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On Regular Cesàro Double Sequence Spaces

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Abstract

In this study, we define the double sequence space $\mathcal{C}es_r$ and examine some properties of this sequence space. Furthermore, we determine the $\beta(r)$-dual of the space $\mathcal{C}es_r$.

Keywords: *Double sequence space, seminormed sequence space, β-dual.*

1 Introduction

By Ω, we denote the set of all real or complex valued double sequences which is a vector space with coordinatewise addition and scalar multiplication. Any vector subspace of Ω is called as a double sequence space. The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u = \left\{ x = (x_{mn}) \in \Omega \, : \, \|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty \right\}$$

which is a Banach space with the norm $\|\cdot\|_\infty$; where $\mathbb{N} = \{1, 2, 3, \ldots\}$. Consider the sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $\ell \in \mathbb{C}$ such that

$$|x_{mn} - \ell| < \varepsilon$$

for all $m, n > n_0$ then we call that the double sequence x is convergent in the Pringsheim's sense to the limit ℓ and write $P - \lim x_{mn} = \ell$; where $\mathbb{C}$ denotes the complex field. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$. So, we may mention the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. By $\mathcal{C}_{bp0}$, we denote the space of the double sequences which are both convergent to zero in the Pringsheim's sense and bounded. A sequence in the space $\mathcal{C}_p$ is said to be regularly convergent if it is a single convergent sequence with respect to each index and denote the set of all such sequences by $\mathcal{C}_r$.

Let λ be the space of double sequences, converging with respect to some linear convergence rule $\upsilon - \lim : \lambda \to \mathbb{C}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $\upsilon - \sum_{ij} x_{ij} = \upsilon - \lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$. Let λ, μ be two spaces of double sequences, converging with respect to the linear convergence rules $\upsilon_1 - \lim$ and $\upsilon_2 - \lim$, respectively, and $A = (a_{mnkl})$ also be a four dimensional matrix of real or complex numbers. Define the set

$$\lambda_A^{(\upsilon_2)} = \left\{ (x_{kl}) \in \Omega : Ax = \left(\upsilon_2 - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and } Ax \in \lambda \right\}. \quad (1)$$

Then, we say, with the notation of (1), that A maps the space λ into the space μ if $\mu \subset \lambda_A^{(\upsilon_2)}$ and denote the set of all four dimensional matrices, mapping the space λ into the space μ, by $(\lambda : \mu)$. It is trivial that for any matrix $A \in (\lambda : \mu)$, $(a_{mnkl})_{k,l \in \mathbb{N}}$ is in the $\beta(\upsilon_2)$-dual $\lambda^{\beta(\upsilon_2)}$ of the space λ for all $m, n \in \mathbb{N}$. An infinite matrix A is said to be $\mathcal{C}_\upsilon$-conservative if $\mathcal{C}_\upsilon \subset (\mathcal{C}_\upsilon)_A$. The characterizations of some four dimensional matrix transformations between double sequence spaces have been given by Robison [16], Hamilton [8] and Zeltser [22].

Lemma 1.1 ([16, 8, 22]) *$A = (a_{mnkl}) \in (\mathcal{C}_r, \mathcal{C}_r)$ if and only if*

$$\sup_{m,n} \sum_{k,l} |a_{mnkl}| < \infty, \quad (2)$$

$$r\text{-}\lim_{m,n} a_{mnkl} = a_{kl} \text{ exists } (k, l \in \mathbb{N}), \quad (3)$$

$$r\text{-}\lim_{m,n} \sum_{k,l} a_{mnkl} = \upsilon \text{ exists}, \quad (4)$$

$$r\text{-}\lim_{m,n} \sum_{k} a_{mnkl_0} = u^{l_0} \text{ and } r\text{-}\lim_{m,n} \sum_{l} a_{mnk_0l} = \upsilon_{k_0} \quad (k_0, l_0 \in \mathbb{N}). \quad (5)$$

The arithmetic (or Cesáro) mean s_{mn} of a double sequence $x = (x_{mn})$ is defined by

$$s_{mn} = \frac{1}{mn} \sum_{j=1}^{m} \sum_{k=1}^{n} x_{jk}, \quad (m, n \in \mathbb{N}).$$

We say that $x = (x_{mn})$ is regularly $(C, 1, 1)$ summable or regularly Cesáro summable to some number ℓ if

$$r - \lim s_{mn} = \ell,$$

where $(C, 1, 1) = (c_{mnkl})$ is a four dimensional matrix defined by

$$c_{mnkl} = \begin{cases} \frac{1}{mn} & , \quad (1 \leq k \leq m \text{ and } 1 \leq l \leq n) \\ 0 & , \quad (\text{otherwise}) \end{cases} \tag{6}$$

(The letter " C " comes from the name " Cesáro ".)

We shall write throughout for simplicity in notation for all $m, n, k, l \in \mathbb{N}$ that

$$\begin{aligned} \triangle_{10} a_{mn} &= a_{mn} - a_{m+1,n} , \\ \triangle_{01} a_{mn} &= a_{mn} - a_{m,n+1} , \\ \triangle_{11} a_{mn} &= \triangle_{01}(\triangle_{10} a_{mn}) = \triangle_{10}(\triangle_{01} a_{mn}). \end{aligned}$$

Now, we may summarize the knowledge given in some document on the double sequence spaces. Móricz [10] proved that the double sequence space $\mathcal{C}_p$ is complete under the pseudonorm $\|x\|_P = \lim_{N\to\infty} \sup_{k,l>N} |x_{kl}|$ and the sets $\mathcal{C}_{bp}$ and $\mathcal{C}_{bp0}$ are Banach spaces under the norm $\|\cdot\|_\infty$. Gökhan and Çolak [5, 6, 7] extended these space to the paranormed double sequence spaces, determined their duals and gave some inclusion relations. The summability of double sequences defining by the product of two complex single sequences, Jardas and Sarapa [9] proved the Silverman-Toeplitz and Steinhaus type theorems for three dimensional matrices. Boos, Leiger and Zeller [3] defined the concept of $\mathcal{V}$-SM-method by the application domain of a matrix sequence $\mathcal{A} = (\mathcal{A}^{(v)})$ of infinite matrices and gave the consistency theory for such type methods and introduce the notions of e, be and c convergence for double sequences. By using the gliding hump method, Zeltser [19] recently characterized the classes of four dimensional matrix mappings from λ into μ; where $\lambda, \mu \in \{\mathcal{C}_e, \mathcal{C}_{be}\}$. Also employing the same arguments, Zeltser [20] gave the theorems determining the necessary and sufficient conditions for $\mathcal{C}_e$-SM and $\mathcal{C}_{be}$-SM-methods to be concervative and coercive. Zeltser [21] considered the dual pairs $\langle E, E^{\beta(v)}\rangle$ of double sequence spaces E and $E^{\beta(v)}$, where $E^{\beta(v)}$ denotes the β-dual of E with respect to v-convergence of double sequences for $v \in \{p, bp, r\}$ and introduced two oscillating properties for a double sequence space E. Also, Zeltser [22] emphasized two types of summability methods of double sequences defined by four dimensional matrices which preserve the regular convergence and the $\mathcal{C}_c$-convergence of double sequences and extends some well-known facts of summability to four dimensional matrices. By using the definitions of limit inferior, limit superior and the core of a double sequence with the notion of the regularity of four dimensional matrices, Patterson [14] proved an invariant core theorem. Also, Patterson [15] determined the sufficient conditions on a four dimensional matrix in order to be stronger than the convergence in the Pringsheim's sense and derives some results concerning with the summability of double sequences. Mursaleen and Edely [11] recently introduced the statistical convergence and Cauchy for double sequences and gave the relation between statistical convergent and strongly Cesàro summable double sequences. Mursaleen [12] and Mursaleen and Edely [13] defined the almost strong regularity of matrices for double sequences and apply these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the M-core of x. Quite recently, Altay and Başar [1] defined some spaces of double sequences. Çakan and Altay [4] investigated statistical core for double sequences and studied an inequality related to the statistical and P-cores of bounded double sequences. Başar and Sever [2] examined some properties of the space $\mathcal{L}_q$. Subramanian and Mishra [17, 18] defined some new double sequence spaces and examined their properties.

In this study, we define the double sequence space $\mathcal{C}es_r$ and examine some properties of this sequence space. Furthermore, we determine the $\beta(r)$-dual of the space $\mathcal{C}es_r$.

2 Regular Cesàro Double Sequence Spaces

In this section, we introduce the set Ces_r consisting of double sequences whose Cesáro transforms are convergent in the regular sense. That is to say that

$$\mathcal{C}es_r = \left\{ (x_{jk}) \in \Omega : \left(\frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk} \right) \in \mathcal{C}_r \right\} = (\mathcal{C}_r)_{(C,1,1)}.$$

We show that Ces_r is a Banach space and is isomorphic to the space $\mathcal{C}_r$.

Theorem 2.1 *The set Ces_r becomes a linear space with the coordinatewise addition and scalar multiplication of double sequences and $\mathcal{C}es_r$ is a Banach space with the norm $\|\cdot\|_{\mathcal{C}es_r}$ defined by*

$$\|x\|_{\mathcal{C}es_r} = \sup_{m,n} \left| \frac{1}{mn} \sum_{j,k=1}^{m,n} x_{jk} \right| \tag{7}$$

and is linearly isomorphic to the space $\mathcal{C}_r$.

Proof. The first part of the theorem is a routine verification and so we omit it.

Now, we may show that $\mathcal{C}es_r$ is a Banach space with norm defined by (7). Let $(x^l)_{l\in\mathbb{N}}$ be any Chauchy sequence in the space $\mathcal{C}es_r$, where $x^l = \left\{x_{mn}^{(l)}\right\}_{m,n=1}^{\infty}$ for every fixed $l \in \mathbb{N}$. Then, for a given $\varepsilon > 0$ there exists a positive integer $n_0(\varepsilon)$ such that

$$\|x^l - x^r\|_{\mathcal{C}es_r} = \sup_{m,n}\left|\frac{1}{mn}\sum_{j,k=1}^{m,n}\left(x_{jk}^l - x_{jk}^r\right)\right| < \varepsilon$$

for all $l, r > n_0(\varepsilon)$ which yields for every m, n that

$$\left|\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}^l - \frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}^r\right| < \varepsilon.$$

This means that $\left(\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}^l\right)_{l\in\mathbb{N}}$ is a Cauchy sequence with complex terms for every fixed $m, n \in \mathbb{N}$. Since $\mathbb{C}$ is complete, it converges, say

$$\lim_{l\to\infty}\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}^l = \frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}. \tag{8}$$

Using these infinitely many limits, we define the sequence $\left(\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}\right)_{m,n\in\mathbb{N}}$. It is seen by (8) that

$$\lim_{l\to\infty}\left\|\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}^l - \frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}\right\|_{\mathcal{C}_r} = 0. \tag{9}$$

Now we show that $x = (x_{jk})$ in $\mathcal{C}es_r$. Let $m, n, p, q \in \mathbb{N}$. Since

$$\begin{aligned}\left|\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk} - \frac{1}{pq}\sum_{j,k=1}^{p,q} x_{jk}\right| \;&\le\; \left|\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk} - \frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}^l\right| \\ &\quad + \left|\frac{1}{mn}\sum_{j,k=1}^{m,n} x_{jk}^l - \frac{1}{pq}\sum_{j,k=1}^{p,q} x_{jk}^l\right| \\ &\quad + \left|\frac{1}{pq}\sum_{j,k=1}^{p,q} x_{jk}^l - \frac{1}{pq}\sum_{j,k=1}^{p,q} x_{jk}\right| \\ &\le\; 3\varepsilon\end{aligned}$$

and for fixed n_0 and $m, p \in \mathbb{N}$,

$$\begin{aligned}\left|\frac{1}{mn_0}\sum_{j,k=1}^{m,n_0} x_{jk} - \frac{1}{pn_0}\sum_{j,k=1}^{p,n_0} x_{jk}\right| &\leq \left|\frac{1}{mn_0}\sum_{j,k=1}^{m,n_0} x_{jk} - \frac{1}{mn_0}\sum_{j,k=1}^{m,n_0} x_{jk}^l\right| \\ &\quad + \left|\frac{1}{mn_0}\sum_{j,k=1}^{m,n_0} x_{jk}^l - \frac{1}{pn_0}\sum_{j,k=1}^{p,n_0} x_{jk}^l\right| \\ &\quad + \left|\frac{1}{pn_0}\sum_{j,k=1}^{p,n_0} x_{jk}^l - \frac{1}{pn_0}\sum_{j,k=1}^{p,n_0} x_{jk}\right| \\ &\leq 3\varepsilon\end{aligned}$$

similarly, for fixed m_0, we can show that the sequence $\left\{\frac{1}{m_0 n}\sum_{j,k=1}^{m_0,n} x_{jk}\right\}_{n\in\mathbb{N}}$ is convergent. This shows that $x \in \mathcal{C}es_r$.

To prove the fact Ces_r and $\mathcal{C}_r$ linearly isomorphic, we should define a lineer bijection between the spaces Ces_r and $\mathcal{C}_r$. Consider the transformation T defined, from Ces_r to $\mathcal{C}_r$ by $x \mapsto Tx$.

$$\begin{aligned} T \ &: \ Ces_r \to \mathcal{C}_r \\ & \quad x \mapsto Tx = \left(\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} x_{jk}\right) = (s_{mn}) = s \end{aligned}$$

(i) Let $x = (x_{jk}), y = (y_{jk}) \in Ces_r$ and $\alpha \in \mathbb{C}$. Then, T is linear since

$$\begin{aligned} T(\alpha x + y) &= \left(\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} \alpha x_{jk} + y_{jk}\right) \\ &= \alpha\left(\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} x_{jk}\right) + \left(\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} y_{jk}\right) \\ &= \alpha Tx + Ty. \end{aligned}$$

(ii) The equation,

$$Tx = \begin{bmatrix} x_{11} & \frac{1}{2}(x_{11}+x_{12}) & \frac{1}{3}(x_{11}+x_{12}+x_{13}) & \dots \\ \frac{1}{2}(x_{11}+x_{21}) & \frac{1}{4}(x_{11}+x_{12}+x_{21}+x_{22}) & \frac{1}{6}(x_{11}+x_{12}+x_{13}+x_{21}+x_{22}+x_{23}) & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m}\sum_{j=1}^{m} x_{j1} & \frac{1}{2m}\sum_{j=1}^{m}\sum_{k=1}^{2} x_{jk} & \frac{1}{3m}\sum_{j=1}^{m}\sum_{k=1}^{3} x_{jk} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = 0$$

leads to the fact that

$$\left.\begin{matrix} x_{11}=0 & x_{12}=0 & x_{13}=0 & \dots \\ x_{21}=0 & x_{22}=0 & x_{23}=0 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{matrix}\right\} \Longrightarrow x = 0$$

and this means that T is a bijection.

(iii) Let us take $s \in \mathcal{C}_r$ and define the sequence $x = (x_{jk})$ via s by

$$x_{jk} = jks_{jk} - (j-1)ks_{j-1,k} - j(k-1)s_{j,k-1} + (j-1)(k-1)s_{j-1,k-1}; \quad (j,k \in \mathbb{N}),$$

where $s_{0,0} = 0$, $s_{0,1} = 0$ and $s_{1,0} = 0$. Since $s \in \mathcal{C}_r$, there exists $L \in \mathbb{C}$ such that $r - \lim_{mn} s_{mn} = L$ and $\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} x_{jk} = s_{mn}$,

$$r\text{-}\lim_{m,n}\left|\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n} x_{jk} - L\right| = 0.$$

Therefore, $x \in Ces_r$. So that, T is surjective.

The conditions (i)-(iii) are satisfied, so T is a linear isomorphism between the linear spaces Ces_r and $\mathcal{C}_r$. This step concludes the proof.

Theorem 2.2 *The space $\mathcal{C}_r$ is subset of the space $\mathcal{C}es_r$.*

Proof. Let $x = (x_{kl}) \in \mathcal{C}_r$. Consider the matrix $(C,1,1) = (c_{mnkl})$, defined by (6). Then $(C,1,1)$ transform of x is

$$\begin{aligned}\{(C,1,1)x\}_{mn} &= \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} c_{mnkl}x_{kl} \\ &= \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} x_{kl} \text{ for all } m,n \in \mathbb{N}.\end{aligned}$$

Since the matrix $(C,1,1)$ is in the class $(\mathcal{C}_r, \mathcal{C}_r)$, the sequence $(C,1,1)x$ is in $\mathcal{C}_r$, hence $x \in Ces_r$. This shows that the inclusion $\mathcal{C}_r \subset Ces_r$ holds.

Let us define the sequence $x = (x_{mn})$ by

$$x_{mn} = \begin{cases} 1 & , \quad (m = n), \\ 0 & , \quad (m \neq n). \end{cases}$$

Since the $(C,1,1)$ transform of x is $s = (s_{mn}) = \left(\frac{\min\{m,n\}}{mn}\right)_{m,n\in\mathbb{N}}$ and $r - \lim s_{mn} = 0$, $x = (x_{mn})$ is in Ces_r but not in $\mathcal{C}_r$. This shows that the inclusion $\mathcal{C}_r \subset Ces_r$ is strict.

3 Dual of the Space Ces_r

In this section, we shall determine the $\beta(r)$-dual of the space Ces_r. Although the β-dual of the spaces of single sequences are unique, the β-duals of the double sequence spaces may be more than one with respect to υ-convergence. The $\beta(\upsilon)$-dual of a double sequence space λ, denoted by $\lambda^{\beta(\upsilon)}$, is defined by

$$\lambda^{\beta(\upsilon)} = \left\{ (a_{ij}) \in \Omega : \upsilon - \sum_{i,j} a_{ij}x_{ij} \text{ exists for all } (x_{ij}) \in \lambda \right\}.$$

It is easy to see for any two spaces λ and μ of double sequences that $\mu^{\beta(\upsilon)} \subset \lambda^{\beta(\upsilon)}$ whenever $\lambda \subset \mu$.

Now, we can give the β-dual of the space Ces_r with respect to the r-convergence using the technique used in [1].

Theorem 3.1 *The $\beta(r)$-dual of the space Ces_r is the set*

$$\Upsilon_{r-r} = \Big\{ a \in \Omega : \sum_{k,l} |kl\triangle_{11}a_{kl}| < \infty,\ (kl\Delta_{10}a_{kl})_k,\ (kl\Delta_{01}a_{kl})_l \in \ell_1, \\ (kla_{kl}) \in \mathcal{M}_u, (a_{kl}) \in \mathcal{CS}_r \Big\},$$

where l_1 and $\mathcal{CS}_r$ denote the space of absolutely summable single sequences and the space of double sequences consisting of all double series whose sequence of partial sums are in the space $\mathcal{C}_r$, respectively.

Proof. Suppose that $x = (x_{kl}) \in Ces_r$. Let us define the sequence $s = (s_{mn})$ as

$$s_{mn} = \frac{1}{mn} \sum_{k=1}^{m} \sum_{l=1}^{n} x_{kl} \quad \text{for all} \quad m, n \in \mathbb{N}.$$

Then, s is in the space $\mathcal{C}_r$, by Teorem 2.1. Let us determine the neccessary and sufficient condition in order to the series

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}x_{kl} \tag{10}$$

is to be r-convergent for a sequence $a = (a_{kl}) \in \Omega$. We obtain m, n^{th} partial sums of the series in (10) that

$$\begin{aligned} z_{mn} &= \textstyle\sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl}x_{kl} \\ &= \textstyle\sum_{k=1}^{m-1} \sum_{l=1}^{n-1} s_{kl}(kl\triangle_{11}a_{kl}) \\ &\quad + \textstyle\sum_{k=1}^{m-1} s_{kn}(kn\triangle_{10}a_{kn}) + \sum_{l=1}^{n-1} s_{ml}(ml\triangle_{01}a_{ml}) \\ &\quad + s_{mn}(mna_{mn}) \end{aligned} \tag{11}$$

for all $m, n \in \mathbb{N}$. (11) may be rewritten by the matrix representation as follows:

$$z_{mn} = \sum_{k=1}^{m} \sum_{l=1}^{n} b_{mnkl}s_{kl} = (Bs)_{mn} \tag{12}$$

for all $m, n \in \mathbb{N}$, where $B = (b_{mnkl})$ is the four dimensional matrix defined by

$$b_{mnkl} = \begin{cases} kl\triangle_{11}a_{kl} & , \quad k \leq m-1 \quad and \quad l \leq n-1 \\ kn\triangle_{10}a_{kn} & , \quad k \leq m-1 \quad and \quad l = n \\ ml\triangle_{01}a_{ml} & , \quad k = m \quad and \quad l \leq n-1 \\ mna_{mn} & , \quad k = m \quad and \quad l = n \\ 0 & , \quad \text{otherwise} \end{cases} \tag{13}$$

We therefore read from the equality (11) that $ax = (a_{kl}x_{kl}) \in \mathcal{CS}_r$ whenever $x = (x_{kl}) \in Ces_r$ if and only if $z = (z_{kl}) \in \mathcal{C}_r$ whenever $s = (s_{kl}) \in \mathcal{C}_r$ which leads to the fact that $B = (b_{mnkl})$, defined by (13), is in the class $(\mathcal{C}_r, \mathcal{C}_r)$. Thus we see from Lemma 1.1 for the matrix B, defined by (13), the conditions (2)-(5) hold. We drive that

$$\sum_{k,l} |kl\triangle_{11}a_{kl}| < \infty, \tag{14}$$

$$\sup_{n} \sum_{k} |kn\triangle_{10}a_{kn}| < \infty \tag{15}$$

$$\sup_{m} \sum_{l} |ml\triangle_{01}a_{ml}| < \infty, \tag{16}$$

$$\sup_{m,n} |mna_{mn}| < \infty \tag{17}$$

and

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl} \text{ exists.} \tag{18}$$

This shows that $Ces_r^{\beta(r)} = \Upsilon_{r-r}$ which completes the proof.

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