$\mathcal{I}_2$-CESÀRO SUMMABILITY OF DOUBLE SEQUENCES OF SETS

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Abstract. In this paper, we defined concept of Wijsman $\mathcal{I}_2$-Cesàro summability and investigate the relationship between the concepts of Wijsman strongly $\mathcal{I}_2$-Cesàro summability, Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman p-strongly $\mathcal{I}_2$-Cesàro summability and Wijsman $\mathcal{I}_2$-statistical convergence of double sequences of sets.

1. INTRODUCTION

The concept of convergence of sequences of real numbers $\mathbb{R}$ has been extended to statistical convergence independently by Fast [10] and Schoenberg [20]. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [14] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence.

Freedman et al. [9] established the connection between the strongly Cesàro summable sequences space and the strongly lacunary summable sequences space. Connor [12] gave the relationships between the concepts of strongly p-Cesàro convergence and statistical convergence of sequences.

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see, [2, 3, 4, 15, 24, 25]). Nuray and Rhoades [15] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [21] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Also, Ulusu and Nuray [22] introduced the concept of Wijsman strongly lacunary summability of sequences of sets.

Kişi and Nuray [13] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence by using ideal. Recently, Ulusu and Kişi [23] studied concept of Wijsman $\mathcal{I}$-Cesàro summability for sequences of sets. Nuray et al. [17] studied the concepts of Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-convergence and Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-Cauchy double sequences of sets. Also, Nuray et al. [16] studied the concepts of Wijsman Cesàro summability and Wijsman lacunary convergence of double sequences of sets and investigate the relationship between them.

2. DEFINITIONS AND NOTATIONS

Now, we recall the basic definitions and concepts (See [1, 2, 5, 6, 7, 8, 11, 13, 14, 16, 17, 18, 19, 23]).

2010 *Mathematics Subject Classification.* 40A05, 40A35.

Key words and phrases. Cesàro summability, statistical convergence, lacunary sequence, $\mathcal{I}_2$-convergence, double sequences of sets, Wijsman convergence.

Let (X,ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x,A) = \inf_{a\in A} \rho(x,a).$$

Throughout the paper we take (X,ρ) be a separable metric space and A, A_{kj} be non-empty closed subsets of X.

The double sequence $\{A_{kj}\}$ is said to be bounded if for each $x \in X$

$$\sup_{k,j} |d(x,A_{kj})| < \infty.$$

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j\to\infty} d(x,A_{kj}) = d(x,A) \quad or \quad \lim_{k,j\to\infty} d(x,A_{kj}) = d(x,A)$$

for each $x \in X$. In this case, we write $W_2 - \lim A_{kj} = A$.

The double sequence $\{A_{kj}\}$ is said to be Wijsman Cesàro summable to A if $\{d(x,A_{kj})\}$ Cesàro summable to $\{d(x,A)\}$; that is, for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x,A_{kj}) = d(x,A).$$

In this case, we write $A_{kj} \stackrel{(W_2\sigma_1)}{\longrightarrow} A$.

The double sequence $\{A_{kj}\}$ is said to be Wijsman strongly Cesàro summable to A if $\{d(x,A_{kj})\}$ strongly Cesàro summable to $\{d(x,A)\}$; that is, for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x,A_{kj}) - d(x,A)| = 0.$$

In this case, we write $A_{kj} \stackrel{[W_2\sigma_1]}{\longrightarrow} A$.

The double sequence $\{A_{kj}\}$ is said to be Wijsman strongly p-Cesàro summable to A if $\{d(x,A_{kj})\}$ strongly p-Cesàro summable to $\{d(x,A)\}$; that is, for each p positive real number and for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x,A_{kj}) - d(x,A)|^p = 0.$$

In this case, we write $A_{kj} \stackrel{[W_2\sigma_p]}{\longrightarrow} A$.

The double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \Big| \{k \le m, j \le n : |d(x,A_{kj}) - d(x,A)| \ge \varepsilon\} \Big| = 0,$$

that is,

$$|d(x,A_{kj}) - d(x,A)| < \varepsilon, \quad \text{a.a. (k,j)}.$$

In this case, we write $st_2 - \lim_W A_k = A$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

i) $\emptyset \in \mathcal{I}$,

ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,

iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

i) $\emptyset \notin \mathcal{F}$,

ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,

iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1 ([14]). *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

The sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-Cesàro summable to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \Big| \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) - d(x, A) \Big| \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $\{A_k\} \overset{C_1(\mathcal{I}_W)}{\to} A$.

The sequence $\{A_k\}$ is Wijsman strongly $\mathcal{I}$-Cesàro summable to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $\{A_k\} \overset{C_1[\mathcal{I}_W]}{\to} A$.

The sequences $\{A_k\}$ is Wijsman p-strongly $\mathcal{I}$-Cesàro summable to A if for each $\varepsilon > 0$, for each p positive real number and for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)|^p \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case, we write $\{A_k\} \overset{C_p[\mathcal{I}_W]}{\to} A$.

The double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}$-convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$,

$$\{(k, j) \in \mathbb{N} \times \mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_{W_2} - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A)$.

The double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A or $S(\mathcal{I}_{W_2})$-convergent to A if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{ (k, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \Big| \{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \Big| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\left(S\left(\mathcal{I}_{W_2}\right)\right)$.

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad as \quad r \to \infty \text{ and } j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad u \to \infty.$$

We use the following notations in the sequel:

$$k_{ru} = k_r j_u, \;\; h_{ru} = h_r \bar{h}_u, \;\; I_{ru} = \{(k,j) : k_{r-1} < k \le k_r \;\; and \;\; j_{u-1} < j \le j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \;\; and \;\; q_u = \frac{j_u}{j_{u-1}}.$$

The double sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta\left[\mathcal{I}_{W_2}\right]$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$A(\varepsilon, x) = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j) \in I_{ru}} |d(x, A_{kj}) - d(x, A)| \ge \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\left(N_\theta\left[\mathcal{I}_{W_2}\right]\right).$

3. MAIN RESULTS

In this section, we defined concepts of Wijsman $\mathcal{I}_2$-Cesàro summability, Wijsman strongly $\mathcal{I}_2$-Cesàro summability and Wijsman p-strongly $\mathcal{I}_2$-Cesàro summability for double sequences of sets. Also, we investigate the relationship between the concepts of Wijsman strongly $\mathcal{I}_2$-Cesàro summability, Wijsman strongly $\mathcal{I}_2$-lacunary convergence, Wijsman p-strongly $\mathcal{I}_2$-Cesàro summability and Wijsman $\mathcal{I}_2$-statistical convergence of double sequences of sets.

Definition 3.1. *The double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \left| \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x, A_{kj}) - d(x, A) \right| \ge \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $\{A_{kj}\} \overset{C_1(\mathcal{I}_{W_2})}{\to} A.$

Definition 3.2. *The double sequence $\{A_{kj}\}$ is Wijsman strongly $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)| \ge \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $\{A_{kj}\} \overset{C_1[\mathcal{I}_{W_2}]}{\to} A.$

Theorem 3.3. *Let θ be a double lacunary sequence. If $\liminf_r q_r > 1, \liminf_u q_u > 1$ then,*

$$\{A_{kj}\} \overset{C_1[\mathcal{I}_{W_2}]}{\to} A \Rightarrow \{A_{kj}\} \overset{N_\theta[\mathcal{I}_{W_2}]}{\to} A.$$

Proof. If $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$. Then, there exist $\lambda, \mu > 0$ such that $q_r \ge 1 + \lambda$ and $q_u \ge 1 + \mu$ for all $r, u \ge 1$, which implies that

$$\frac{k_r j_u}{h_r \bar{h}_u} \le \frac{(1+\lambda)(1+\mu)}{\lambda\mu} \quad \text{and} \quad \frac{k_{r-1} j_{u-1}}{h_r \bar{h}_u} \le \frac{1}{\lambda\mu}.$$

Let $\varepsilon > 0$ and for each $x \in X$ we define the set

$$S = \left\{ (k_r, j_u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{k_r j_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x, A_{is}) - d(x, A)| < \varepsilon \right\}.$$

We can easily say that $S \in \mathcal{F}(\mathcal{I}_2)$, which is a filter of the ideal $\mathcal{I}_2$. Then, we have

$$\begin{aligned}
\frac{1}{h_r \overline{h_u}} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| &= \frac{1}{h_r \overline{h_u}} \sum_{i,s=1,1}^{k_r, j_u} |d(x, A_{is}) - d(x, A)| \\
&\quad - \frac{1}{h_r \overline{h_u}} \sum_{i,s=1,1}^{k_{r-1}, j_{u-1}} |d(x, A_{is}) - d(x, A)| \\
&= \frac{k_r j_u}{h_r \overline{h_u}} . \left(\frac{1}{k_r j_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x, A_{is}) - d(x, A)| \right) \\
&\quad - \frac{k_{r-1} j_{u-1}}{h_r \overline{h_u}} . \left(\frac{1}{k_{r-1} j_{u-1}} \sum_{i,s=1,1}^{k_{r-1}, j_{u-1}} |d(x, A_{is}) - d(x, A)| \right) \\
&\leq \left(\frac{(1+\lambda)(1+\mu)}{\lambda\mu} \right) \varepsilon - \left(\frac{1}{\lambda\mu} \right) \varepsilon'
\end{aligned}$$

for each $x \in X$ and for each $(k_r, j_u) \in S$. Choose $\eta = \left(\frac{(1+\lambda)(1+\mu)}{\lambda\mu} \right) \varepsilon - \left(\frac{1}{\lambda\mu} \right) \varepsilon'$. Therefore,

$$\left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h_u}} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| \ < \eta \right\} \in \mathcal{F}(\mathcal{I}_2)$$

and it completes the proof. □

Theorem 3.4. *Let θ be a double lacunary sequence. If $\limsup_r q_r < \infty, \limsup_u q_u < \infty$ then,*

$$\{A_{kj}\} \stackrel{N_\theta[\mathcal{I}_{W_2}]}{\rightarrow} A \Rightarrow \{A_{kj}\} \stackrel{C_1[\mathcal{I}_{W_2}]}{\rightarrow} A.$$

Proof. If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then there exists $M, N > 0$ such that $q_r < M$ and $q_u < N$ for all $r, u \geq 1$. Let $\{A_{kj}\} \stackrel{N_\theta[\mathcal{I}_{W_2}]}{\rightarrow} A$ and for $\varepsilon_1, \varepsilon_2 > 0$ define the sets T and R such that

$$T = \left\{ (r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h_u}} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| < \varepsilon_1 \right\}$$

and

$$R = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x, A)| \ < \varepsilon_2 \right\},$$

for each $x \in X$. Let

$$A_{tv} = \frac{1}{h_t \overline{h_v}} \sum_{(i,s)\in I_{tv}} |d(x, A_{is}) - d(x, A)| < \varepsilon_1$$

for each $x \in X$ and for all $(t, v) \in T$. It is obvious that $T \in \mathcal{F}(\mathcal{I}_2)$. Choose m, n is any integer with $k_{r-1} < m < k_r$ and $j_{u-1} < n < j_u$, where $(r, u) \in T$.

Then, for each $x \in X$ we have

$$
\begin{aligned}
\frac{1}{mn}\sum_{k,j=1,1}^{m,n}|d(x,A_{kj})-d(x,A)| &\leq \frac{1}{k_{r-1}j_{u-1}}\sum_{i,s=1,1}^{k_r,j_u}|d(x,A_{is})-d(x,A)| \\
&= \frac{1}{k_{r-1}j_{u-1}}\left(\sum_{(i,s)\in I_{11}}|d(x,A_{is})-d(x,A)|\right. \\
&\quad + \sum_{(i,s)\in I_{12}}|d(x,A_{is})-d(x,A)| \\
&\quad + \sum_{(i,s)\in I_{21}}|d(x,A_{is})-d(x,A)| \\
&\quad + \sum_{(i,s)\in I_{22}}|d(x,A_{is})-d(x,A)| \\
&\quad \left. + \cdots + \sum_{(i,s)\in I_{ru}}|d(x,A_{is})-d(x,A)|\right) \\
&= \frac{k_1 j_1}{k_{r-1}j_{u-1}}\left(\frac{1}{h_1\overline{h}_1}\sum_{(i,s)\in I_{11}}|d(x,A_{is})-d(x,A)|\right) \\
&\quad + \frac{k_1(j_2-j_1)}{k_{r-1}j_{u-1}}\left(\frac{1}{h_1\overline{h}_2}\sum_{(i,s)\in I_{12}}|d(x,A_{is})-d(x,A)|\right) \\
&\quad + \frac{(k_2-k_1)j_1}{k_{r-1}j_{u-1}}\left(\frac{1}{h_1\overline{h}_2}\sum_{(i,s)\in I_{21}}|d(x,A_{is})-d(x,A)|\right) \\
&\quad + \frac{(k_2-k_1)(j_2-j_1)}{k_{r-1}j_{u-1}}\left(\frac{1}{h_1\overline{h}_2}\sum_{(i,s)\in I_{22}}|d(x,A_{is})-d(x,A)|\right) \\
&\quad + \ldots + \frac{(k_r-k_{r-1})(j_u-j_{u-1})}{k_{r-1}j_{u-1}}\left(\frac{1}{h_r\overline{h}_u}\sum_{(i,s)\in I_{ru}}|d(x,A_{is})-d(x,A)|\right) \\
&= \frac{k_1 j_1}{k_{r-1}j_{u-1}}A_{11} + \frac{k_1(j_2-j_1)}{k_{r-1}j_{u-1}}A_{12} + \frac{(k_2-k_1)j_1}{k_{r-1}j_{u-1}}A_{21} \\
&\quad + \frac{(k_2-k_1)(j_2-j_1)}{k_{r-1}j_{u-1}}A_{22} + \ldots + \frac{(k_r-k_{r-1})(j_u-j_{u-1})}{k_{r-1}j_{u-1}}A_{ru} \\
&\leq \left(\sup_{(t,v)\in T} A_{tv}\right)\frac{k_r j_u}{k_{r-1}j_{u-1}} \\
&< \varepsilon_1 \cdot M \cdot N.
\end{aligned}
$$

Choose $\varepsilon_2 = \frac{\varepsilon_1}{M \cdot N}$ and in view of the fact that

$$\bigcup \left\{ (m,n) : k_{r-1} < m < k_r, j_{u-1} < n < j_u,\ (r,u) \in T \right\} \subset R,$$

where $T \in \mathcal{F}(\mathcal{I}_2)$, it follows from our assumption on θ that the set R also belongs to $F(\mathcal{I}_2)$ and this completes the proof of the theorem. □

We have the following Theorem by Theorem 3.3 and Theorem 3.4.

Theorem 3.5. *Let θ be a double lacunary sequence. If $1 < \liminf_r q_r < \limsup_r q_r < \infty$ and $1 < \liminf_u q_u < \limsup_u q_u < \infty$ then,*

$$\{A_{kj}\} \overset{C_1[\mathcal{I}_{W_2}]}{\rightarrow} A \Leftrightarrow \{A_{kj}\} \overset{N_\theta[\mathcal{I}_{W_2}]}{\rightarrow} A.$$

Definition 3.6. *The double sequences $\{A_{kj}\}$ is Wijsman p-strongly $\mathcal{I}_2$-Cesàro summable to A if for every $\varepsilon > 0$, for each p positive real number and for each $x \in X$,*

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x,A)|^p \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $\{A_{kj}\} \overset{C_p[\mathcal{I}_{W_2}]}{\rightarrow} A$.

Theorem 3.7. *If $\{A_{kj}\}$ is Wijsman p-strongly $\mathcal{I}_2$-Cesàro summable to A then, $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A.*

Proof. Let $\{A_{kj}\} \overset{C_p[\mathcal{I}_{W_2}]}{\rightarrow} A$ and $\varepsilon > 0$ given. Then, for each $x \in X$ we have

$$\begin{aligned} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x,A)|^p &\geq \sum_{\substack{k,j=1,1 \\ \left| d\left(x, A_{kj}\right) - d(x,A) \right| \geq \varepsilon}}^{m,n} |d(x, A_{kj}) - d(x,A)|^p \\ &\geq \varepsilon^p . \left| \{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x,A)| \geq \varepsilon \} \right| \end{aligned}$$

and so

$$\frac{1}{\varepsilon^p \cdot mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x,A)|^p \geq \frac{1}{mn} \left| \{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x,A)| \geq \varepsilon \} \right|.$$

So for a given $\delta > 0$ and for each $x \in X$

$$\begin{aligned} &\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x,A)| \geq \varepsilon \} \right| \geq \delta \right\} \\ &\subseteq \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x, A_{kj}) - d(x,A)|^p \geq \varepsilon^p \cdot \delta \right\} \in \mathcal{I}_2. \end{aligned}$$

Therefore, $\{A_k\} \overset{S(\mathcal{I}_{W_2})}{\rightarrow} A$. □

Theorem 3.8. *Let $\{A_{kj}\} \in L_\infty$. If $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A then, $\{A_{kj}\}$ is Wijsman p-strongly $\mathcal{I}_2$-Cesàro summable to A.*

Proof. Suppose that $\{A_{kj}\}$ is bounded and $\{A_{kj}\} \stackrel{S(\mathcal{I}_{W_2})}{\rightarrow} A$. Then, there is a $M > 0$ such that

$$|d(x, A_{kj}) - d(x, A)| \leq M$$

for each $x \in X$ and for all k, j. Then, for given $\varepsilon > 0$ and for each $x \in X$ we have

$$\begin{aligned}
\frac{1}{mn}\sum_{k,j=1,1}^{m,n}|d(x,A_{kj})-d(x,A)|^p &= \frac{1}{mn}\sum_{\substack{k,j=1,1\\ \left|d(x,A_{kj})-d(x,A)\right|\geq\varepsilon}}^{m,n}|d(x,A_{kj})-d(x,A)|^p \\
&\quad +\frac{1}{mn}\sum_{\substack{k,j=1,1\\ \left|d(x,A_{kj})-d(x,A)\right|<\varepsilon}}^{m,n}|d(x,A_{kj})-d(x,A)|^p \\
&\leq \frac{1}{mn}M^p\cdot\big|\{k\leq m, j\leq n : |d(x,A_{kj})-d(x,A)|\geq\varepsilon\big| \\
&\quad +\frac{1}{mn}\varepsilon^p\cdot\big|\{k\leq m, j\leq n : |d(x,A_{kj})-d(x,A)|<\varepsilon\}\big| \\
&\leq \frac{M^p}{mn}\big|\{k\leq m, j\leq n : |d(x,A_{kj})-d(x,A)|\geq\varepsilon\}\big|+\varepsilon^p.
\end{aligned}$$

Then, for any $\delta > 0$ and for each $x \in X$,

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{mn}\sum_{k,j=1,1}^{m,n}|d(x,A_{kj})-d(x,A)|^p\geq\delta\right\}$$

$$\subseteq\left\{(m,n)\in\mathbb{N}\times\mathbb{N} : \frac{1}{mn}\big|\{k\leq m, j\leq n : |d(x,A_{kj})-d(x,A)|\geq\varepsilon\}\big|\geq\frac{\delta^p}{M^p}\right\}\in\mathcal{I}_2.$$

Therefore $\{A_k\} \stackrel{C_p[\mathcal{I}_{W_2}]}{\rightarrow} A$. □

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