Lacunary Statistical Convergence of Double Sequences in Fuzzy n-Normed Spaces

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Öz

Reel sayı dizilerinin yakınsaklık kavramı genişletilerek istatistiksel yakınsaklık kavramı ortaya çıktı. Daha sonra Mursaleen ve Edely tarafından bu kavram çift diziler için genişletildi. Fuzzy sayı dizileri için klasik anlamda yakınsaklık kavramı ilk defa Matloka tarafından çalışıldı ve fuzzy sayı dizilerinin bazı temel teoremlerin ispatlandı. Türkmen ve Çınar lacunary istatistiksel yakınsaklık kavramını fuzzy normlu uzaylarda çalıştı. Mohiuddine ve arkadaşları ise fuzzy normlu uzaylarda çift diziler için istatistiksel yakınsaklık kavramını çalıştı. Bu çalışmada daha önce fuzzy normlu uzaylarda tanımlanan lacunary istatistiksel yakınsaklığın uygulamalarını, çift diziler fuzzy n-normlu uzayları kullanarak yeniden tanımlayacağız. Bunun için ilk olarak lacunary toplanabilme daha sonra ise lacunary istatistiksel yakınsaklık kavramlarını tanımlayacağız. Daha sonra bu tanımlar arasındaki ilişkiler teoremle verilip ispatlanacaktır.

Anahtar Kelimeler: Fuzzy n-normlu uzay, lacunary dizi, istatistiksel yakınsaklık, çift dizi.

Abstract

The concept of convergence of real sequences has been extended to statistical convergence and later this concept was extended to the double sequences by Mursaleen and Edely. The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka and proved some basic theorems for sequences of fuzzy numbers. Türkmen and Çınar studied lacunary statistical convergence in fuzzy normed linear spaces. Mohiuddine et al. studied statistical convergence of double sequences in fuzzy normed spaces. In this paper, the application of lacunary statistical convergence of double sequences, previously described in fuzzy normed spaces, has been redefined using fuzzy n-norm. In this study, first of all, the concept of lacunary summable has been introduced and then the definition of lacunary statistical convergence and the basic theorems related to this convergence have been introduced for double sequences in fuzzy n-normed spaces. Then the relations between lacunary summable and lacunary statistical convergence have examined and some theorems are given together with the proofs.

Keywords: Fuzzy n-normed space, lacunary sequence, statistical convergence, double sequence.

1. Introduction and Background

The concept of convergence of real sequences has been extended to statistical convergence independently by Fast (1951) and Schoenberg (1959). The concept of ordinary convergence of a sequence of fuzzy numbers was firstly introduced by Matloka (1986) and proved some basic theorems for sequences of fuzzy numbers. Nanda (1989) studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. Şençimen and Pehlivan (2008) introduced the notions of statistically convergent sequence and statistically Cauchy sequence in a fuzzy normed linear space. Reddy and Srinivas (2015) studied statistical convergence in fuzzy n-normed linear spaces and also many authors studied in fuzzy normed spaces and in fuzzy n-normed spaces Felbin (1992), Narayan and Vijayabalaji (2005), Türkmen and Çınar (2018), Türkmen and Efe (2013). Türkmen and Çınar (2017) presented analogues in fuzz normed linear spaces of the results given by Fridy and Orhan (1993) and Türkmen and Dündar (2018) studied lacunary statistical convergence of double sequences in fuzzy normed linear spaces.

Fuzzy sets are considered with respect to a nonempty base set X of elements of interest. The essential idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0,1]$, with $u(x) = 0$ corresponding to nonmembership, $0 < u(x) < 1$ to partial membership, and $u(x) = 1$ to full membership. According to Zadeh (1965), a fuzzy subset of X is a nonempty subset $\{(x, u(x)) : x \in X\}$ of $X \times [0,1]$ for some function $u: X \to [0,1]$. The function u itself is often used for the fuzzy set.

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:

1. u is normal, that is, there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$;

2. u is fuzzy convex, that is, for $x, y \in \mathbb{R}$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)]$;

3. u is upper semicontinuous;

4. $supp\ u = cl\{x \in \mathbb{R}: u(x) > 0\}$, or denoted by $[u]_0$, is compact.

Now, we recall the basic definitions and concepts (see (Bag and Samanta, 2008: Debnath, 2012: Fast, 1951: Felbin, 1992: Fridy, 1985: Fridy and Orhan, 1993: Mohiuddine et al., (2012): Mursaleen and Edely, 2003: Narayan and Vijayabalaji, 2005: Nuray, 1998: Nuray and Savaş, 1995: Pringsheim, 1900: Šalát, 1980: Şençimen and Pehlivan, 2008: Steinhaus, 1951: Türkmen, 2018a: Türkmen, 2018b)).

Let $L(\mathbb{R})$ be set of all fuzzy numbers. If $u \in L(\mathbb{R})$ and $u(t) = 0$ for $t < 0$, then u is called a non-negative fuzzy number. We have written $L^*(\mathbb{R})$ by the set of all non-negative fuzzy numbers. We can say that $u \in L^*(\mathbb{R})$ if and only if $u_\alpha^- \geq 0$ for each $\alpha \in [0,1]$. Clearly we have $\tilde{0} \in L(\mathbb{R})$. For $u \in L(\mathbb{R})$, the α level set of u is defined by

$$[u]_\alpha = \begin{cases} \{x \in \mathbb{R}: u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ suppu, & \text{if } \alpha = 0. \end{cases}$$

Some arithmetic operations for $\alpha -$level sets are defined as follows: $u, v \in L(\mathbb{R})$ and $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^+]$, $\alpha \in (0,1]$. Then

$[u \oplus v]_\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+]$, $[u \ominus v]_\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ - v_\alpha^-]$

$[u \otimes v]_\alpha = [u_\alpha^-.v_\alpha^-, u_\alpha^+.v_\alpha^+]$, $\left[\tilde{1} \oslash u\right]_\alpha = \left[\frac{1}{u_\alpha^+}, \frac{1}{u_\alpha^-}\right] u_\alpha^- > 0$

For $u, v \in L(\mathbb{R})$, the supremum metric on $L(\mathbb{R})$ is defined as

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\}.$$

It is known that D is a metric on $L(\mathbb{R})$ and $(L(\mathbb{R}), D)$ is a complete metric space.

A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to the fuzzy number x_0, if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $D(x_k, x_0) < \varepsilon$ for $k > k_0$ and a sequence $x = (x_k)$ of fuzzy numbers convergens to levelwise to x_0 if and only if $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^-$ and $\lim_{k\to\infty} [x_k]_\alpha = [x_0]_\alpha^+$, where $[x_k]_\alpha = [(x_k)_\alpha^-, (x_k)_\alpha^+]$ and $[x_0]_\alpha = [(x_0)_\alpha^-, (x_0)_\alpha^+]$, for every $\alpha \in (0,1)$.

Let X be a vector space over $\mathbb{R}$, $\|.\|: X \to L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) $: [0,1] \times [0,1] \to [0,1]$ be symetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$.

The quadruple $(X, \|.\|, L, R)$ is called fuzzy normed linear space (briefly$(X, \|.\|)FNS$) and $\|.\|$ a fuzzy norm if the following axioms are satisfied

1. $\|x\| = \tilde{0}$ iff $x = 0$,
2. $\|rx\| = |r|.\|x\|$ for $x \in X, r \in \mathbb{R}$,
3. For all $x, y \in X$

(a) $\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$, whenever $s \leq \|x\|_1^-, t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$,

(b) $\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$, whenever $s \geq \|x\|_1^-, t \geq \|y\|_1^-$ and $s + t \geq \|x + y\|_1^-$.

Let $(X, \|.\|_C)$ be an ordinary normed linear space. Then, a fuzzy norm $\|.\|$ on X can be obtained by

$$\|x\|(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq a\|x\|_C \text{ or } \quad t \geq b\|x\|_C \\ \dfrac{t}{(1-a)\|x\|_C} - \dfrac{a}{1-a} & a\|x\|_C \leq t \leq \|x\|_C \\ \dfrac{-t}{(b-1)\|x\|_C} + \dfrac{b}{b-1} & \|x\|_C \leq t \leq b\|x\|_C \end{cases}$$

where $\|x\|_C$ is the ordinary norm of x $(\neq 0)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $\|x\| = \tilde{0}$. Hence, $(X, \|.\|)$ is a fuzzy normed linear space.

Let us consider the topological structure of an FNS $(X, \|.\|)$. For any $\varepsilon > 0, \alpha \in [0,1]$ and $x \in X$, the $(\varepsilon, \alpha) -$ neighborhood of x is the set $\mathcal{N}_x(\varepsilon, \alpha) = \{y \in X: \|\ x - y\ \|_\alpha^+ < \varepsilon\}$.

Let $(X, \|.\|)$ be an FNS. A sequence $(x_n)_{n=1}^\infty$ in X is convergent to $x \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FN}{\to} x$, provided that $(D) - \lim_{n\to\infty} \|x_n - x\| = \tilde{0}$; i.e., for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that $D(\|x_n - x\|, \tilde{0}) < \varepsilon$ for all $n \geq N(\varepsilon)$. This means that for every $\varepsilon > 0$ there is an $N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N(\varepsilon)$, $\sup_{\alpha\in[0,1]} \|x_n - x\|_\alpha^+ = \|x_n - x\|_0^+ < \varepsilon$.

Let $(X, \|.\|)$ be an FNS. A sequence (x_k) in X is statistically convergent to $L \in X$ with respect to the fuzzy norm on X and we denote by $x_n \overset{FS}{\rightarrow} x$, provided that for each $\varepsilon > 0$, we have $\delta(\{k \in \mathbb{N}: D(\|x_k - L\|, \tilde{0}) \geq \varepsilon\}) = 0$. This implies that for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \{k \in \mathbb{N}: \|x_k - L\|_0^+ \geq \varepsilon\}$$

has natural density zero; namely, for each $\varepsilon > 0$, $\|x_k - L\|_0^+ < \varepsilon$ for almost all k.

Let $n \in \mathbb{N}$ and let X be a real linear space of dimension $d \geq n$. A real valued function $\|\cdot,\cdot,\ldots,\cdot\|$ on $\underbrace{X \times X \times \cdots \times X}_{n}$ satisfying the following conditions:

nN_1: $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

nN_2: $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

nN_3: $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in \mathbb{R}$,

nN_4: $\|y + z, x_2, \ldots, x_n\| \leq \|y, x_2, \ldots, x_n\| + \|z, x_2, \ldots, x_n\|$ for all $y, z, x_2, \ldots, x_n \in X$,

then the function $\|\cdot,\cdot,\ldots,\cdot\|$ is called an $n-$norm on X and pair $(X, \|\cdot,\cdot,\ldots,\cdot\|)$ is called $n-$normed space.

Let X be a real linear space of dimension d, where $2 \leq d < \infty$. Let $\|\cdot,\cdot,\ldots,\cdot\|: X^n \to L^*(\mathbb{R})$ and the mappings $L; R$ (respectively, left norm and right norm) $: [0,1] \times [0,1] \to [0,1]$ be symetric, nondecreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$ then the quadruple $(X, \|\cdot,\cdot,\ldots,\cdot\|, L, R)$ is called fuzzy $n-$normed linear space (briefly$(X, \|\cdot,\cdot,\ldots,\cdot\|) FnNS$) and $\|\cdot,\cdot,\ldots,\cdot\|$ a fuzzy $n-$norm if the following axioms are satisfied for every $y, x_1, x_2, \ldots, x_n \in X$ and $s, t \in \mathbb{R}$

fnN_1: $\|x_1, x_2, \ldots, x_n\| = \tilde{0}$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent vectors,

fnN_2: $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

fnN_3: $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for all $\alpha \in \mathbb{R}$,

fnN_4: $\|x_1 + y, x_2, \ldots, x_n\|(s + t) \geq L(\|x_1, x_2, \ldots, x_n\|(s), \|y, x_2, \ldots, x_n\|(t))$ whenever $s \leq \|x_1, x_2, \ldots, x_n\|_1^-$, $t \leq \|y, x_2, \ldots, x_n\|_1^-$ and $s + t \leq \|x_1 + y, x_2, \ldots, x_n\|_1^-$,

fnN_5: $\|x_1 + y, x_2, \ldots, x_n\|(s + t) \leq R(\|x_1, x_2, \ldots, x_n\|(s), \|y, x_2, \ldots, x_n\|(t))$ whenever $s \geq \|x_1, x_2, \ldots, x_n\|_1^-$, $t \geq \|y, x_2, \ldots, x_n\|_1^-$ and $s + t \geq \|x_1 + y, x_2, \ldots, x_n\|_1^-$,

where $[\|x_1, x_2, \ldots, x_n\|]_\alpha = [\|x_1, x_2, \ldots, x_n\|_\alpha^-, \|x_1, x_2, \ldots, x_n\|_\alpha^+]$ for $x_1, x_2, \ldots, x_n \in X, 0 \leq \alpha \leq 1$ and $\inf_{\alpha \in [0,1]} \|x_1, x_2, \ldots, x_n\|_\alpha^- > 0$. Hence the norm $\|\cdot,\cdot,\ldots,\cdot\|$ is called fuzzy $n-$norm on X and pair $(X, \|\cdot,\cdot,\ldots,\cdot\|)$ is called fuzzy $n-$normed space.

Let $(X, \|\cdot,\cdot,\ldots,\cdot\|)$ be fuzzy $n-$normed space. A sequence $\{x_k\}$ in X is said to be convergent to an element $x \in X$ with respect to the fuzzy $n-$norm on X if for every $\varepsilon > 0$ and for every $z_2, z_3, \ldots, z_n \neq 0$, $z_2, z_3, \ldots, z_n \in X$, $\exists$ a number $N = N(\varepsilon, z_2, z_3, \ldots, z_n)$ such that $D(\|x_k - x, z_2, z_3, \ldots, z_n\|, \tilde{0}) < \varepsilon$,$\forall k \geq N$ or equivalently $(D) - \lim_{k \to \infty} \|x_k - x, z_2, z_3, \ldots, z_n\| = \tilde{0}$.

Let $(X, \|\cdot,\cdot,\ldots,\cdot\|)$ be fuzzy $n-$normed space. A sequence $\{x_k\}$ in X is said to be statistically convergent to an element $x \in X$ with respect to the fuzzy $n-$norm on X if for every $\varepsilon > 0$ and for every $z_2, z_3, \ldots, z_n \neq 0$, $z_2, z_3, \ldots, z_n \in X$, we have $\delta(\{k \in \mathbb{N}: D(\|x_k - x, z_2, z_3, \ldots, z_n\|, \tilde{0}) \geq \varepsilon\}) = 0$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Let $(X, \|.\|)$ be an FNS and $\theta = \{k_r\}$ be lacunary sequence. A sequence $x = (x_k)_{k \in \mathbb{N}}$ in X is said to be lacunary summable with respect to fuzzy norm on X if there is an $L \in X$ such that

$$\lim_{r \to \infty} \frac{1}{h_r} \left(\sum_{k \in I_r} D(\| x_k - L \|, \tilde{0}) \right) = 0.$$

In this case, we can write $x_k \to L((N_\theta)_{FN})$ or $x_k \overset{(N_\theta)_{FN}}{\rightarrow} L$ and

$$(N_\theta)_{FN} = \left\{ x = (x_k): \lim_{r \to \infty} \frac{1}{h_r} \left(\sum_{k \in I_r} D(\|x_k - L\|, \tilde{0}) \right) = 0, \text{for some } L \right\}.$$

A sequence $x = (x_k)$ in X is said to be lacunary statistically convergent or FS_θ −convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : D(\|x_k - L\|, \tilde{0}) \geq \varepsilon\}\right| = 0$$

where $|A|$ denotes the number of elements of the set $A \subseteq \mathbb{N}$. In this case, we write $x_k \overset{FS_\theta}{\rightarrow} L$ or $x_k \rightarrow L(FS_\theta)$ or $FS_\theta - \lim_{k\to\infty} x_k = L$. This implies that for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in I_r : \|x_k - L\|_0^+ \geq \varepsilon\}$ has natural density zero, namely, for each $\varepsilon > 0$, $\|x_k - L\|_0^+ < \varepsilon$, for almost all k.

A double sequence $x = (x_{jk})$ is said to be Pringsheim's convergent (or P-convergent) if for given $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - l| < \varepsilon$, whenever $j, k > N$. We shall write this as $\lim_{j,k\to\infty} x_{jk} = l$, where j and k tending to infinity independent of each other.

A double sequence $x = (x_{jk})$ is said to be bounded if there exists a positive real number M such that for all $k, j \in \mathbb{N}$, $|x_{jk}| < M$, that is, $\| x \|_\infty = \sup_{k,j} |x_{jk}| < \infty$. We let the set of all bounded double sequences by l_∞.

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j, k) \in K$ such that $j \leq m$, $k \leq n$. If the sequence $\{\frac{K_{mn}}{m.n}\}$ has a limit in Pringsheim's sense then we say that K has double natural density and is denoted by

$$\delta_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}$$

A double sequence $x = (x_{jk})$ is said to be statistically convergent to the number l if for each $\varepsilon > 0$, the set $\{(j, k): j \leq m \text{and} k \leq n,\ |x_{jk} - l| \geq \varepsilon\}$ has double natural density zero. In this case, we write $st_2 - \lim_{j,k} x_{jk} = l$.

Let $(X, \|.\|)$ be an FNS. Then a double sequence (x_{jk}) is said to be convergent to $x \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$ there exist a number $N = N(\varepsilon)$ such that

$$\|x_{jk} - x\|_0^+ < \varepsilon, \text{for all } j, k \geq N.$$

In this case, we write $x_{jk} \overset{FN_2}{\rightarrow} x$.

Let $(X, \|.\|)$ be an FNS. A double sequence (x_{jk}) is said to be statistically convergent to $x \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$,

$$\delta_2\left(\left\{(j, k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - x\|_0^+ \geq \varepsilon\right\}\right) = 0.$$

Namely, for each $\varepsilon > 0$, $\|x_{jk} - x\|_0^+ < \varepsilon$ for almost all j, k. In this case, we write $FS_2 - \lim \|x_{jk} - x\|_0^+ = \tilde{0}$ or $x_{jk} \overset{FS_2}{\rightarrow} x$.

Let $(X, \|.\|)$ be an FNS. Then a double sequence (x_{jk}) is said to be statistically Cauchy with respect to the fuzzy norm on X if for every $\varepsilon > 0$, there exist $N(\varepsilon)$ and $M(\varepsilon)$ such that for all $j, p \geq N$ and $k, q \geq M$,

$$\delta_2\left(\left\{(j, k) \in \mathbb{N} \times \mathbb{N} : j \leq n \text{ and } k \leq m, \|x_{jk} - x_{pq}\|_0^+ \geq \varepsilon\right\}\right) = 0.$$

The double sequence $[y]$ is a double subsequence of the double sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j, k_j}$, then y is formed by

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & - \\ z_9 & z_8 & z_7 & - \\ - & - & - & - \end{array}$$

The double sequence $\theta_2 = \{(k_r, j_u)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that $k_0 = 0$, $h_r = k_r - k_{r-1} \rightarrow \infty$ and $j_0 = 0, \bar{h}_u = j_u - j_{u-1} \rightarrow \infty$, as $r, u \rightarrow \infty$.

We use following notations in the sequel: $k_{ru} = k_r j_u$, $h_{ru} = h_r \bar{h}_u$, $I_{ru} = \{(k, j): k_{r-1} < k \leq k_r$ and $j_{u-1} < j \leq j_u\}$, $q_r = \frac{k_r}{k_{r-1}}$ and $\bar{q}_u = \frac{j_u}{j_{u-1}}$.

Let θ_2 be a double lacunary sequence. The double sequence $x = (x_{kj})$ is S''_{θ_2}-convergent to L provided that for every $\varepsilon > 0$,

$$P - \lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\{(k,j) \in I_{ru}: \left|x_{kj} - L\right| \geq \varepsilon\}\right| = 0.$$

In this case, write $S''_{\theta_2} - \lim x = L$ or $x_{k,l} \to L(S''_{\theta_2})$. Let $(X, \|.\|)$ be an FNS.

A double sequence (x_{jk}) is said to be statistically convergent to $x \in X$ with respect to the fuzzy norm on X if for every $\varepsilon > 0$,

$$\delta_2\left(\left\{(j,k) \in \mathbb{N}\times\mathbb{N}: \left\|x_{jk} - x\right\|_0^+ \geq \varepsilon\right\}\right) = 0.$$

In this case, we write $FS_2 - \lim\left\|x_{jk} - x\right\| = \tilde{0}$ or $x_{jk} \overset{FS_2}{\to} x$.

A double sequence $x = (x_{mn})$ in X is said to be lacunary statistically convergent or FS_{θ_2}-convergent to $L \in X$ with respect to fuzzy norm on X if for each $\varepsilon > 0$

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} |\{(m,n) \in I_{ru}: \|x_{mn} - L\|_0^+) \geq \varepsilon\}| = 0.$$

In this case, we write $x_{mn} \to L(FS_{\theta_2})$ or $FS_{\theta_2} - \lim_{m,n\to\infty} x_{mn} = L$ or $x_{mn} \overset{FS_{\theta_2}}{\to} L$

2. Main Result

In this section, we introduce the concepts of lacunary summable, lacunary statistically convergence and lacunary statistically Cauchy sequence in fuzzy n-normed spaces. Also, we investigate some properties and relationships between these concepts.

Throughout the paper, we consider $(X, \|\cdot,\ldots,\cdot\|)$ be a fuzzy n-normed space (brifly FnNS) and $\theta_2 = \{(k_r, j_u)\}$ be a double lacunary sequence. And also we will get $z_2, z_3, \ldots, z_n \in X$.

Definition 2.1 *A double sequence $x = (x_{mn})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ in X is said to be lacunary summable with respect to fuzzy n-norm on X if there is an $L \in X$ such that*

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left(\sum_{(m,n)\in I_{ru}} D(\| x_{mn} - L, z_2, z_3, \ldots, z_n \|, \tilde{0}) \right) = 0.$$

In this case, we write $x_{mn} \to L\left(\left(N_{\theta_2}\right)_{FnN}\right)$ or $x_{mn} \overset{(N_{\theta_2})_{FN}}{\to} L$ and

$\left(N_{\theta_2}\right)_{FN} = \{(x_{mn}): \lim_{r,u\to\infty} \frac{1}{h_{ru}} (\sum_{(m,n)\in I_{ru}} D(\|x_{mn} - L, z_2, z_3, \ldots, z_n\|, \tilde{0})) = 0$, for some $L\}$

Definition 2.2 *A double sequence $x = (x_{mn})$ in X is said to be lacunary statistically convergent or FnS_{θ_2}-convergent to $L \in X$ with respect to fuzzy n-norm on X if for each $\varepsilon > 0$*

$$\lim_{r,u\to\infty} \frac{1}{h_{ru}} \left|\{(m,n) \in I_{ru}: D(\|x_{mn} - L, z_2, z_3, \ldots, z_n\|, \tilde{0}) \geq \varepsilon\}\right| = 0. \quad (2.1)$$

In this case, we write $FnS_{\theta_2} - \lim_{m,n\to\infty} x_{mn} = L$ or $x_{mn} \to L(FnS_{\theta_2})$ or $x_{mn} \overset{FnS_{\theta_2}}{\to} L$. This implies that, for each $\varepsilon > 0$, the set

$$K(\varepsilon) = \{(m,n) \in I_{ru}: \|x_{mn} - L, z_2, z_3, \ldots, z_n\|_0^+ \geq \varepsilon\}$$

has natural density zero, namely, for each $\varepsilon > 0$, $\|x_{mn} - L, z_2, z_3, \ldots, z_n\|_0^+ < \varepsilon$ for almost all (m,n). In terms of neighborhoods, we have $x_{mn} \overset{FnS_{\theta_2}}{\to} L$ if for every $\varepsilon > 0$,

$$\delta_2(\{(m,n) \in I_{ru}: x_{mn} \notin \mathcal{N}_L(\varepsilon, 0)\}) = 0,$$

that is, for each $\varepsilon > 0$, $(x_{mn}) \in \mathcal{N}_L(\varepsilon, 0)$ for almost all (m,n).

A useful interpretation of the above definition is the following;

$$x_{mn} \overset{FnS_{\theta_2}}{\to} L \Leftrightarrow FnS_{\theta_2} - \lim\|x_{mn} - L, z_2, z_3, \ldots, z_n\|_0^+ = 0.$$

Note that $FnS_{\theta_2}-\lim\|x_{mn}-L,z_2,z_3,\dots,z_n\|_0^+=0$ implies that $FnS_{\theta_2}-\lim\|x_{mn}-L,z_2,z_3,\dots,z_n\|_\alpha^-=FnS_{\theta_2}-\lim\|x_{mn}-L,z_2,z_3,\dots,z_n\|_\alpha^+=0$, for each $\alpha\in[0,1]$, since $0\leq\|x_{mn}-L,z_2,z_3,\dots,z_n\|_\alpha^-\leq\|x_{mn}-L,z_2,z_3,\dots,z_n\|_\alpha^+\leq\|x_{mn}-L,z_2,z_3,\dots,z_n\|_0^+$ holds for every $m,n\in\mathbb{N}$ and for each $\alpha\in[0,1]$.

The set of all lacunary statistically convergent double sequence with respect to fuzzy n-norm on X will be denoted by $FnS_{\theta_2}=\{x:\text{for some } L,\ FnS_{\theta_2}-\lim x=L\}$.

Theorem 2.3 *We have the following statements for every $x=(x_{mn})$ double sequences:*

(i) $x_{mn}\to L\left(\left(N_{\theta_2}\right)_{FnN}\right)\Rightarrow x_{mn}\to L\left(FnS_{\theta_2}\right)$.

(ii) $\left(N_{\theta_2}\right)_{FnN}$ is a proper subset of FnS_{θ_2}.

Proof. (i) If $x_{mn}\to L\left(\left(N_{\theta_2}\right)_{FnN}\right)$, then for given $\varepsilon>0$

$$\sum_{(m,n)\in I_{ru}} D\left(\|x_{mn}-L,z_2,z_3,\dots,z_n\|,\tilde{0}\right)$$

$$\geq \sum_{\substack{(m,n)\in I_{ru}\\ D(\|x_{mn}-L,z_2,z_3,\dots,z_n\|,\tilde{0})>\varepsilon}} D\left(\|x_{mn}-L,z_2,z_3,\dots,z_n\|,\tilde{0}\right)$$

$$\geq \varepsilon.\left|\left\{(m,n)\in I_{ru}: D\left(\|x_{mn}-L,z_2,z_3,\dots,z_n\|,\tilde{0}\right)\geq\varepsilon\right\}\right|.$$

Therefore, we have

$$\lim_{r,u\to\infty}\frac{1}{h_{ru}}\left|\left\{(m,n)\in I_{ru}: D\left(\|x_{mn}-L,z_2,z_3,\dots,z_n\|,\tilde{0}\right)\geq\varepsilon\right\}\right|=0.$$

This implies that $x_{mn}\to L\left(FnS_{\theta_2}\right)$.

(ii) In order to indicate that the inclusion $\left(N_{\theta_2}\right)_{FnN}\subseteq FnS_{\theta_2}$ in (i) is proper, let a double lacunary sequence θ_2 be given and define a sequence $x=(x_{mn})$ as follows:

$$x_{mn}=\begin{cases}(m,n),\text{if } k_{r-1}<m<k_{r-1}+\left[\sqrt{h_r}\right], j_{u-1}<n<j_{u-1}+\left[\sqrt{\bar{h}_u}\right],\\ (0,0)\quad \text{, otherwise.}\end{cases}$$

for r,u=1,2,... . Note that, $x=(x_{mn})$ is not bounded. We have, for every $\varepsilon>0$ and for each $x\in X$,

$$\frac{1}{h_r\bar{h}_u}\left|\left\{(m,n)\in I_{ru}: D\left(\|x_{mn}-0,z_2,z_3,\dots,z_n\|,\tilde{0}\right)\geq\varepsilon\right\}\right|$$

$$=\frac{\left[\sqrt{h_r}\right]\left[\sqrt{\bar{h}_u}\right]}{h_r h_u}\to 0,\quad \text{as}\quad r,u\to\infty.$$

That is, $x_{mn}\to 0\left(FnS_{\theta_2}\right)$. On the other hand

$$\frac{1}{h_r\bar{h}_u}\sum_{(m,n)\in I_{ru}} D\left(\|x_{mn}-0,z_2,z_3,\dots,z_n\|,\tilde{0}\right)$$

$$=\frac{1}{h_r\bar{h}_u}\sum_{(m,n)\in I_{ru}}\|x_{mn},z_2,z_3,\dots,z_n\|_0^+$$

$$=\frac{1}{h_r\bar{h}_u}\cdot\frac{\left[\sqrt{h_r}\right].\left(\left[\sqrt{h_r}\right]+1\right)\left[\sqrt{\bar{h}_u}\right].\left(\left[\sqrt{\bar{h}_u}\right]+1\right)}{4}$$

$$\to\frac{1}{4}\neq 0.$$

Hence, $x_{mn}\nrightarrow 0\left(\left(N_{\theta_2}\right)_{FnN}\right)$.

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