ASYMPTOTICALLY $\mathcal{I}$-INVARIANT EQUIVALENCE OF SEQUENCES DEFINED BY A MODULUS FUNCTION

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Abstract. In this paper, we introduce the concepts of strongly asymptotically ideal invariant equivalence, f-asymptotically ideal invariant equivalence, strongly f-asymptotically ideal invariant equivalence and asymptotically ideal invariant statistical equivalence for sequences. Also, we investigate some relationships among them.

1. Introduction

Throughout the paper $\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a real sequence has been extended to statistical convergence independently by Fast [1], Schoenberg [24] and studied by many authors. The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [2] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of $\mathbb{N}$.

Several authors including Raimi [17], Schaefer [23], Mursaleen and Edely [7], Mursaleen [9], Savaş [18, 19], Nuray and Savaş [11], Pancaroğlu and Nuray [13] and some authors have studied invariant convergent sequences. The concept of strongly σ-convergence was defined by Mursaleen [8]. Savaş and Nuray [20] introduced the concepts of σ-statistical convergence and lacunary σ-statistical convergence and gave some inclusion relations. Nuray et al. [12] defined the concepts of σ-uniform density of a subset A of the set $\mathbb{N}$, $\mathcal{I}_\sigma$-convergence and investigated relationships between $\mathcal{I}_\sigma$-convergence and invariant convergence also $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Pancaroḡlu and Nuray [13] studied Statistical lacunary invariant summability. Recently, Nuray and Ulusu [25] investigated lacunary $\mathcal{I}$-invariant convergence and lacunary $\mathcal{I}$-invariant Cauchy sequence of real numbers.

Marouf [6] peresented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [14] presented asymptotically statistical equivalent sequences for nonnegative summability matrices. Patterson and Savaş [15, 22] introduced asymptotically lacunary statistically equivalent sequences and also asymptotically $\sigma\theta$-statistical equivalent sequences. Ulusu [26, 27] studied asymptotically ideal invariant equivalence and asymptotically lacunary $\mathcal{I}_\sigma$-equivalence.

Modulus function was introduced by Nakano [10]. Maddox [5], Pehlivan [16] and many authors used a modulus function f to define some new concepts and inclusion theorems. Kumar and Sharma [3] studied lacunary equivalent sequences by ideals and modulus function.

Now, we recall the basic concepts and some definitions and notations (See [2, 4, 5, 6, 12, 14, 16]).

Let σ be a mapping of the positive integers into itself. A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

(1) $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1...)$,
(3) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

The mappings ϕ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In case

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σ is translation mappings $\sigma(n)=n+1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

A sequence $x=(x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\Big|\big\{k\leq n:|x_k-L|\geq\varepsilon\big\}\Big|=0,$$

where the vertical bars indicate the number of elements in the enclosed set.

A family of sets $\mathcal{I}\subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset\in\mathcal{I}$, (ii) For each $A,B\in\mathcal{I}$ we have $A\cup B\in\mathcal{I}$, (iii) For each $A\in\mathcal{I}$ and each $B\subseteq A$ we have $B\in\mathcal{I}$.

An ideal is called nontrivial if $\mathbb{N}\notin\mathcal{I}$ and nontrivial ideal is called admissible if $\{n\}\in\mathcal{I}$ for each $n\in\mathbb{N}$. Throughout the paper we let $\mathcal{I}$ be an admissible ideal.

Let $A\subseteq\mathbb{N}$ and

$$s_m=\min_n\Big|A\cap\big\{\sigma(n),\sigma^2(n),...,\sigma^m(n)\big\}\Big| \quad\text{and}\quad S_m=\max_n\Big|A\cap\big\{\sigma(n),\sigma^2(n),...,\sigma^m(n)\big\}\Big|.$$

If the limits $\underline{V}(A)=\lim\limits_{m\to\infty}\frac{s_m}{m}$ and $\overline{V}(A)=\lim\limits_{m\to\infty}\frac{S_m}{m}$ exist then, they are called a lower σ-uniform density and an upper σ-uniform density of the set A, respectively. If $\underline{V}(A)=\overline{V}(A)$, then $V(A)=\underline{V}(A)=\overline{V}(A)$ is called the σ-uniform density of A.

Denote by $\mathcal{I}_\sigma$ the class of all $A\subseteq\mathbb{N}$ with $V(A)=0$.

A sequence $x=(x_k)$ is said to be $\mathcal{I}_\sigma$-convergent to L if for every $\varepsilon>0$, the set $A_\varepsilon=\big\{k:|x_k-L|\geq\varepsilon\big\}$ belongs to $\mathcal{I}_\sigma$, i.e., $V(A_\varepsilon)=0$. It is denoted by $\mathcal{I}_\sigma-\lim x_k=L$.

The two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically equivalent if

$$\lim_k\frac{x_k}{y_k}=1$$

(denoted by $x\sim y$).

The two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically statistical equivalent of multiple L if for every $\varepsilon>0$,

$$\lim_n\frac{1}{n}\left|\left\{k\leq n:\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\right|=0$$

(denoted by $x\overset{S_L}{\sim}y$) and simply asymptotically statistical equivalent if $L=1$.

The two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ if for every $\varepsilon>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\in\mathcal{I}$$

(denoted by $x_k\overset{\mathcal{I}(\omega)}{\sim}y_k$) and simply strongly asymptotically equivalent with respect to the ideal $\mathcal{I}$, if $L=1$.

The two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically $\mathcal{I}_\sigma$-equivalent of multiple L if for every $\varepsilon>0$, $A_\varepsilon=\left\{k\in\mathbb{N}:\left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\in\mathcal{I}_\sigma$, i.e., $V(A_\varepsilon)=0$. It is denoted by $x_k\overset{[\mathcal{I}_\sigma^L]}{\sim}y_k$.

A function $f:[0,\infty)\to[0,\infty)$ is called a modulus if

(1) $f(x)=0$ if and if only if $x=0$,
(2) $f(x+y)\le f(x)+f(y)$,
(3) f is increasing,
(4) f is continuous from the right at 0.

A modulus may be unbounded (for example $f(x)=x^p$, $0<p<1$) or bounded (for example $f(x)=\frac{x}{x+1}$).

Let f be modulus function. The two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that, for every $\varepsilon>0$,

$$\left\{k\in\mathbb{N}: f\left(\left|\frac{x_k}{y_k}-L\right|\right)\ge\varepsilon\right\}\in\mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(f)}{\sim} y_k$) and simply f-asymptotically $\mathcal{I}$-equivalent if $L=1$.

Let f be modulus function. The two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly f-asymptotically equivalent of multiple L with respect to the ideal $\mathcal{I}$ provided that, for every $\varepsilon>0$

$$\left\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right)\ge\varepsilon\right\}\in\mathcal{I}$$

(denoted by $x_k \overset{\mathcal{I}(\omega_f)}{\sim} y_k$)) and simply strongly f-asymptotically $\mathcal{I}$-equivalent if $L=1$.

Lemma 1. [16] *Let f be a modulus and $0<\delta<1$. Then, for each $x\ge\delta$ we have $f(x)\le 2f(1)\delta^{-1}x$.*

2. Main Results

Definition 2.1. *The sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly asymptotically $\mathcal{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,*

$$\left\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} \left|\frac{x_k}{y_k}-L\right|\ge\varepsilon\right\}\in\mathcal{I}_\sigma$$

(denoted by $x_k \overset{[\mathcal{I}_\sigma^L]}{\sim} y_k$) and simply strongly asymptotically $\mathcal{I}$-invariant equivalent if $L=1$.

Definition 2.2. *Let f be a modulus function. The sequences $x=(x_k)$ and $y=(y_k)$ are said to be f-asymptotically $\mathcal{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,*

$$\left\{k\in\mathbb{N}: f\left(\left|\frac{x_k}{y_k}-L\right|\right)\ge\varepsilon\right\}\in\mathcal{I}_\sigma$$

(denoted by $x_k \overset{\mathcal{I}_\sigma^L(f)}{\sim} y_k$) and simply f-asymptotically $\mathcal{I}$-invariant equivalent if $L=1$.

Definition 2.3. *Let f be a modulus function. The sequences $x=(x_k)$ and $y=(y_k)$ are said to be strongly f-asymptotically $\mathcal{I}$-invariant equivalent of multiple L if for every $\varepsilon>0$,*

$$\left\{n\in\mathbb{N}: \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_k}{y_k}-L\right|\right)\ge\varepsilon\right\}\in\mathcal{I}_\sigma$$

(denoted by $x_k \overset{[\mathcal{I}_\sigma^L(f)]}{\sim} y_k$)) and simply strongly f-asymptotically $\mathcal{I}$-invariant equivalent if $L=1$.

Theorem 2.1. *Let f be a modulus function. Then,*

$$x_k \overset{[\mathcal{I}_\sigma^{\mathcal{L}}]}{\sim} y_k \Rightarrow x_k \overset{[\mathcal{I}_\sigma^{\mathcal{L}}(f)]}{\sim} y_k.$$

Proof. Let $x_k \overset{[\mathcal{I}_\sigma^{\mathcal{L}}]}{\sim} y_k$ and $\varepsilon > 0$ be given. Choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for $0 \leq t \leq \delta$. Then, for $m = 1, 2, \ldots$, we can write

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) &= \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \leq \delta}}^{n} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \\ &+ \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| > \delta}}^{n} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \end{aligned}$$

and so by Lemma 1

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) < \varepsilon + \left(\frac{2f(1)}{\delta}\right)\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|$$

uniformly in m. Thus, for every any $\gamma > 0$

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \geq \gamma\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \frac{(\gamma - \varepsilon)\delta}{2f(1)}\right\},$$

uniformly in m. Since $x_k \overset{[\mathcal{I}_\sigma^{L}]}{\sim} y_k$, it follows the later set and hence, the first set in above expression belongs to $\mathcal{I}_\sigma$. This proves that $x_k \overset{[\mathcal{I}_\sigma^{L}(f)]}{\sim} y_k$. □

Definition 2.4. *The sequences x_k and y_k are said to be asymptotically $\mathcal{I}$-invariant statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $\gamma > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I}_\sigma$$

(denoted by $x_k \overset{\mathcal{I}(\mathcal{S}_\sigma)}{\sim} y_k$) and simply asymptotically $\mathcal{I}$-invariant statistical equivalent if $L = 1$.

Theorem 2.2. *Let f be a modulus function. Then,*

$$x_k \overset{[\mathcal{I}_\sigma^{L}(f)]}{\sim} y_k \Rightarrow x_k \overset{\mathcal{I}(\mathcal{S}_\sigma)}{\sim} y_k.$$

Proof. Assume that $x_k \overset{[\mathcal{I}_\sigma^{L}(f)]}{\sim} y_k$ and $\varepsilon > 0$ be given. Since for $m = 1, 2, \ldots$,

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) &\geq \frac{1}{n} \sum_{\substack{k=1 \\ \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon}}^{n} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \\ &\geq f(\varepsilon).\frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon\right\}\right| \end{aligned}$$

it follows that for any $\gamma > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\gamma}{f(\varepsilon)}\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n} f\left(\left|\frac{x_{\sigma^k(m)}}{y_{\sigma^k(m)}} - L\right|\right) \geq \gamma\right\},$$

uniformly in m. Since $x_k \overset{[\mathcal{I}_\sigma^{L}(f)]}{\sim} y_k$, so the last set belongs to $\mathcal{I}_\sigma$. But then by the definition of an ideal, the first set belongs to $\mathcal{I}_\sigma$ and therefore $x_k \overset{\mathcal{I}(\mathcal{S}_\sigma)}{\sim} y_k$ □

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