Asymptotically Lacunary $\mathcal{I}_2^{\sigma}$-Equivalence for Double Set Sequences Defined by Modulus Functions

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Abstract

Fast (1951) and Schoenberg (1959), independently, introduced concept of statistical convergence and many authors studied some properties of this concepts. Mursaleen and Edely (2003) extended this concept to the double sequences. Kostyrko et al. (2000) defined $\mathcal{I}$ of subset of $\mathbb{N}$ (natural numbers) and investigated $\mathcal{I}$-convergence with some properties and proved theorems about $\mathcal{I}$-convergence. The idea of $\mathcal{I}_2$-convergence and some properties of this convergence were studied by Das et al. (2008). Nuray and Rhoades (2011) defined the idea of statistical convergence of set sequence and investigated some theorems about this notion and importance properties. After, several authors extended the convergence of real numbers sequences to convergence of sequences of sets and investigated it's characteristic in summability.

Several authors have studied invariant convergent sequences. Tortop and Dündar (2018) introduced $\mathcal{I}_2$-invariant convergence of double set sequences. Asymptotically equivalent and some properties of equivalence studied by several authors. Ulusu and Gülle (2019) introduced the concept of asymptotically $\mathcal{I}_\sigma$-equivalence of sequences of sets. Recently, Dündar et al. (in review) studied asymptotically ideal invariant equivalence of double sequences. Many authors studied asymptotic equivalence using f modulus function. Recently, Akın, Dündar and Ulusu (2018) defined and studied asymptotically lacunary $\mathcal{I}$-invariant statistical equivalence for sequences of sets defined by a modulus function. Kumar and Sharma (2012) studied $\mathcal{I}_\theta$-equivalent sequences using a modulus function f.

In this study, first, we present the concepts of strongly asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalence, f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalence, strongly f-asymptotically $\mathcal{I}_2^{\sigma\theta}$-equivalence for double sequences of sets. Then, we investigated some properties and relationships among this new concepts.

Key Words: Asymptotic Equivalence, Lacunary Invariant Convergence, $\mathcal{I}_2$-Convergence, Wijsman Convergence, Modulus Function.

Özet

Fast (1951) ve Schoenberg (1959) istatistiksel yakınsaklık kavramını tanımladı. Daha sonra birçok yazar bu kavram ve özellikleri ile ilgili çalışmalar yaptı. Mursaleen ve Edely (2003) istatistiksel yakınsaklık kavramını çift dizilere taşımıştır. . Kostyrko vd. (2000) $\mathbb{N}$'ın(doğal sayılar) alt kümesi $\mathcal{I}$ ideal kavramını tanımladı. Sonra yeni tanımlanan $\mathcal{I}$ yakınsaklık kavramı ile ilgili bazı özellikleri inceleyip teoremleri ispatladı. $\mathcal{I}_2$- yakınsaklık kavramı ve bu kavramın bazı özellikleri Das vd. (2008) tarafından incelendi. Nuray ve Rhoades (2012) küme dizileri için istatistiksel yakınsaklık kavramını tanımlayıp bu kavramla ilgili bazı özellikleri ve teoremleri inceledi. Daha sonra birçok yazar tarafından reel sayı dizilerinin yakınsaklığı küme dizilerinin yakınsaklığına genişletilerek çalışmalar yapılmıştır.

Birçok yazar invariant yakınsaklık ile ilgili çalışmalar yapmıştır. Tortop ve Dündar (2018) çift küme dizileri için $\mathcal{I}_2$-invariant yakınsaklık kavramını tanımladı. Aşimptotik denklik ve bu denkliğin özellikleri bazı yazarlar tarafından çalışılmıştır. Ulusu ve Gülle (2019) küme dizilerinin asimptotik $\mathcal{I}_\sigma$-denkliği kavramını tanımladı. Son zamanlarda Dündar vd. (incelemede) çift dizilerin asimptotik ideal invariant denkliği ile ilgili çalışma yaptı. Modülüs fonksiyonu ilk defa Nakano (1953) tarafından tanımlandı. Birçok yazar f modülüs fonksiyonunu kullanarak asimptotik denklik ile ilgili çalışmalar yaptı. Akın, Dündar ve Ulusu (2018) küme dizileri için f modülüs fonksiyonunu kullanarak asimptotik lacunary $\mathcal{I}$-invariant istatistiksel denklik kavramını tanımlayıp bu kavram ile ilgili çalışmalar yaptı. Modülüs fonsiyonunu kullanılarak lacunary ideal denk diziler ile ilgili Kumar ve Sharma (2012) tarafından bir çalışma yapıldı.

Bu çalışmada çift küme dizileri için kuvvetli asimptotik $\mathcal{I}_2^{\sigma\theta}$-denklik, , f-asimptotik $\mathcal{I}_2^{\sigma\theta}$-denklik, kuvvetli f-asimptotik $\mathcal{I}_2^{\sigma\theta}$-denklik tanımları yapıldı. Daha sonra tanımlanan bu yeni kavramların özellikleri ve bu kavramlar arasındaki ilişkiler incelendi.

Key Words: Asimptotik Denklik, Lacunary Invariant Yakınsaklık, $\mathcal{I}_2$-Yakınsaklık, Wijsman Yakınsaklık, Modülüs Fonksiyonu.

Introduction and Definitions

Recently, the statistical convergence extended to ideal convergence of real numbers and some important properties about ideal convergence investigated by many mathematicians. Kostyrko et al. [11] defined $\mathcal{J}$ of subset of $\mathbb{N}$ (positive integers) and investigated $\mathcal{J}$-convergence with some properties and proved theorems about $\mathcal{J}$-convergence. The idea of $\mathcal{J}_2$-convergence and some properties of this convergence were studied by Das et al. [4].

Several authors have studied invariant convergent sequences (see, [17,18, 21, 26, 27, 29, 34, 35, 36, 37, 39, 44]). Nuray et al. [22] defined the notions of invariant uniform density of subsets E of $\mathbb{N}$, $\mathcal{J}_\sigma$-convergence and investigated relationships between $\mathcal{J}_\sigma$-convergence and σ-convergence also $\mathcal{J}_\sigma$-convergence and $[V_\sigma]_p$-convergence. Tortop and Dündar [41] introduced $\mathcal{J}_2$-invariant convergence of double set sequences. Akın [25] studied Wijsman lacunary $\mathcal{J}_2$-invariant convergence of double sequences of sets.

Several authors define some new concepts and give inclusion theorems using a modulus function f (see, [8, 9, 14, 19, 30, 33]). Kumar and Sharma [12] studied $\mathcal{J}_\theta$-equivalent sequences using a modulus function f. Kişi et al. [10] introduced f-asymptotically $\mathcal{J}_\theta$-equivalent set sequences. Akın and Dündar [23] and Akın et al. [24] give definitions of f-asymptotically $\mathcal{J}_\sigma$ and $\mathcal{J}_{\sigma\theta}$-statistical equivalence of set sequences. Dündar and Akın [5] studied f-asymptotically $\mathcal{J}_2^\sigma$-equivalence for double set sequences.

Now, we recall the basic concepts and some definitions and notations (See [1, 2, 3, 10, 11, 13, 14, 15, 22, 27, 28, 31, 33, 36, 42, 43, 45, 46, 47, 48]).

Let two non-negative sequences $u = (u_k)$ and $v = (v_k)$. If $\lim_{k\to\infty} \frac{u_k}{v_k} = 1$, then $u = (u_k)$ and $v = (v_k)$ are said to be asymptotically equivalent (denoted by $u \sim v$).

Let (Y, ρ) be a metric space, $y \in Y$ and any non-empty subset C of Y, then we define the distance from y to C by

$$d(y, C) = \inf_{c\in C} \rho(y, c).$$

This after, we let (Y, ρ) be a metric space and C, D, C_k and D_k $(k = 1,2,\dots)$ be non-empty closed subsets of Y.

A sequence $\{C_k\}$ is Wijsman convergent to C if $\lim_{k\to\infty} d(y, C_k) = d(y, C)$ for each $y \in Y$. In this instance, it is showed by $W - \lim_{k\to\infty} C_k = C$.
If $\sup_k d(y, C_k) < \infty$ for each $y \in Y$, then $\{C_k\}$ is bounded and we write $\{C_k\} \in L_\infty$.

Let C_k, $D_k \subseteq Y$ such that $d(y, C_k) > 0$ and $d(y, D_k) > 0$ for each $y \in Y$. The sequences $\{C_k\}$ and $\{D_k\}$ are asymptotically equivalent if for each $y \in Y$, $\lim_{k\to\infty} \frac{d(y,C_k)}{d(y,D_k)} = 1$ (denoted by $C_k \sim D_k$).

Let C_k, $D_k \subseteq Y$ such that $d(y, C_k) > 0$ and $d(y, D_k) > 0$ for each $y \in Y$. If for every $\varepsilon > 0$ and each $y \in Y$, $\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(y,C_k)}{d(y,D_k)} - L \right| \geq \varepsilon \right\} \right| = 0$, then $\{C_k\}$ and $\{D_k\}$ are asymptotically statistical equivalent of multiple L (denoted by $C_k \overset{WS_L}{\sim} D_k$) and if $L = 1$, then $\{C_k\}$ and $\{D_k\}$ are asymptotically statistical equivalent.

Let $\sigma: \mathbb{N} \to \mathbb{N}$ be a mapping and ϕ be a continuous linear functional on the space of real bounded sequences (ℓ_∞). ϕ is an invariant mean or a σ-mean, if the following conditions hold:

1. $\phi(u) \geq 0$, when the sequence $u = (u_n)$ has $u_n \geq 0$, for all n,
2. $\phi(e) = 1$, where $e = (1,1,1\dots)$,
3. $\phi(u_{\sigma(n)}) = \phi(u)$ for all $u \in \ell_\infty$.

Suppose that the mappings ϕ are injective and such that $\sigma^m(j) \neq j$, for all $j, m \in \mathbb{N}$, where $\sigma^m(j)$ is the mth iterate of σ at j. Therefore, for all $u \in c$ $\phi(u)$ equals to $\lim u$ which is the extension of the limit functional on c, where $c = \{x = (x_k): \lim_k x_k exists\}$.

If the equality $\sigma(j) \neq j + 1$ exists, then σ-mean is called a Banach limit, generally.

Now, we give the definition of ideal. $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal, provided that the followings hold:

(i) $\emptyset \in \mathcal{I}$, (ii) For each $E, F \in \mathcal{I}$ we have $E \cup F \in \mathcal{I}$, (iii) For each $E \in \mathcal{I}$ and each $F \subseteq E$ we have $F \in \mathcal{I}$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a ideal. $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial provided that $\mathbb{N} \notin \mathcal{I}$. Also, for non-trivial ideal and for each $n \in \mathbb{N}$ provided that $\{n\} \in \mathcal{I}$, then $\mathcal{I}$ is admissible ideal. After that, we consider that $\mathcal{I}$ is an admissible ideal.

Let $C \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{mk} := \min_{i,j} |C \cap \{(\sigma(i), \sigma(j)), (\sigma^2(i), \sigma^2(j)), \ldots, (\sigma^m(i), \sigma^k(j))\}|$$

and

$$S_{mk} := \max_{i,j} |C \cap \{(\sigma(i), \sigma(j)), (\sigma^2(i), \sigma^2(j)), \ldots, (\sigma^m(i), \sigma^k(j))\}|.$$

If the limits $\underline{V_2}(C) := \lim_{m,k\to\infty} \frac{s_{mk}}{mk}$ and $\overline{V_2}(C) := \lim_{m,k\to\infty} \frac{S_{mk}}{mk}$ exists then $\underline{V_2}(C)$ is called a lower and $\overline{V_2}(C)$ is called an upper σ-uniform density of the set C, respectively. If $\underline{V_2}(C) = \overline{V_2}(C)$, then $V_2(C) = \underline{V_2}(C) = \overline{V_2}(C)$ is called the σ-uniform density of C. Denote by $\mathcal{I}_2^\sigma$ the class of all $C \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2(C) = 0$.

This after, let C_{ij}, D_{ij}, C, D be any nonempty closed subsets of Y.

If for each $y \in Y$,

$$\lim_{m,k\to\infty} \frac{1}{mk} \sum_{i,j=1,1}^{m,k} d(y, C_{\sigma^i(s),\sigma^j(t)}) = d(y, C),$$

uniformly in s, t then, the double sequence $\{C_{ij}\}$ is said to be invariant convergent to C in Y.

If for every $\gamma > 0$,

$$A(\gamma, y) = \{(i,j): |d(y, C_{ij}) - d(y, C)| \geq \gamma\} \in \mathcal{I}_2^\sigma$$

that is, $V_2(A(\gamma, y)) = 0$ then, the double sequence $\{C_{ij}\}$ is said to be Wijsman $\mathcal{I}_2$-invariant convergent or $\mathcal{I}_{W_2}^\sigma$-convergent to C, In this instance, we write $C_{ij} \to C(\mathcal{I}_{W_2}^\sigma)$ and by $\mathcal{I}_{W_2}^\sigma$ we will denote the set of all Wijsman $\mathcal{I}_2^\sigma$-convergent double sequences of sets.

For non-empty closed subsets C_{ij}, D_{ij} of Y define $d(y; C_{ij}, D_{ij})$ as follows:

$$d(y; C_{ij}, D_{ij}) = \begin{cases} \frac{d(y, C_{ij})}{d(y, D_{ij})} & , \quad y \notin C_{ij} \cup D_{ij} \\ L & , \quad y \in C_{ij} \cup D_{ij}. \end{cases}$$

If for every $\gamma > 0$ and for each $y \in Y$,

$$\left\{(m,k): \in \mathbb{N} \times \mathbb{N}: \frac{1}{mk} \sum_{i,j=1,1}^{m,k} |d(y; C_{ij}, D_{ij}) - L| \geq \gamma\right\} \in \mathcal{I}_2^\sigma$$

then, double sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be strongly asymptotically $\mathcal{I}_2^\sigma$-equivalent of multiple L (denoted by $C_{ij} \overset{[W_{\mathcal{I}_2^\sigma}^L]}{\sim} D_{ij}$) and if $L = 1$, then $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be strongly asymptotically $\mathcal{I}_2^\sigma$-equivalent.
If following conditions hold for $f: [0, \infty) \to [0, \infty)$ function, then it is called a modulus function:

1. $f(u) = 0$ if and if only if $u = 0$,
2. $f(u + v) \leq f(u) + f(v)$
3. f is increasing
4. f is continuous from the right at 0.

This after, we let f as a modulus function.

The modulus function f may be unbounded (for instance $f(u) = u^q,\ 0 < q < 1$) or bounded (for example $f(u) = \frac{u}{u+1}$).

If for every $\gamma > 0$ and for each $y \in Y$,

$$\{(i,j) \in \mathbb{N} \times \mathbb{N}: f(|d(y; C_{ij}, D_{ij}) - L|) \geq \gamma\} \in \mathcal{I}_2^\sigma,$$

then the double sequences $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be f-asymptotically $\mathcal{I}_2^\sigma$-equivalent of multiple L (denoted by $C_{ij} \overset{W_{\mathcal{I}_2^\sigma}^L(f)}{\sim} D_{ij}$) and if $L = 1$, then $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be f-asymptotically $\mathcal{I}_2^\sigma$-equivalent.

If for every $\gamma > 0$ and for each $y \in Y$,

$$\left\{(m,k) \in \mathbb{N} \times \mathbb{N}: \frac{1}{mk} \sum_{i,j=1,1}^{mk} f(|d(y; C_{ij}, D_{ij}) - L|) \geq \gamma\right\} \in \mathcal{I}_2^\sigma$$

then, $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be strongly f-asymptotically $\mathcal{I}_2^\sigma$-equivalent of multiple L (denoted by $C_{ij} \overset{[W_{\mathcal{I}_2^\sigma}^L(f)]}{\sim} D_{ij}$) and if $L = 1$, then $\{C_{ij}\}$ and $\{D_{ij}\}$ are said to be strongly f-asymptotically $\mathcal{I}_2^\sigma$-equivalent.

A double sequence $\theta_2 = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \ \ and \ \ j_0 = 0, \bar{h}_u = j_u - j_{u-1} \to \infty, \qquad as \ \ r, u \to \infty.$$

We use the following notations afterwards:

$$k_{ru} = k_r j_u, \ \ h_{ru} = h_r \bar{h}_u, \ \ \ I_{ru} = \{(k,j): k_{r-1} < k \leq k_r \quad and \quad j_{u-1} < j \leq j_u\}.$$

After this, we take $\theta_2 = \{(k_r, j_u)\}$ as a double lacunary sequence.

Let $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence, $C \subseteq \mathbb{N} \times \mathbb{N}$ and

$$s_{ru} := \min_{m,n} \left| C \cap \left\{ \left(\sigma^k(m), \sigma^j(n)\right) : (k,j) \in I_{ru} \right\} \right|$$

and

$$S_{ru} := \max_{m,n} \left| C \cap \left\{ \left(\sigma^k(m), \sigma^j(n)\right) : (k,j) \in I_{ru} \right\} \right|.$$

If the limits $\underline{V_2^\theta}(C) := \lim_{r,u\to\infty} \frac{s_{ru}}{h_{ru}}$ and $\overline{V_2^\theta}(C) := \lim_{r,u\to\infty} \frac{S_{ru}}{h_{ru}}$ exist, then they are called a lower lacunary σ-uniform density and an upper lacunary σ-uniform density of the set C, respectively. If $\underline{V_2^\theta}(C) = \overline{V_2^\theta}(C)$, then $V_2^\theta(C) = \underline{V_2^\theta}(C) = \overline{V_2^\theta}(C)$ is called the lacunary σ-uniform density of C.
Denote by $\mathcal{I}_2^{\sigma\theta}$ the class of all $C \subseteq \mathbb{N} \times \mathbb{N}$ with $V_2^\theta(C) = 0$.

After this, we take $\mathcal{I}_2^{\sigma\theta}$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$.

Lemma 1 [33] Let f be a modulus and $0 < \delta < 1$. Then, for each $\mathrm{u} \geq \gamma$ we have $\mathrm{f(u)} \leq 2\mathrm{f}(1)\gamma^{-1}\mathrm{u}$.

Method

In the proofs of the theorems obtained in this study, used frequently in mathematics,

i.Direct proof method,

ii. Reverse proof method,

iii.Contrapositive method,

iv. Induction method

methods were used as needed.

Main Results

Definition 2.1. *If for every* $\gamma > 0$ *and each* $y \in Y$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} |d(y; C_{kj}, D_{kj}) - L| \geq \gamma\right\} \in \mathcal{J}_2^{\sigma\theta},$$

then the double sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be strongly asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent of multiple L denoted by

$$C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim} D_{kj}$$

and if $L = 1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be strongly asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent.

Definition 2.2. *If for every* $\gamma > 0$ *and each* $y \in Y$,

$$\{(k,j) \in \mathbb{N} \times \mathbb{N}: f(|d(y; C_{kj}, D_{kj}) - L|) \geq \gamma\} \in \mathcal{J}_2^{\sigma\theta},$$

then the double sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be f-asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent of multiple L denoted by $C_{kj} \overset{W^L_{\mathcal{J}_2^{\sigma\theta}}(f)}{\sim} D_{kj}$

and if $L = 1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be f-asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent.

Definition 2.3 *If for every* $\gamma > 0$ *and each* $y \in Y$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{ru}} \sum_{(k,j)\in I_{ru}} f(|d(y; C_{kj}, D_{kj}) - L|) \geq \gamma\right\} \in \mathcal{J}_2^{\sigma\theta},$$

then the double sequences $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be strongly f-asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent of multiple L denoted by

$$C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim} D_{kj}$$

and if $L = 1$, then $\{C_{kj}\}$ and $\{D_{kj}\}$ are said to be strongly f-asymptotically $\mathcal{J}_2^{\sigma\theta}$-equivalent.

Theorem 2.1. For each $y \in Y$, we have

$$C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim} D_{kj} \Rightarrow C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim} D_{kj}.$$

Theorem 2.2. If $\lim_{z\to\infty} \frac{f(z)}{z} = \alpha > 0$, then

$$C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}]}{\sim} D_{kj} \Leftrightarrow C_{kj} \overset{[W^L_{\mathcal{J}_2^{\sigma\theta}}(f)]}{\sim} D_{kj}.$$

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