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On $\mathcal{I}$-Convergence of Sequences of Functions in 2-Normed Spaces

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Abstract. In this paper, we study concepts of convergence and ideal convergence of sequence of functions and investigate relationships between them and some properties such as linearity in 2-normed spaces. Also, we prove a decomposition theorem for ideal convergent sequences of functions in 2-normed spaces.

Keywords: Ideal; Filter; Sequence of functions; $\mathcal{I}$-Convergence; 2-normed spaces.

1. Introduction, Definitions and Notations

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [8] and Schoenberg [29].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [20] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$ [8, 9]. Gökhan et al. [13] introduced the notion of pointwise and uniform statistical convergent of double sequences of real-valued functions. Gezer and Karakuş [12] investigated $\mathcal{I}$-pointwise and uniform convergence and $\mathcal{I}^*$-pointwise and uniform convergence of function sequences and they examined the relation between them. Baláz et al. [2] investigated $\mathcal{I}$-convergence and $\mathcal{I}$-continuity of real functions. Balcerzak et al. [3] studied statistical convergence and ideal convergence for sequences of functions Dündar and Altay [5, 6] studied the concepts of pointwise and uniformly $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [7] investigated some results of $\mathcal{I}_2$-convergence of double sequences of functions.

The concept of 2-normed spaces was initially introduced by Gähler [10, 11] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [17] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Şahiner et al. [31] and Gürdal [19] studied $\mathcal{I}$-convergence in 2-normed spaces. Gürdal and Açık [18] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [27] presented various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathcal{I}$-equistatistically convergence and study $\mathcal{I}$-equistatistically convergence of sequences of functions. Recently, Savaş and Gürdal [28] concerned with $\mathcal{I}$-convergence of sequences of functions in random 2-normed spaces and introduce the concepts of ideal uniform convergence and ideal pointwise convergence in the topology induced by random 2-normed spaces, and gave some basic properties of these concepts. Arslan and Dündar [1] investigated the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions in 2-normed spaces. Also, Yegül and Dündar [33] studied statistical convergence of sequence of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [4, 21, 22, 26, 30, 32]).

Now, we recall the concept of 2-normed space, ideal convergence and some fundamental definitions and notations (see [2, 3, 8, 9, 14–20, 23–25, 27, 31]).

If $K \subseteq \mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the cardinality of K_n. The natural density of K is given by $\delta(K) = \lim_n \frac{1}{n}|K_n|$, if it exists.

The number sequence $x = (x_k)$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set

$$K = K(\varepsilon) := \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

has natural density zero; in this case, we write $st - \lim x = L$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,

(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,

(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$, for each $x \in X$.

Example 1.1. Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then, $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ convergence is the usual convergence.

Throughout the paper, we let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 1.2. [20] *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class $\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$ is a filter on X, called the filter associated with $\mathcal{I}$.*

A sequence (f_n) of functions is said to be $\mathcal{I}$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if for every $\varepsilon > 0$ and each $x \in D$, $\{n : |f_n(x) - f(x) \geq \varepsilon|\} \in \mathcal{I}$. In this case, we will write $f_n \xrightarrow{\mathcal{I}} f$ on D.

A sequence (f_n) of functions is said to be $\mathcal{I}^*$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if $\forall \varepsilon > 0$ and $\forall x \in D$, $\exists K_x \notin \mathcal{I}$ and $\exists n_0 = n_0(\varepsilon, x) \in K_x$: $\forall n \geq n_0$ and $n \in K_x$, $|f_n(x) - f(x)| < \varepsilon$.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x, y\| = 0$ if and only if x and y are linearly dependent.
(ii) $\|x, y\| = \|y, x\|$.
(iii) $\|\alpha x, y\| = |\alpha| \|x, y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim_{n\to\infty} \|x_n - L, y\| = 0$, for every $y \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}$-convergent to $L \in X$, if for each $\varepsilon > 0$ and each nonzero $z \in X$, $A(\varepsilon, z) = \{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\} \in \mathcal{I}$. In this case, we write $\mathcal{I} - \lim_{n\to\infty} \|x_n - L, z\| = 0$ or $\mathcal{I} - \lim_{n\to\infty} \|x_n, z\| = \|L, z\|$.

A sequence (x_n) in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be $\mathcal{I}^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathcal{F}$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ such that $\lim_{n\to\infty} \|x_{m_k} - L, z\| = 0$, for each nonzero $z \in X$.

Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. $\{f_n\}$ is said to be convergent to f if $f_n(x) \stackrel{\|.,.\|_Y}{\longrightarrow} f(x)$ for each $x \in X$. We write $f_n \stackrel{\|.,.\|_Y}{\longrightarrow} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x), z\| < \varepsilon.$$

Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. $\{f_n\}$ is said to be $\mathcal{I}$-pointwise convergent to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, $A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \in \mathcal{I}$ or $\mathcal{I} - \lim_{n\to\infty} \|f_n(x) - f(x), z\|_Y = 0$ (in $(Y, \|.,.\|_Y)$), for each $x \in X$. In this case, we write $f_n \stackrel{\|.,.\|_Y}{\longrightarrow}_{\mathcal{I}} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall \varepsilon > 0)(\exists M \in \mathcal{I})(\forall n_0 \in \mathbb{N}\backslash M)(\forall x \in X)(\forall n \geq n_0)$$
$$\|f_n(x) - f(x), z\| \leq \varepsilon.$$

Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. $\{f_n\}$ is said to be pointwise $\mathcal{I}^*$-convergent to f, if there exists a set $M \in \mathcal{F}(\mathcal{I})$, (i.e., $\mathbb{N}\backslash M \in \mathcal{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, such that for each $x \in X$ and each nonzero $z \in Y$ $\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|$ and we write $\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$ or $f_n \stackrel{\mathcal{I}^*}{\to} f$.

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_i \Delta B_i$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{i=1}^{\infty} B_i \in \mathcal{I}$.

Now we begin with quoting the lemmas due to Arslan and Dündar [1] which are needed throughout the paper.

Lemma 1.3. [1] *Let X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. For each $x \in X$ and each nonzero $z \in Y$,*

$$\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \textit{ implies } \mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|.$$

Lemma 1.4. [1] *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal having the property (AP), X and Y be two 2-normed spaces, $\{f_n\}$ be a sequence of functions and f be a function from X to Y. If the sequence of functions $\{f_n\}$ is $\mathcal{I}$-convergent, then it is $\mathcal{I}^*$-convergent.*

2. Main Results

In this paper, we study concepts of convergence, $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence of functions and investigate relationships between them and some properties such as linearity in 2-normed spaces.

Throughout the paper, we let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal, X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f, g be two functions from X to Y.

Theorem 2.1. *For each $x \in X$ and each nonzero $z \in Y$ we have*

$$\lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \ \ \textit{implies} \ \ \mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|.$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$, therefore, there exists a positive integer $k_0 = k_o(\varepsilon, x)$ such that $\|f_n(x) - f(x), z\| < \varepsilon$, whenever $n \geq k_0$. This implies that the set

$$A(\varepsilon, z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z \geq \varepsilon\|\} \subset \{1, 2, ..., (k_0 - 1)\}.$$

Since $\mathcal{I}$ be an admissible ideal and $\mathcal{I}_f \subset \mathcal{I}$, $\{1, 2, ..., (k_0 - 1)\} \in \mathcal{I}$. Hence, it is clear that $A(\varepsilon, z) \in \mathcal{I}$ and consequently we have

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. ■

Theorem 2.2. *If $\mathcal{I}$-limit of any sequence of functions $\{f_n\}$ exists, then it is unique.*

Proof. Let a sequence $\{f_n\}$ of functions and f, g be two functions from X to Y. Assume that

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x_0), z\| = \|f(x_0), z\| \text{ and } \mathcal{I} - \lim_{n\to\infty} \|f_n(x_0), z\| = \|g(x_0), z\|,$$

where $f(x_0) \neq g(x_0)$ for a $x_0 \in X$ and each nonzero $z \in Y$. Since $f(x_0) \neq g(x_0)$, so we may suppose that $f(x_0) \geq g(x_0)$. Select $\varepsilon = \frac{f(x_0)-g(x_0)}{3}$, so that the neighborhoods $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ of points $f(x_0)$ and $g(x_0)$, respectively are disjoints. Since for $x_0 \in X$ and each nonzero $z \in Y$,

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x_0), z\| = \|f(x_0), z\| \text{ and } \mathcal{I} - \lim_{n\to\infty} \|g_n(x_0), z\| = \|g(x_0), z\|,$$

we have

$$\begin{aligned} A(\varepsilon, z) &= \{n \in \mathbb{N} : \|f_n(x_0) - f(x_0), z\| \geq \varepsilon\} \in \mathcal{I}, \\ B(\varepsilon, z) &= \{n \in \mathbb{N} : \|f_n(x_0) - g(x_0), z\| \geq \varepsilon\} \in \mathcal{I}. \end{aligned}$$

This implies that the sets

$$\begin{aligned} A^c(\varepsilon, z) &= \{n \in \mathbb{N} : \|f_n(x_0) - f(x_0), z\| < \varepsilon\}, \\ B^c(\varepsilon, z) &= \{n \in \mathbb{N} : \|f_n(x_0) - g(x_0), z\| < \varepsilon\} \end{aligned}$$

belong to $\mathcal{F}(\mathcal{I})$ and $A^c(\varepsilon, z) \cap B^c(\varepsilon, z)$ is a nonempty set in $\mathcal{F}(\mathcal{I})$ for $x_0 \in X$ and each nonzero $z \in Y$. Since $A^c(\varepsilon, z) \cap B^c(\varepsilon, z) \neq \emptyset$, we obtain a contradiction on the fact that the neighborhoods $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$ and $(g(x_0)-\varepsilon, g(x_0)+\varepsilon)$ of points $f(x_0)$ and $g(x_0)$, respectively are disjoints. Hence, it is clear that for $x_0 \in X$ and each nonzero $z \in Y$, $\|f_n(x_0), z\| = \|g_n(x_0), z\|$ and consequently we have $\|f_n(x), z\| = \|g_n(x), z\|$, (i.e., $f = g$), for each $x \in X$ and each nonzero $z \in Y$. ∎

Theorem 2.3. *For each $x \in X$ and each nonzero $z \in Y$,*

(i) *If $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$ and $\mathcal{I} - \lim_{n\to\infty} \|g_n(x), z\| = \|g(x), z\|$, then $\mathcal{I} - \lim_{n\to\infty} \|f_n(x) + g_n(x), z\| = \|f(x) + g(x), z\|$.*

(ii) *$\mathcal{I} - \lim_{n\to\infty} \|c.f_n(x), z\| = \|c.f(x), z\|$, $c \in \mathbb{R}$.*

(iii) *$\mathcal{I} - \lim_{n\to\infty} \|f_n(x).g_n(x), z\| = \|f(x).g(x), z\|$.*

Proof. (i) Let $\varepsilon > 0$ be given. Since

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ and } \mathcal{I} - \lim_{n\to\infty} \|g_n(x), z\| = \|g(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. Therefore,

$$A\Big(\frac{\varepsilon}{2}, z\Big) = \Big\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}$$

and

$$B\Big(\frac{\varepsilon}{2}, z\Big) = \Big\{n \in \mathbb{N} : \|g_n(x) - g(x), z\| \geq \frac{\varepsilon}{2}\Big\} \in \mathcal{I}$$

and by the definition of ideal we have

$$A\Big(\frac{\varepsilon}{2}, z\Big) \cup B\Big(\frac{\varepsilon}{2}, z\Big) \in \mathcal{I}.$$

Now, for each $x \in X$ and each nonzero $z \in Y$ we define the set

$$C(\varepsilon, z) = \{n \in \mathbb{N} : \|(f_n(x) + g_n(x)) - (f(x) + g(x)), z\| \geq \varepsilon\}$$

and it is sufficient to prove that $C(\varepsilon, z) \subset A\big(\frac{\varepsilon}{2}, z\big) \cup B\big(\frac{\varepsilon}{2}, z\big)$. Let $n \in C(\varepsilon, z)$. Then for each $x \in X$ and each nonzero $z \in Y$, we have

$$\varepsilon \leq \|\big(f_n(x)+g_n(x)\big) - \big(f(x)+g(x)\big), z\| \leq \|f_n(x) - f(x), z\| + \|g_n(x) - g(x), z\|.$$

As both of $\{\|f_n(x) - f(x), z\|, \|g_n(x) - g(x), z\|\}$ can not be (together) strictly less than $\frac{\varepsilon}{2}$ and therefore either

$$\|f_n(x) - f(x), z\| \geq \frac{\varepsilon}{2} \quad or \quad \|g_n(x) - g(x), z\| \geq \frac{\varepsilon}{2},$$

for each $x \in X$ and each nonzero $z \in Y$. This shows that $n \in A\big(\frac{\varepsilon}{2}, z\big)$ or $n \in B\big(\frac{\varepsilon}{2}, z\big)$ and so we have $n \in A\Big(\frac{\varepsilon}{2}, z\Big) \cup B\Big(\frac{\varepsilon}{2}, z\Big)$. Hence, $C(\varepsilon, z) \subset A\big(\frac{\varepsilon}{2}, z\big) \cup B\big(\frac{\varepsilon}{2}, z\big)$.

(ii) Let $c \in \mathbb{R}$ and $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. If $c = 0$, there is nothing to prove, so we assume $c \neq 0$. Then,

$$\left\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \frac{\varepsilon}{|c|}\right\} \in \mathcal{I},$$

for each $x \in X$ and each nonzero $z \in Y$ and by the definition we have

$$\{n \in \mathbb{N} : \|c.f_n(x) - c.f(x), z\| \geq \varepsilon\} = \left\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \frac{\varepsilon}{|c|}\right\}.$$

Hence, the right side of above equality belongs to $\mathcal{I}$ and so

$$\mathcal{I} - \lim_{n\to\infty} \|c.f_n(x), z\| = \|c.f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$.

(iii) Since $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$,

$$\{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq 1\} \in \mathcal{I},$$

and so

$$A = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < 1\} \in \mathcal{F}(\mathcal{I}),$$

for $\varepsilon = 1 > 0$. Also, for any $n \in A, \|f_n(x), z\| < 1 + \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$. Let $\varepsilon > 0$ be given. Chose $\delta > 0$ such that

$$0 < 2\delta < \frac{\varepsilon}{\|f(x), z\| + \|g(x), z\| + 1},$$

for each $x \in X$ and each nonzero $z \in Y$. It follows from the assumption that,

$$\begin{aligned} B &= \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \delta\} \in \mathcal{F}(\mathcal{I}), \\ C &= \{n \in \mathbb{N} : \|g_n(x) - g(x), z\| < \delta\} \in \mathcal{F}(\mathcal{I}) \end{aligned}$$

for each $x \in X$ and each nonzero $z \in Y$. Since $\mathcal{F}(\mathcal{I})$ is a filter, therefore $A \cap B \cap C \in \mathcal{F}(\mathcal{I})$. Then, for each $n \in A \cap B \cap C$ we have

$$\begin{aligned} &\|f_n(x).g_n(x) - f(x).g(x), z\| \\ &= \|f_n(x).g_n(x) - f_n(x).g(x) + f_n(x).g(x) - f(x).g(x), z\| \\ &\leq \|f_n(x), z\|.\|g_n(x) - g(x), z\| + \|g(x), z\|.\|f_n(x) - f(x), z\| \\ &< (\|f(x), z\| + 1).\delta + (\|g(x), z\|).\delta \\ &= (\|f(x), z\| + \|g(x), z\| + 1).\delta \\ &< \varepsilon \end{aligned}$$

and so, we have $\{n \in \mathbb{N} : \|f_n(x).g_n(x) - f(x).g(x), z\| \geq \varepsilon\} \in \mathcal{I}$, for each $x \in X$ and each nonzero $z \in Y$. This completes the proof. ■

Theorem 2.4. *Let X, Y be two 2-normed spaces, $\{f_n\}$, $\{g_n\}$ and $\{h_n\}$ be sequences of functions and k be a function from X to Y. For each $x \in X$ and each nonzero $z \in Y$, if*

(i) $\{f_n\} \leq \{g_n\} \leq \{h_n\}$, *for every* $n \in K$, *where* $\mathbb{N} \supseteq K \in \mathcal{F}(\mathcal{I})$ *and*
(ii) $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|k(x), z\|$ *and* $\mathcal{I} - \lim_{n\to\infty} \|h_n(x), z\| = \|k(x), z\|$,

then $\mathcal{I} - \lim_{n\to\infty} \|g_n(x), z\| = \|k(x), z\|$.

Proof. Let $\varepsilon > 0$ be given. By condition (ii) we have

$$\{n \in \mathbb{N} : \|f_n(x) - k(x), z\| \geq \varepsilon\} \in \mathcal{I} \text{ and } \{n \in \mathbb{N} : \|h_n(x) - k(x), z\| \geq \varepsilon\} \in \mathcal{I},$$

for each $x \in X$ and each nonzero $z \in Y$. This implies that the sets

$$P = \{n \in \mathbb{N} : \|f_n(x) - k(x), z\| < \varepsilon\} \text{ and } R = \{n \in \mathbb{N} : \|h_n(x) - k(x), z\| < \varepsilon\}$$

belong to $\mathcal{F}(\mathcal{I})$, for each $x \in X$ each nonzero $z \in Y$. Let

$$Q = \{n \in \mathbb{N} : \|g_n(x) - k(x), z\| < \varepsilon\},$$

for each $x \in X$ and each nonzero $z \in Y$. It is clear that the set $P \cap R \cap K \subset Q$. Since $P \cap R \cap K \in \mathcal{F}(\mathcal{I})$ and $P \cap R \cap K \subset Q$, then from the property of filter, we have $Q \in \mathcal{F}(\mathcal{I})$ and so

$$\{n \in \mathbb{N} : \|g_n(x) - k(x), z\| \geq \varepsilon\} \in \mathcal{I},$$

for each $x \in X$ and each nonzero $z \in Y$. ■

Theorem 2.5. *For each* $x \in X$ *and each nonzero* $z \in Y$, *we let*

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ and } \mathcal{I} - \lim_{n\to\infty} \|g_n(x), z\| = \|g(x), z\|.$$

Then, for every $n \in K$ *we have*

(i) *If* $f_n(x) \geq 0$ *then,* $f(x) \geq 0$ *and*
(ii) *If* $f_n(x) \leq g_n(x)$ *then* $f(x) \leq g(x)$, *where* $K \subseteq \mathbb{N}$ *and* $K \in \mathcal{F}(\mathcal{I})$.

Proof. (i) Suppose that $f(x) < 0$. Select $\varepsilon = -\frac{f(x)}{2}$, for each $x \in X$. Since $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$, so there exists the set M such that

$$M = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \varepsilon\} \in \mathcal{F}(\mathcal{I}),$$

for each $x \in X$ and each nonzero $z \in Y$. Since $M, K \in \mathcal{F}(\mathcal{I})$, $M \cap K$ is a nonempty set in $\mathcal{F}(\mathcal{I})$. So we can find out a point n_0 in K such that

$$\|f_{n_0}(x) - f(x), z\| < \varepsilon.$$

Since $f(x) < 0$ and $\varepsilon = \frac{-f(x)}{2}$ for each $x \in X$, we have $f_{n_0}(x) \leq 0$. This is a conradiction to the fact that $f_n(x) > 0$ for every $n \in K$. Hence, we have $f(x) > 0$, for each $x \in X$.

(ii) Suppose that $f(x) > g(x)$. Select $\varepsilon = \frac{f(x)-g(x)}{3}$ for each $x \in X$. So that the neighborhoods $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ of $f(x)$ and $g(x)$, respectively, are disjoints. Since for each $x \in X$ and each nonzero $z \in Y$,

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ and } \mathcal{I} - \lim_{n\to\infty} \|g_n(x), z\| = \|g(x), z\|$$

and $\mathcal{F}(\mathcal{I})$ is a filter on $\mathbb{N}$, therefore we have

$$\begin{aligned} A &= \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| < \varepsilon\} \in \mathcal{F}(\mathcal{I}), \\ B &= \{n \in \mathbb{N} : \|g_n(x) - g(x), z\| < \varepsilon\} \in \mathcal{F}(\mathcal{I}). \end{aligned}$$

This implies that $\emptyset \neq A \cap B \cap K \in \mathcal{F}(\mathcal{I})$. There exists a point n_0 in K such that

$$\|f_n(x) - f(x), z\| < \varepsilon \text{ and } \|g_n(x) - g(x), z\| < \varepsilon.$$

Since $f(x) > g(x)$ and $\varepsilon = \frac{f(x)-g(x)}{3}$ for each $x \in X$, we have $f_{n_0}(x) > g_{n_0}(x)$. This is a contradiction to the fact $f_n(x) \leq g_n(x)$ for every $n \in K$. Thus, we have $f(x) \leq g(x)$, for each $x \in X$. ∎

Theorem 2.6. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal having the property (AP). Then, for each $x \in X$ and each nonzero $z \in Y$, the following conditions are equivalent:*

(i) $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$.

(ii) *There exist $\{g_n\}$ and $\{h_n\}$ to be two sequences of functions from X to Y such that $f_n(x) = g_n(x) + h_n(x)$, $\lim_{n\to\infty} \|g_n(x), z\| = \|f(x), z\|$ and supp $h_n(x) \in \mathcal{I}$, where supp $h_n(x) = \{n \in \mathbb{N} : h_n(x) \neq 0\}$.*

Proof. (i)⇒(ii): $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. Then, by Lemma 1.4 there exists a set $M \in \mathcal{F}(\mathcal{I})$, (i.e., H$=\mathbb{N}\backslash M \in \mathcal{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, such that for each $x \in X$ and each nonzero $z \in Y$,

$$\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|.$$

Let us define the sequence $\{g_n\}$ by

$$g_n(x) = \begin{cases} f_n(x) & \text{if } n \in M, \\ f(x) & \text{if } n \in \mathbb{N}\backslash M. \end{cases} \tag{1}$$

It is clear that $\{g_n\}$ is a sequence of functions and $\lim_{n\to\infty} \|g_n(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$. Also let

$$h_n(x) = f_n(x) - g_n(x), \quad n \in \mathbb{N}, \tag{2}$$

for each $x \in X$. Since

$$\{n \in \mathbb{N} : f_n(x) \neq g_n(x)\} \subset \mathbb{N}\backslash M \in \mathcal{I},$$

for each $x \in X$, so we have

$$\{n \in \mathbb{N} : h_n(x) \neq 0\} \in \mathcal{I}.$$

It follows that *supp* $h_n(x) \in \mathcal{I}$ and by (1) and (2) we get $f_n(x) = g_n(x) + h_n(x)$, for each $x \in X$.

(ii)$\Rightarrow$(i): Suppose that there exist two sequences $\{g_n\}$ and $\{h_n\}$ such that

$$f_n(x) = g_n(x) + h_n(x), \ \lim_{n\to\infty} \|g_n(x), z\| = \|f(x), z\| \text{ and } supp\ \ h_n(x) \in \mathcal{I}, \quad (3)$$

for each $x \in X$ and each nonzero $z \in Y$, where *supp* $h_n(x) = \{n \in \mathbb{N} : h_n(x) \neq 0\}$. We will show that $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$. Define $M = \{n_k\}$ to be a subset of $\mathbb{N}$ such that

$$M = \{n \in \mathbb{N} : h_n(x) = 0\} = \mathbb{N} \backslash supp\ \ h_n(x) \quad (4)$$

Since

$$supp\ \ h_n(x) = \{n \in \mathbb{N} : h_n(x) \neq 0\} \in \mathcal{I},$$

from (3) and (4) we have $M \in \mathcal{F}(\mathcal{I})$, $f_n(x) = g_n(x)$ if $n \in M$. Hence, we conclude that there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, $M \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|,$$

and so $\mathcal{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. By Lemma 1.3, it follows that $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. This completes the proof. ∎

Corollary 2.7. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal having the property (AP). Then, $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$ if and only if there exist $\{g_n\}$ and $\{h_n\}$ be two sequences of functions from X to Y such that*

$$f_n(x) = g_n(x) + h_n(x), \quad \lim_{n\to\infty} \|g_n(x), z\| = \|f(x), z\| \ \ \textit{and}$$
$$\mathcal{I} - \lim_{n\to\infty} \|h_n(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$.

Proof. Let $\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|$ and $\{g_n\}$ be a sequence defined by (1). Consider the sequence

$$h_n(x) = f_n(x) - g_n(x), \ \ n \in \mathbb{N} \quad (5)$$

for each $x \in X$. Then, we have

$$\lim_{n\to\infty} \|g_n(x), z\| = \|f(x), z\|$$

and since $\mathcal{I}$ is an admissible ideal so

$$\mathcal{I} - \lim_{n\to\infty} \|g_n(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. By Theorem 2.3 and by (5) we have

$$\mathcal{I} - \lim_{n\to\infty} \|h_n(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$.

Now let $f_n(x) = g_n(x) + h_n(x)$, where

$$\lim_{n\to\infty} \|g_n(x), z\| = \|f(x), z\| \text{ and } \mathcal{I} - \lim_{n\to\infty} \|h_n(x), z\| = 0,$$

for each $x \in X$ and each nonzero $z \in Y$. Since $\mathcal{I}$ is an admissible ideal so

$$\mathcal{I} - \lim_{n\to\infty} \|g_n(x), z\| = \|f(x), z\|$$

and by Theorem 2.3 we get

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$. ■

Remark 2.8. In Theorem 2.6, if (ii) is satisfied then the admissible ideal $\mathcal{I}$ need not have the property (AP). Since for each $x \in X$ and each nonzero $z \in Y$,

$$\{n \in \mathbb{N} : \|h_n(x), z\| \geq \varepsilon\} \subset \{n \in \mathbb{N} : h_n(x) \neq 0\} \in \mathcal{I},$$

for each $\varepsilon > 0$, we have

$$\mathcal{I} - \lim_{n\to\infty} \|h_n(x), z\| = 0.$$

Hence, we have

$$\mathcal{I} - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$.

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