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On Some Double Cesàro Sequence Spaces

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Abstract. In this study, we define the double Cesàro sequence spaces Ces_p, Ces_{bp} and Ces_{bp0} and examine some properties of those sequence spaces. Furthermore, we determine the $\beta(bp)$-duals of the spaces Ces_{bp} and Ces_p.

1. Introduction

By Ω, we denote the set of all complex valued double sequences, i.e.,

$$\Omega = \{x = (x_{mn}) : x_{mn} \in \mathbb{C} \text{ for all } m, n \in \mathbb{N}\},$$

which is a vector space with co-ordinatewise addition and scalar multiplication of double sequences, where $\mathbb{N}$ and $\mathbb{C}$ denote the set of positive integers and the complex field, respectively. Any vector subspace of Ω is called as a double sequence space. The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u = \left\{ x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty \right\}$$

which is a Banach space with the norm $\|\cdot\|_\infty$. Consider the sequence $x = (x_{mn}) \in \Omega$ and $\ell \in \mathbb{C}$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$|x_{mn} - \ell| < \varepsilon$$

for all $m, n > n_0$ then we call that the double sequence x is convergent in the Pringsheim's sense to ℓ and write $P - \lim_{m,n} x_{mn} = \ell$. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. So, we may mention the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. By $\mathcal{C}_{bp0}$, we denote the space of the double sequences which are both convergent to zero in the Pringsheim's sense and bounded.

Let λ be the space of double sequences, converging with respect to some linear convergence rule $\upsilon - \lim : \lambda \to \mathbb{C}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $\upsilon - \sum_{ij} x_{ij} = \upsilon - \lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$. Let λ, μ be two spaces of double sequences, converging with respect to the linear

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convergence rules $\upsilon_1 - \lim$ and $\upsilon_2 - \lim$, respectively, and $A = (a_{mnkl})$ also be a four dimensional matrix of complex numbers. Define the set

$$\lambda_A^{(\upsilon_2)} = \left\{ (x_{kl}) \in \Omega : Ax = \left(\upsilon_2 - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and } Ax \in \lambda \right\}. \tag{1}$$

Then, we say, with the notation of (1), that A maps the space λ into the space μ if $\mu \subset \lambda_A^{(\upsilon_2)}$ and denote the set of all four dimensional matrices, mapping the space λ into the space μ, by $(\lambda : \mu)$. It is trivial that for any matrix $A \in (\lambda : \mu)$, $(a_{mnkl})_{k,l \in \mathbb{N}}$ is in the $\beta(\upsilon_2)$-dual $\lambda^{\beta(\upsilon_2)}$ of the space λ for all $m, n \in \mathbb{N}$. An infinite matrix A is said to be $\mathcal{C}_\upsilon$-conservative if $\mathcal{C}_\upsilon \subset (\mathcal{C}_\upsilon)_A$. The characterizations of some four dimensional matrix transformations between double sequence spaces have been given by Robison [16], Hamilton [8] and Zeltser [22].

Lemma 1.1 ([8, 16, 22]). *$A = (a_{mnkl}) \in (\mathcal{C}_{bp} : \mathcal{C}_{bp})$ if and only if*

$$\sup_{m,n} \sum_{k,l} |a_{mnkl}| < \infty, \tag{2}$$

$$bp - \lim_{m,n} a_{mnkl} = a_{kl} \text{ exists } (k, l \in \mathbb{N}), \tag{3}$$

$$bp - \lim_{m,n} \sum_{k,l} a_{mnkl} = v \text{ exists} \tag{4}$$

$$bp - \lim_{m,n} \sum_{k} \left| a_{mnkl_0} - a_{kl_0} \right| = 0 \text{ and } bp - \lim_{m,n} \sum_{l} \left| a_{mnk_0 l} - a_{k_0 l} \right| = 0 \ \ (k_0, l_0 \in \mathbb{N}). \tag{5}$$

The arithmetic (or Cesáro) mean s_{mn} of a double sequence $x = (x_{mn})$ is defined by

$$s_{mn} = \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk}, \quad (m, n \in \mathbb{N}).$$

We say that $x = (x_{mn})$ is $(C, 1, 1)$ summable or Cesáro summable to some number ℓ if

$$P - \lim s_{mn} = \ell,$$

where $(C, 1, 1) = (c_{mnkl})$ is a four dimensional matrix defined by

$$c_{mnkl} = \begin{cases} \frac{1}{mn} & , \ (1 \le k \le m \text{ and } 1 \le l \le n) \\ 0 & , \ (\text{otherwise}) \end{cases}$$

(The letter " C " comes from the name " Cesáro ".)

We shall write throughout for simplicity in notation for all $m, n, k, l \in \mathbb{N}$ that

$$\begin{aligned} \triangle_{10} a_{mn} &= a_{mn} - a_{m+1,n} \, , \\ \triangle_{01} a_{mn} &= a_{mn} - a_{m,n+1} \, , \\ \triangle_{11} a_{mn} &= \triangle_{01}(\triangle_{10} a_{mn}) = \triangle_{10}(\triangle_{01} a_{mn}). \end{aligned}$$

Now, we may summarize the knowledge given in some document on the double sequence spaces. Móricz [10] proved that the double sequence space $\mathcal{C}_p$ is complete under the pseudonorm $\|x\|_P = \lim_{N \to \infty} \sup_{k,l > N} |x_{kl}|$ and the sets $\mathcal{C}_{bp}$ and $\mathcal{C}_{bp0}$ are Banach spaces under the norm $\| \cdot \|_\infty$. Gökhan and Çolak [5–7] extended these spaces to the paranormed double sequence spaces, determined their duals and gave some inclusion relations. Considering the summability of double sequences defining by the product of two complex single sequences, Jardas and Sarapa [9] proved the Silverman-Toeplitz and Steinhaus type theorems for three dimensional matrices. Boos et al. [3] defined the concept of $\mathcal{V}$-SM-method by the application domain of a matrix sequence $\mathcal{A} = (\mathcal{A}^{(v)})$ of infinite matrices and gave the consistency theory for such type methods and introduce the notions of e, be and c convergence for double sequences. By using the gliding hump method, Zeltser [19] recently characterized the classes of four dimensional matrix mappings from λ into μ; where $\lambda, \mu \in \{C_e, C_{be}\}$. Also employing the same arguments, Zeltser [20] gave the theorems determining the necessary and sufficient conditions for C_e-SM and C_{be}-SM-methods to be conservative and coercive. Zeltser [21] considered the dual pairs $\langle E, E^{\beta(v)}\rangle$ of double sequence spaces E and $E^{\beta(v)}$, where $E^{\beta(v)}$ denotes the β-dual of E with respect to v-convergence of double sequences for $v \in \{p, bp, r\}$ and introduced two oscillating properties for a double sequence space E. Also, Zeltser [22] emphasized two types of summability methods of double sequences defined by four dimensional matrices which preserve the regular convergence and the C_c-convergence of double sequences and extended some well-known facts of summability to four dimensional matrices. By using the definitions of limit inferior, limit superior and the core of a double sequence with the notion of the regularity of four dimensional matrices, Patterson [14] proved an invariant core theorem. Also, Patterson [15] determined the sufficient conditions on a four dimensional matrix in order to be stronger than the convergence in the Pringsheim's sense and derived some results concerning with the summability of double sequences. Mursaleen and Edely [11] recently introduced the statistical convergence and Cauchy for double sequences and gave the relation between statistical convergent and strongly Cesàro summable double sequences. Mursaleen [12] and Mursaleen and Edely [13] defined the almost strong regularity of matrices for double sequences and apply these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the M-core of x. Quite recently, Altay and Başar [1] defined some spaces of double sequences. Çakan and Altay [4] investigated statistical core for double sequences and studied an inequality related to the statistical and P-cores of bounded double sequences. Başar and Sever [2] examined some properties of the space $\mathcal{L}_q$. Subramanian and Misra [17, 18] defined some new double sequence spaces and examined their properties.

In this study, we define the Cesàro spaces Ces_p, Ces_{bp} and Ces_{bp0} of double sequences and examine some properties of these sequence spaces. Furthermore, we determine the $\beta(bp)$-duals of the spaces Ces_{bp} and Ces_p.

2. Some Double Cesàro Sequence Spaces

In this section, we introduce the sets Ces_p, Ces_{bp} and Ces_{bp0} consisting of double sequences whose Cesáro transforms of order one are convergent in the Pringsheim's sense, convergent in the Pringsheim's sense and bounded, and null in the Pringsheim's sense and bounded, respectively. We show that Ces_p is a complete seminormed linear space and isomorphic to the space C_p. Also we establish that Ces_{bp} and Ces_{bp0} are Banach spaces and they are isomorphic to the spaces C_{bp} and C_{bp0}, respectively. We give two inclusion theorems related to the space Ces_{bp}.

The Cesàro spaces Ces_p, Ces_{bp} and Ces_{bp0} of double sequences are defined, with the notation of (1), as follows:

$$Ces_p = \left\{(x_{jk}) \in \Omega : \left(\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}\right) \in C_p\right\} = (C_p)_{(C,1,1)},$$

$$Ces_{bp} = \left\{(x_{jk}) \in \Omega : \left(\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}\right) \in C_{bp}\right\} = (C_{bp})_{(C,1,1)}$$

and

$$Ces_{bp0} = \left\{(x_{jk}) \in \Omega : \left(\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}\right) \in C_{bp0}\right\} = (C_{bp0})_{(C,1,1)}.$$

Theorem 2.1. *The set Ces_p becomes a linear space with the coordinatewise addition and scalar multiplication of double sequences and Ces_p is a complete seminormed with*

$$\|x\|_{Ces_p} = \lim_{i\to\infty} \sup_{m,n\geq i} \left| \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk} \right| \tag{6}$$

which is linearly isomorphic to the space C_p.

Proof. The first part of the theorem is a routine verification and so we omit it.

Now, we show that Ces_p is a complete seminormed with the seminorm defined by (6). Let $(x^l)_{l\in\mathbb{N}}$ be any Chauchy sequence in the space Ces_p, where $x^l = \left\{x_{mn}^{(l)}\right\}_{m,n=1}^{\infty}$ for every fixed $l \in \mathbb{N}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$\|x^l - x^r\|_{Ces_p} = \lim_{i\to\infty} \sup_{m,n\geq i} \left| \frac{1}{mn} \sum_{j,k=1}^{m,n} \left(x_{jk}^l - x_{jk}^r\right) \right| < \varepsilon$$

for all $l, r > n_0(\varepsilon)$ which yields for every $m, n \geq i_0$ that

$$\left| \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}^l - \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}^r \right| < \varepsilon.$$

This means that $\left(\frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}^l\right)_{l\in\mathbb{N}}$ is a Chauchy sequence with complex terms for every $m, n \geq i_0$. Since $\mathbb{C}$ is complete, it converges, say

$$\lim_{l\to\infty} \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}^l = \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}. \tag{7}$$

Using these infinitely many limits, we define the sequence $\left(\frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}\right)$. It is seen by (7) that

$$\lim_{l\to\infty} \left\| \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}^l - \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk} \right\|_{C_p} = 0. \tag{8}$$

Now we can show that $x = (x_{jk}) \in Ces_p$. Let $m, n, p, q > i_0$. Since

$$\begin{aligned} \left| \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk} - \frac{1}{pq} \sum_{j,k=1}^{p,q} x_{jk} \right| &\leq \left| \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk} - \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}^l \right| + \left| \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk}^l - \frac{1}{pq} \sum_{j,k=1}^{p,q} x_{jk}^l \right| \\ &\quad + \left| \frac{1}{pq} \sum_{j,k=1}^{p,q} x_{jk}^l - \frac{1}{pq} \sum_{j,k=1}^{p,q} x_{jk} \right| \\ &\leq 3\varepsilon, \end{aligned}$$

we have that $x \in Ces_p$.

To prove the fact Ces_p and C_p linearly isomorphic, we should define a lineer bijection between the spaces Ces_p and C_p. Consider the transformation T defined from Ces_p to C_p by

$$\begin{aligned} T \ &: \ Ces_p \to C_p \\ &x \mapsto Tx = \left(\frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk} \right) = (s_{mn}) = s. \end{aligned}$$

(i) Let $x = (x_{jk}), y = (y_{jk}) \in Ces_p$ and $\alpha \in \mathbb{C}$. Then, since

$$\begin{aligned} T(\alpha x + y) &= \left(\frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha x_{jk} + y_{jk} \right) \\ &= \alpha \left(\frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk} \right) + \left(\frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} y_{jk} \right) \\ &= \alpha Tx + Ty, \end{aligned}$$

T is linear.

(ii) The equality,

$$Tx = \begin{bmatrix} x_{11} & \frac{1}{2}(x_{11}+x_{12}) & \frac{1}{3}(x_{11}+x_{12}+x_{13}) & \cdots \\ \frac{1}{2}(x_{11}+x_{21}) & \frac{1}{4}(x_{11}+x_{12}+x_{21}+x_{22}) & \frac{1}{6}(x_{11}+x_{12}+x_{13}+x_{21}+x_{22}+x_{23}) & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m}\sum_{j=1}^{m} x_{j1} & \frac{1}{2m}\sum_{j=1}^{m}\sum_{k=1}^{2} x_{jk} & \frac{1}{3m}\sum_{j=1}^{m}\sum_{k=1}^{3} x_{jk} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = 0$$

yields that

$$\left. \begin{array}{cccccccccc} x_{11} & = & 0 & x_{12} & = & 0 & x_{13} & = & 0 & \cdots \\ x_{21} & = & 0 & x_{22} & = & 0 & x_{23} & = & 0 & \cdots \\ \vdots & & & \vdots & & & \vdots & & & \vdots \end{array} \right\} \Longrightarrow x = 0.$$

This means that T is a bijection.

(iii) Let us take $s = (s_{jk}) \in C_p$ and define the sequence $x = (x_{jk})$ via s by

$$x_{jk} = jks_{jk} - (j-1)ks_{j-1,k} - j(k-1)s_{j,k-1} + (j-1)(k-1)s_{j-1,k-1}$$

for all $j, k \in \mathbb{N}$; where $s_{0,0} = 0$, $s_{0,1} = 0$ and $s_{1,0} = 0$. Since $s \in C_p$, there exists $L \in \mathbb{C}$ such that $P - \lim_{mn} s_{mn} = L$ and $\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} x_{jk} = s_{mn}$, one can easily see that

$$P - \lim_{m,n} \left| \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk} - L \right| = 0.$$

Therefore, $x \in Ces_p$. That is to say that T is surjective.

Since the conditions (i)-(iii) are satisfied, T is a linearly isomorphism between the Cesàro spaces Ces_p and C_p of double sequences. This step concludes the proof. □

We give the following theorem without proof, since its proof is similar to that of Theorem 2.1:

Theorem 2.2. *The sets Ces_{bp} and Ces_{bp0} become a linear space with the coordinatewise addition and scalar multiplication of double sequences. Ces_{bp} and Ces_{bp0} are Banach spaces with the norm*

$$\|x\|_\infty = \sup_{m,n \geq 1} \left| \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk} \right|$$

which are linearly isomorphic to the spaces C_{bp} and C_{bp0}, respectively.

Theorem 2.3. *The inclusion $C_{bp} \subset Ces_{bp}$ strictly holds.*

Proof. Let $x = (x_{kl}) \in C_{bp}$. Then, since the matrix $(C, 1, 1)$ is in the class $(C_{bp} : C_{bp})$, the sequence $s = (s_{mn}) = (C, 1, 1)x$ is in the space C_{bp} which means that $x \in Ces_{bp}$. This shows that the inclusion $C_{bp} \subset Ces_{bp}$ holds.

Let us define the sequence $x = (x_{mn})$ by

$$x_{mn} = \begin{cases} 1 & , \quad (m = n), \\ 0 & , \quad (m \neq n), \end{cases}$$

for all $m, n \in \mathbb{N}$. Since the $(C, 1, 1)$-transform of x is $s = (s_{mn})$ with $s_{mn} = (\min\{m, n\})/mn$ for all $m, n \in \mathbb{N}$ and $P - \lim_{m,n} s_{mn} = 0$, $x = (x_{mn})$ is in Ces_{bp} but not in C_{bp}. This example shows that the inclusion $C_{bp} \subset Ces_{bp}$ is strict. □

Theorem 2.4. *The inclusion $C_p \subset Ces_{bp}$ strictly holds.*

Proof. This is similar to the proof of Theorem 2.3. So we omit the detail. □

3. $\beta(bp)$-Duals of the Spaces Ces_{bp} and Ces_p

In this section, we determine the $\beta(bp)$-duals of the Cesàro spaces Ces_{bp} and Ces_p. Although the β-duals of the spaces of single sequences are unique, the β-duals of the double sequence spaces may be more than one with respect to υ-convergence. The $\beta(\upsilon)$-duals $\lambda^{\beta(\upsilon)}$ of a double sequence space λ is defined by

$$\lambda^{\beta(\upsilon)} = \left\{ (a_{ij}) \in \Omega : \upsilon - \sum_{i,j} a_{ij} x_{ij} \text{ exists for all } (x_{ij}) \in \lambda \right\}.$$

It is easy to see for any two spaces λ, μ of double sequences that $\mu^{\beta(\upsilon)} \subset \lambda^{\beta(\upsilon)}$ whenever $\lambda \subset \mu$.

Now, we determine the β-dual of the Cesàro space Ces_{bp} with respect to the bp-convergence using the technique given in [1].

Theorem 3.1. *Define the set Υ_{bp-bp} by*

$$\Upsilon_{bp-bp} = \Big\{ a = (a_{kl}) \in \Omega : \sum_{k,l} |kl\triangle_{11} a_{kl}| < \infty,\ (kl\Delta_{10} a_{kl})_k,\ (kl\Delta_{01} a_{kl})_l \in \ell_1, (kla_{kl}) \in \mathcal{M}_u, (a_{kl}) \in \mathcal{CS}_{bp} \Big\},$$

where l_1 and $\mathcal{CS}_{bp}$ denote the space of absolutely summable single sequences and the space of double sequences consisting of all double series whose sequence of partial sums are in the space C_{bp}, respectively. Then the following statements hold:

(i) The $\beta(bp)$-dual of the space Ces_{bp} is the set Υ_{bp-bp}.
(ii) The $\beta(bp)$-dual of the space Ces_p is the set Υ_{bp-bp}.

Proof. (i) Suppose that $x = (x_{kl}) \in Ces_{bp}$. Then, $s = (C, 1, 1)x$ is in the space C_{bp}, by Theorem 2.2. Let us determine the necessary and sufficient condition in order to the series

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl} x_{kl} \tag{9}$$

is to be bp-convergent for a sequence $a = (a_{kl}) \in \Omega$. We obtain m, n^{th} partial sums of the series in (9) that

$$\begin{aligned} z_{mn} &= \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} x_{kl} \\ &= \sum_{k=1}^{m-1} \sum_{l=1}^{n-1} s_{kl}(kl\triangle_{11} a_{kl}) + \sum_{k=1}^{m-1} s_{kn}(kn\triangle_{10} a_{kn}) + \sum_{l=1}^{n-1} s_{ml}(ml\triangle_{01} a_{ml}) + s_{mn}(mna_{mn}) \end{aligned} \tag{10}$$

for all $m, n \in \mathbb{N}$. (10) can be rewritten by the matrix representation as follows:

$$z_{mn} = \sum_{k=1}^{m} \sum_{l=1}^{n} b_{mnkl} s_{kl} = (Bs)_{mn} \tag{11}$$

for all $m, n \in \mathbb{N}$, where $B = (b_{mnkl})$ is the four dimensional matrix defined by

$$b_{mnkl} = \begin{cases} kl\triangle_{11}a_{kl} & , \quad (k \leq m-1 \quad and \quad l \leq n-1), \\ kn\triangle_{10}a_{kn} & , \quad (k \leq m-1 \quad and \quad l = n), \\ ml\triangle_{01}a_{ml} & , \quad (k = m \quad and \quad l \leq n-1), \\ mna_{mn} & , \quad (k = m \quad and \quad l = n), \\ 0 & , \quad (\text{otherwise}). \end{cases} \tag{12}$$

We therefore read from the equality (10) that $ax = (a_{kl}x_{kl}) \in \mathcal{CS}_{bp}$ whenever $x = (x_{kl}) \in Ces_{bp}$ if and only if $z = (z_{kl}) \in \mathcal{C}_{bp}$ whenever $s = (s_{kl}) \in \mathcal{C}_{bp}$ which leads to the fact that $B = (b_{mnkl})$, defined by (12), is in the class $(\mathcal{C}_{bp} : \mathcal{C}_{bp})$. Thus we see from Lemma 1.1 that the following conditions

$$\sup_{m,n\geq 1} \sum_{k=1}^{m} \sum_{l=1}^{n} |b_{mnkl}| \quad = \quad \sup_{m,n\geq 1} \left\{ \sum_{k=1}^{m-1} \sum_{l=1}^{n-1} |kl\triangle_{11}a_{kl}| + \sum_{k=1}^{m-1} |kn\triangle_{10}a_{kn}| + \sum_{l=1}^{n-1} |ml\triangle_{01}a_{ml}| + |mna_{mn}| \right\} < \infty,$$

$$P - \lim_{m,n} b_{mnkl} = kl\triangle_{11}a_{kl},$$

$$P - \lim_{m,n} \sum_{k,l} b_{mnkl} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}$$

and

$$P - \lim_{m,n} \sum_{k} |b_{mnkl} - b_{kl}| = P - \lim_{m,n} \sum_{k=m}^{\infty} |kl\Delta_{11}a_{kl}| = 0,$$

$$P - \lim_{m,n} \sum_{k} |b_{mnkl} - b_{kl}| = P - \lim_{m,n} \sum_{l=n}^{\infty} |kl\Delta_{11}a_{kl}| = 0$$

hold for the matrix B, defined by (12). Therefore, we derive from the conditions (2)-(5) that

$$\sum_{k,l} |kl\triangle_{11}a_{kl}| < \infty, \tag{13}$$

$$\sup_{n} \sum_{k} |kn\triangle_{10}a_{kn}| < \infty, \tag{14}$$

$$\sup_{m} \sum_{l} |ml\triangle_{01}a_{ml}| < \infty, \tag{15}$$

$$\sup_{m,n} |mna_{mn}| < \infty, \tag{16}$$

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl} \text{ exists.} \tag{17}$$

This shows that $Ces_{bp}^{\beta(bp)} = \Upsilon_{bp-bp}$ which completes the proof of Part (i).

(ii) Since the proof is similar to that of Part (i), to avoid undue repetition in the statements, we leave the detail to the reader. □

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