - x_*\| \leq r \tag{2.2}$$

is satisfied. In addition, we can write $x_{mn} \xrightarrow{r-\mathcal{I}_2} x_*$ if the inequality $\|x_{mn} - x_*\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all (m, n).

Now, we give the definitions of $\mathcal{I}_2$-cluster point of a double sequence and of $\mathcal{I}_2$-boundedness for a double sequence.

$c \in \mathbb{R}^n$ is called a $\mathcal{I}_2$-cluster point of a double sequence $x = (x_{mn})$ provided that

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - c\| < \varepsilon\} \notin \mathcal{I}_2$$

for every $\varepsilon > 0$. We denote the set of all $\mathcal{I}_2$-cluster points of the double sequence $x = (x_{mn})$ by $\mathcal{I}_2(\Gamma_x)$.

A double sequence $x = (x_{mn})$ is said to be $\mathcal{I}_2$-bounded if there exists a positive real number M such that $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn}\| \geq M\} \in \mathcal{I}_2$.

Remark 2.2. r-convergence implies r-$\mathcal{I}_2$-convergence as depending the roughness degree.

Here r is called the roughness degree. If we take $r = 0$, then we obtain the ordinary $\mathcal{I}_2$-convergence of a double sequence.

In general, the r-$\mathcal{I}_2$-limit of a double sequence may not be unique for the roughness degree $r > 0$. So we have to consider the so-called rough $\mathcal{I}_2$-limit set of a double sequence $x = (x_{mn})$, which is defined by

$$\mathcal{I}_2 - \mathrm{LIM}^r x := \{x_* \in \mathbb{R}^n : x_{mn} \xrightarrow{r-\mathcal{I}_2} x_*\}.$$

A double sequence $x = (x_{mn})$ is said to be r-$\mathcal{I}_2$-convergent if $\mathcal{I}_2 - \mathrm{LIM}^r x \neq \emptyset$.

As noted above, we cannot say that the r-$\mathcal{I}_2$-limit of a double sequence is unique for the roughness degree $r > 0$. The following theorem is related to this claim.

Theorem 2.3. *We have $diam(\mathcal{I}_2 - \mathrm{LIM}^r x) \leq 2r$, for any sequence $x = (x_{mn})$. In general, $diam(\mathcal{I}_2 - \mathrm{LIM}^r x)$ has no smaller bound.*

Proof. Suppose that $diam(\mathcal{I}_2 - \mathrm{LIM}^r x) = \sup\{\|y - z\| : y, z \in \mathcal{I}_2 - \mathrm{LIM}^r x\} > 2r$. Then there exist $y, z \in \mathcal{I}_2 - \mathrm{LIM}^r x$ such that $\|y - z\| > 2r$. Take $\varepsilon \in (0, \frac{\|y-z\|}{2} - r)$. Since $y, z \in \mathcal{I}_2 - \mathrm{LIM}^r x$, for every $\varepsilon > 0$, we have

$$A_1(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - y\| \geq r + \varepsilon\} \in \mathcal{I}_2$$

and

$$A_2(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - z\| \geq r + \varepsilon\} \in \mathcal{I}_2.$$

In this case, we have

$$A_1^c(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - y\| < r + \varepsilon\} \in \mathcal{F}(\mathcal{I}_2)$$

and

$$A_2^c(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - z\| < r + \varepsilon\} \in \mathcal{F}(\mathcal{I}_2).$$

Using the properties of $\mathcal{F}(\mathcal{I}_2)$, $A_1(\varepsilon)^c \cap A_2(\varepsilon)^c$ is non-empty and we get

$$(A_1^c(\varepsilon) \cap A_2^c(\varepsilon)) \in \mathcal{F}(\mathcal{I}_2).$$

Thus, we can write

$$\|y - z\| \leq \|x_{mn} - y\| + \|x_{mn} - z\| < 2(r + \varepsilon) < 2(r + \frac{\|y - z\|}{2} - r) = \|y - z\|,$$

for all $(m, n) \in A_1(\varepsilon)^c \cap A_2(\varepsilon)^c$, which is a contradiction.

Now, for proof of the second part of the theorem, consider a double sequence $x = (x_{mn})$ such that $\mathcal{I}_2$-lim $x_{mn} = x_*$. Let $\varepsilon > 0$. Then, we can write

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| \geq \varepsilon\} \in \mathcal{I}_2.$$

Thus, we have

$$\|x_{mn} - y\| \leq \|x_{mn} - x_*\| + \|x_* - y\| \leq \|x_{mn} - x_*\| + r,$$

for each $y \in \overline{B_r}(x_*) := \{y \in \mathbb{R}^n : \|y - x_*\| \leq r\}$. Then, we get

$$\|x_{mn} - y\| < r + \varepsilon,$$

for each $(m,n) \in \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| < \varepsilon\}$. Since the double sequence $x = (x_{mn})$ is $\mathcal{I}_2$-convergent to x_*, we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| < \varepsilon\} \in \mathcal{F}(\mathcal{I}_2).$$

Thus, we have $y \in \mathcal{I}_2 - \text{LIM}^r x$, and we can write

$$\mathcal{I}_2 - \text{LIM}^r x = \overline{B_r}(x_*). \tag{2.3}$$

Since $diam(\overline{B_r}(x_*)) = 2r$, this shows that, in general, the upper bound $2r$ of the diameter of the set $\mathcal{I}_2 - \text{LIM}^r x$ cannot be decreased further. □

Now we give some topological and geometrical properties of the r-$\mathcal{I}_2$-limit set of a double sequence.

Theorem 2.4. *The r-$\mathcal{I}_2$-limit set of a double sequence $x = (x_{mn})$ is closed.*

Proof. If $\mathcal{I}_2 - \text{LIM}^r x = \emptyset$, then there is nothing to prove. Suppose that $\mathcal{I}_2 - \text{LIM}^r x \neq \emptyset$. In this case we can select an arbitrary sequence $(y_{mn}) \subseteq \mathcal{I}_2 - \text{LIM}^r x$ such that $\lim_{m,n\to\infty} y_{mn} = y_*$. We must show that $y_* \in \mathcal{I}_2 - \text{LIM}^r x$.

Let $\varepsilon > 0$ be given. Since $y_{mn} \to y_*$, there exists $k = k_\varepsilon \in \mathbb{N}$ such that

$$\|y_{mn} - y_*\| < \varepsilon, \quad \text{for all } m,n > k.$$

Now select an an $m_0, n_0 \in \mathbb{N}$ such that $m_0, n_0 \geq k$. Then we can write

$$\|y_{m_0 n_0} - y_*\| < \varepsilon.$$

On the other hand, since $(y_{mn}) \subseteq \mathcal{I}_2 - \text{LIM}^r x$, we have $y_{m_0 n_0} \in \mathcal{I}_2 - \text{LIM}^r x$, that is,

$$A(\varepsilon) = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - y_{m_0 n_0}\| \geq r + \varepsilon\right\} \in \mathcal{I}_2. \tag{2.4}$$

Now, let us show that the inclusion

$$A^c(\varepsilon) \subseteq A^c(2\varepsilon) \tag{2.5}$$

holds, where $A(2\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - y_*\| \geq r + 2\varepsilon\}$. Take $(k,l) \in A^c(\varepsilon)$. Then we have

$$\|x_{kl} - y_{m_0 n_0}\| < r + \varepsilon,$$

and hence

$$\|x_{kl} - y_*\| \leq \|x_{kl} - y_{m_0 n_0}\| + \|y_{m_0 n_0} - y_*\| < r + 2\varepsilon,$$

that is, $(k,l) \in A^c(2\varepsilon)$, which proves (2.5). So, we have

$$A(2\varepsilon) \subseteq A(\varepsilon).$$

Because of $A(\varepsilon) \in \mathcal{I}_2$ by (2.4), we have $A(2\varepsilon) \in \mathcal{I}_2$ (i.e., $y_* \in \mathcal{I}_2 - \text{LIM}^r x$), which completes the proof. □

Theorem 2.5. *The r-$\mathcal{I}_2$-limit set of a double sequence $x = (x_{mn})$ is convex.*

Proof. Assume that $y_1, y_2 \in \mathcal{I}_2 - \text{LIM}^r x$ for the sequence $x = (x_{mn})$ and let $\varepsilon > 0$ be given. Define

$$A_1(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - y_1\| \geq r + \varepsilon\} \quad \textit{and}$$
$$A_2(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - y_2\| \geq r + \varepsilon\}.$$

Because of $y_1, y_2 \in \mathcal{I}_2 - \text{LIM}^r x$, we have $A_1(\varepsilon), A_2(\varepsilon) \in \mathcal{I}_2$. Hence, we have

$$\|x_{mn} - [(1-\lambda)y_1 + \lambda y_2]\| = \|(1-\lambda)(x_{mn} - y_1) + \lambda(x_{mn} - y_2)\| < r + \varepsilon,$$

for each $(m,n) \in A_1^c(\varepsilon) \cap A_2^c(\varepsilon)$ and each $\lambda \in [0,1]$. Because of $(A_1^c(\varepsilon) \cap A_2^c(\varepsilon)) \in \mathcal{F}(\mathcal{I}_2)$, by definition $\mathcal{F}(\mathcal{I}_2)$ we get

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - [(1-\lambda)y_1 + \lambda y_2]\| \geq r + \varepsilon\} \in \mathcal{I}_2,$$

that is,

$$[(1-\lambda)y_1 + \lambda y_2] \in \mathcal{I}_2 - \text{LIM}^r x,$$

which proves the convexity of the set $\mathcal{I}_2 - \text{LIM}^r x$. □

Theorem 2.6. *Suppose $r > 0$. Then a double sequence $x = (x_{mn})$ is r-$\mathcal{I}_2$-convergent to x_* if and only if there exists a sequence $y = (y_{mn})$ such that*

$$\mathcal{I}_2 - \lim y = x_* \quad \text{and} \quad \|x_{mn} - y_{mn}\| \leq r, \quad \text{for each } m,n \in \mathbb{N}. \tag{2.6}$$

Proof. Assume that $x = (x_{mn})$ is r-$\mathcal{I}_2$-convergent to x_*. Then, by (2.2) we have

$$\mathcal{I}_2 - \limsup \|x_{mn} - x_*\| \leq r. \tag{2.7}$$

Now, define

$$y_{mn} = \begin{cases} x_*, & \textit{if } \|x_{mn} - x_*\| \leq r \\ x_{mn} + r\dfrac{x_* - x_{mn}}{\|x_{mn} - x_*\|}, & \textit{otherwise} \end{cases}.$$

Then, we have

$$\|y_{mn} - x_*\| = \begin{cases} 0, & \textit{if } \|x_{mn} - x_*\| \leq r \\ \|x_{mn} - x_*\| - r, & \textit{otherwise} \end{cases},$$

and, by definition of y_{mn},

$$\|x_{mn} - y_{mn}\| \leq r \tag{2.8}$$

for all $m, n \in \mathbb{N}$. By (2.7) and the definition of y_{mn}, we get

$$\mathcal{I}_2 - \limsup \|y_{mn} - x_*\| = 0,$$

which implies that $\mathcal{I}_2 - \lim y_{mn} = x_*$.

Assume that (2.6) holds. Because of $\mathcal{I}_2 - \lim y = x_*$, we have

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|y_{mn} - x_*\| \geq r + \varepsilon\} \in \mathcal{I}_2,$$

for each $\varepsilon > 0$. Now, define the set

$$B(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| \geq r + \varepsilon\}.$$

It is easy to see that the inclusion

$$B(\varepsilon) \subseteq A(\varepsilon)$$

holds. Because of $A(\varepsilon) \in \mathcal{I}_2$, we get $B(\varepsilon) \in \mathcal{I}_2$. Hence, $x = (x_{mn})$ is r-$\mathcal{I}_2$-convergent to x_*. □

Lemma 2.7. *For an arbitrary $c \in \mathcal{I}_2(\Gamma_x)$ of a double sequence $x = (x_{mn})$, we have*

$$\|x_* - c\| \leq r \text{ for all } x_* \in \mathcal{I}_2 - \mathrm{LIM}^r x.$$

Proof. Assume on the contrary that there exist a point $c \in \mathcal{I}_2(\Gamma_x)$ and $x_* \in \mathcal{I}_2 - \mathrm{LIM}^r x$ such that $\|x_* - c\| > r$. Define $\varepsilon := \frac{\|x_* - c\| - r}{3}$. Hence, we can write

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - c\| < \varepsilon\} \subseteq \{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| \geq r + \varepsilon\}. \tag{2.9}$$

Because of $c \in \mathcal{I}_2(\Gamma_x)$, we get

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - c\| < \varepsilon\} \notin \mathcal{I}_2.$$

But from definition of $\mathcal{I}_2$-convergence, since

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| \geq r + \varepsilon\} \in \mathcal{I}_2,$$

so by (2.9) we have

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - c\| < \varepsilon\} \in \mathcal{I}_2,$$

which contradicts with the fact $c \in \mathcal{I}_2(\Gamma_x)$. This completed the proof of the lemma. □

Theorem 2.8. (i) *If $c \in \mathcal{I}_2(\Gamma_x)$, then*

$$\mathcal{I}_2 - \text{LIM}^r x \subseteq \overline{B}_r(c). \tag{2.10}$$

(ii)

$$\mathcal{I}_2 - \text{LIM}^r x = \bigcap_{c \in \mathcal{I}_2(\Gamma_x)} \overline{B}_r(c) = \{x_* \in \mathbb{R}^n : \mathcal{I}_2(\Gamma_x) \subseteq \overline{B}_r(x_*)\}. \tag{2.11}$$

Proof. (i) If $c \in \mathcal{I}_2(\Gamma_x)$ then by Lemma 2.7, we have

$$\|x_* - c\| \leq r, \text{ for all } x_* \in \mathcal{I}_2 - \text{LIM}^r x,$$

otherwise we get

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - x_*\| \geq r + \varepsilon\} \notin \mathcal{I}, \text{ for } \varepsilon := \frac{\|x_* - c\| - r}{3}.$$

Because of c is an $\mathcal{I}_2$-cluster point of (x_{mn}), this contradicts with the fact that $x_* \in \mathcal{I}_2 - \text{LIM}^r x$.

(ii) From (2.10), we have

$$\mathcal{I}_2 - \text{LIM}^r x \subseteq \bigcap_{c \in \mathcal{I}_2(\Gamma_x)} \overline{B}_r(c). \tag{2.12}$$

Now, let

$$y \in \bigcap_{c \in \mathcal{I}_2(\Gamma_x)} \overline{B}_r(c).$$

Then we have $\|y - c\| \leq r$, for all $c \in \mathcal{I}_2(\Gamma_x)$, which is equivalent to $\mathcal{I}_2(\Gamma_x) \subseteq \overline{B}_r(y)$, i.e.,

$$\bigcap_{c \in \mathcal{I}_2(\Gamma_x)} \overline{B}_r(c) \subseteq \{x_* \in \mathbb{R}^n : \mathcal{I}_2(\Gamma_x) \subseteq \overline{B}_r(x_*)\}. \tag{2.13}$$

Now, let $y \notin \mathcal{I}_2 - \text{LIM}^r x$. Then, there exists an $\varepsilon > 0$ such that

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - y\| \geq r + \varepsilon\} \notin \mathcal{I}_2$$

which implies the existence of an $\mathcal{I}_2$-cluster point c of the sequence x with $\|y - c\| \geq r + \varepsilon$, that is,

$$\mathcal{I}_2(\Gamma_x) \not\subseteq \overline{B}_r(y) \quad \text{and} \quad y \notin \{x_* \in \mathbb{R}^n : \mathcal{I}_2(\Gamma_x) \subseteq \overline{B}_r(x_*)\}.$$

Hence, $y \in \mathcal{I}_2 - \text{LIM}^r x$ follows from $y \in \{x_* \in \mathbb{R}^n : \mathcal{I}_2(\Gamma_x) \subseteq \overline{B}_r(x_*)\}$, i.e.,

$$\{x_* \in \mathbb{R}^n : \mathcal{I}_2(\Gamma_x) \subseteq \overline{B}_r(x_*)\} \subseteq \mathcal{I}_2 - \text{LIM}^r x. \tag{2.14}$$

Therefore, the inclusions (2.12)–(2.14) ensure that (2.11) holds that is,

$$\mathcal{I}_2 - \text{LIM}^r x = \bigcap_{c \in \mathcal{I}_2(\Gamma_x)} \overline{B}_r(c) = \{x_* \in \mathbb{R}^n : \mathcal{I}_2(\Gamma_x) \subseteq \overline{B}_r(x_*)\}.$$ □

Finally we give the relation between the set of $\mathcal{I}_2$-cluster points and the set of rough $\mathcal{I}_2$-limit points of a double sequence.

Theorem 2.9. *Let $x = (x_{mn})$ be an $\mathcal{I}_2$-bounded sequences. If $r \geq diam(\mathcal{I}_2(\Gamma_x))$, then we have $\mathcal{I}_2(\Gamma_x) \subseteq \mathcal{I}_2 - \mathrm{LIM}^r x$.*

Proof. Let $c \notin \mathcal{I}_2 - \mathrm{LIM}^r x$. Then there exist an $\varepsilon > 0$ such that

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : \|x_{mn} - c\| \geq r + \varepsilon\} \notin \mathcal{I}_2. \tag{2.15}$$

Since $x = (x_{mn})$ is $\mathcal{I}_2$-bounded and from the inequality (2.15), there exists an $\mathcal{I}_2$-cluster point c_1 such that

$$\|c - c_1\| > r + \varepsilon_1,$$

where $\varepsilon_1 := \frac{\varepsilon}{2}$. So we get

$$diam(\mathcal{I}_2(\Gamma_x)) > r + \varepsilon_1,$$

which proves the theorem. □

The converse of this theorem is also holds, i.e., if $\mathcal{I}_2(\Gamma_x) \subseteq \mathcal{I}_2 - \mathrm{LIM}^r x$, then we have $r \geq diam(\mathcal{I}_2(\Gamma_x))$.

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