On Quasi-Lacunary Invariant Convergence of Sequences of Sets

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Abstract: In this study, we give definitions of Wijsman quasi-lacunary invariant convergence, Wijsman strongly quasi-lacunary invariant convergence and Wijsman quasi-lacunary invariant statistically convergence for sequences of sets. We also examine the existence of some relations among these definitions and some convergence types for sequences of sets given in [7, 14], too.

1. INTRODUCTION AND BACKGROUNDS

The concept of statistical convergence was firstly introduced by Fast [4] and this concept has been studied by Šalát [18], Fridy [5] and many others, too.
A sequence $x = (x_k)$ is statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \Big| \big\{ k \leq n : |x_k - L| \geq \varepsilon \big\} \Big| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set.
By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this study the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.
Then, Fridy and Orhan [6] defined lacunary statistical convergence of a sequence using the lacunary sequence concept as follows:
Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is lacunary statistically convergent to L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} \Big| \big\{ k \in I_r : |x_k - L| \geq \varepsilon \big\} \Big| = 0.$$

Several authors have studied on the concepts of invariant mean and invariant convergent (see, [9–11, 17, 19, 22]).
Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

1. $\phi(x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,
2. $\phi(e) = 1$, where $e = (1, 1, 1, \ldots)$ and
3. $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

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The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n+1$, the σ-mean is often called a Banach limit.

The space of lacunary strong σ-convergent sequences L_θ was defined by Savaş [20] as below:

$$L_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_{\sigma^k(m)} - L| = 0, \text{ uniformly in } m \right\}.$$

Pancaroğlu and Nuray [15] introduced the concept of lacunary invariant summability as follows:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is said to be lacunary invariant summable to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} x_{\sigma^k(m)} = L,$$

uniformly in m.

The concept of lacunary σ-statistically convergent sequence was defined by Savaş and Nuray in [21] as below:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $x = (x_k)$ is $S_{\sigma\theta}$-convergent to L if for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |x_{\sigma^k(m)} - L| \geq \varepsilon \right\} \right| = 0,$$

uniformly in m.

Let X be any non-empty set and $\mathbb{N}$ be the set of natural numbers. The function

$$f : \mathbb{N} \to P(X)$$

is defined by $f(k) = A_k \in P(X)$ for each $k \in \mathbb{N}$, where $P(X)$ is power set of X. The sequence $\{A_k\} = (A_1, A_2, \ldots)$, which is the range's elements of f, is said to be sequences of sets.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, the distance from x to A is defined by

$$d(x, A) = \inf_{a\in A} \rho(x, a).$$

Throughout the paper we take (X, ρ) as a metric space and A, A_k as any non-empty closed subsets of X.

There are different convergence notions for sequence of sets. One of them handled in this paper is the concept of Wijsman convergence (see, [1–3, 12, 16, 25, 26]).

A sequence $\{A_k\}$ is said to be Wijsman convergent to A if for each $x \in X$,

$$\lim_{k\to\infty} d(x, A_k) = d(x, A)$$

and denoted by $A_k \stackrel{W}{\to} A$.

A sequence $\{A_k\}$ is said to be bounded if for each $x \in X$, $\sup_k \{d(x, A_k)\} < \infty$.

The set of all bounded sequences of sets is denoted by L_∞.

The concepts of Wijsman lacunary summability, Wijsman strongly lacunary summability and Wijsman lacunary statistical convergence were introduced by Ulusu and Nuray [23, 24].

Using the invariant mean concept, the concepts of Wijsman lacunary invariant convergence, Wijsman strongly lacunary invariant convergence and Wijsman lacunary invariant statistical convergence were also defined by Pancaroğlu and Nuray [16] as follows:

Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} d(x, A_{\sigma^k(m)}) = d(x, A)$$

uniformly in m.

A sequence $\{A_k\}$ is said to be Wijsman strongly lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \left| d(x, A_{\sigma^k(m)}) - d(x, A) \right| = 0$$

uniformly in m.

A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant statistically convergent to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |d(x, A_{\sigma^k(m)}) - d(x, A)| \geq \varepsilon \right\} \right| = 0$$

uniformly in m.

The idea of quasi-almost convergence in a normed space was introduced by Hajduković [8]. Then, Nuray [13] studied concepts of quasi-invariant convergence and quasi-invariant statistical convergence in a normed space. Recently, Gülle and Ulusu [7] introduced the concept of Wijsman strongly quasi-invariant convergence for sequences of sets as below:

A sequence $\{A_k\}$ is said to be Wijsman strongly quasi-invariant convergent to A if for each $x \in X$,

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} \left| d_x(A_{\sigma^k(np)}) - d_x(A) \right| = 0$$

uniformly in n where $d_x(A_{\sigma^k(np)}) = d(x, A_{\sigma^k(np)})$ and $d_x(A) = d(x,A)$. It is denoted by $A_k \stackrel{[WQV_\sigma]}{\longrightarrow} A$.

2. MAIN RESULTS

In this study, we give definitions of Wijsman quasi-lacunary invariant convergence, Wijsman strongly quasi-lacunary invariant convergence and Wijsman quasi-lacunary invariant statistically convergence for sequences of sets. We also examine the existence of some relations among these definitions and some convergence types for sequences of sets given in [7, 14], too.

Definition 2.1 Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is said to be Wijsman quasi-lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r\to\infty} \left| \frac{1}{h_r} \sum_{k\in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| = 0$$

uniformly in n. In this case, we write $A_k \stackrel{WQV_{\sigma\theta}}{\longrightarrow} A$.

Theorem 2.2 *If a sequence $\{A_k\}$ is Wijsman lacunary invariant convergent to A, then $\{A_k\}$ is Wijsman quasi-lacunary invariant convergent to A.*

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman lacunary invariant convergent to A. Then, for each $x \in X$ and every $\varepsilon > 0$ there exists an integer $r_0 > 0$ such that for all $r > r_0$

$$\left| \frac{1}{h_r} \sum_{k\in I_r} d_x(A_{\sigma^k(m)}) - d_x(A) \right| < \varepsilon,$$

for all m. If m is taken as $m = nr$, then we have

$$\left| \frac{1}{h_r} \sum_{k\in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| < \varepsilon,$$

for all n. Since $\varepsilon > 0$ is an arbitrary, the limit is taken for $r \to \infty$ we can write

$$\left| \frac{1}{h_r} \sum_{k\in I_r} d_x(A_{\sigma^k(nr)}) - d_x(A) \right| \longrightarrow 0$$

for all n. That is, the sequence $\{A_k\}$ is Wijsman quasi-lacunary invariant convergent to A. □

Definition 2.3 Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is *Wijsman quasi-lacunary invariant statistically convergent to* A if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon \right\} \right| = 0$$

uniformly in n. In this case, we write $A_k \stackrel{WQS_{\sigma\theta}}{\longrightarrow} A$.

Theorem 2.4 *If a sequence $\{A_k\}$ is Wijsman lacunary invariant statistically convergent to A, then $\{A_k\}$ is Wijsman quasi-lacunary invariant statistically convergent to A.*

Proof. Suppose that the sequence $\{A_k\}$ is Wijsman lacunary invariant statistically convergent to A. In this case, when $\delta > 0$ is given, for each $x \in X$ and for every $\varepsilon > 0$ there exists an integer $r_0 > 0$ such that for all $r > r_0$

$$\frac{1}{h_r} \left| \left\{ k \in I_r : |d_x(A_{\sigma^k(m)}) - d_x(A)| \geq \varepsilon \right\} \right| < \delta,$$

for all m.
If m is taken as $m = nr$, then we have

$$\frac{1}{h_r}\left|\left\{k\in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| < \delta,$$

for all n. Since $\delta > 0$ is an arbitrary, we have

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\left\{k\in I_r : |d_x(A_{\sigma^k(nr)}) - d_x(A)| \geq \varepsilon\right\}\right| = 0,$$

for all n which means that $\{A_k\}$ is Wijsman quasi-lacunary invariant statistically convergent to A. □

Definition 2.5 Let $\theta = \{k_r\}$ be a lacunary sequence. A sequence $\{A_k\}$ is Wijsman strongly quasi-lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right| = 0$$

uniformly in n. In this case, we write $A_k \stackrel{[WQV_{\sigma\theta}]}{\longrightarrow} A$.

Theorem 2.6 *For any lacunary sequence* $\theta = \{k_r\}$,

$$A_k \stackrel{[WQV_{\sigma\theta}]}{\longrightarrow} A \Leftrightarrow A_k \stackrel{[WQV_{\sigma}]}{\longrightarrow} A.$$

Proof. Let $A_k \stackrel{[WQV_{\sigma\theta}]}{\longrightarrow} A$ and $\varepsilon > 0$ is given. Then, there exists an integer r_0 such that for each $x \in X$

$$\frac{1}{h_r}\sum_{k=0}^{h_r-1}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right| < \varepsilon$$

for $r \geq r_0$ and $nr = k_{r-1} + 1 + w$, $w \geq 0$. Let $p \geq h_r$. Thus, p can be written as $p = \alpha.h_r + \theta$ where $0 \leq \theta \leq h_r$ and α is an integer. Since $p \geq h_r$, $\alpha \geq 1$. Then,

$$\begin{aligned}
\frac{1}{p}\sum_{k=0}^{p-1}\left|d_x(A_{\sigma^k(np)}) - d_x(A)\right| &\leq \frac{1}{p}\sum_{k=0}^{(\alpha+1)h_r-1}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right| \\
&= \frac{1}{p}\sum_{j=0}^{\alpha}\sum_{k=jh_r}^{(j+1)h_r-1}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right| \\
&\leq \frac{1}{p}\varepsilon h_r(\alpha+1) \\
&\leq \frac{2\alpha h_r\varepsilon}{p} \quad (\alpha \geq 1).
\end{aligned}$$

For $\frac{h_r}{p} \leq 1$ and since $\frac{\alpha h_r}{p} \leq 1$

$$\frac{1}{p}\sum_{k=0}^{p-1}\left|d_x(A_{\sigma^k(np)}) - d_x(A)\right| \leq 2\varepsilon,$$

that is, $A_k \stackrel{[WQV_{\sigma}]}{\longrightarrow} A$.

Let $A_k \stackrel{[WQV_{\sigma}]}{\longrightarrow} A$ and $\varepsilon > 0$ is given. Then, there exists $P > 0$ such that for each $x \in X$

$$\frac{1}{p}\sum_{k=0}^{p-1}\left|d_x(A_{\sigma^k(np)}) - d_x(A)\right| < \varepsilon$$

for all $p > P$. Since $\theta = \{k_r\}$ is a lacunary sequence, a number $R > 0$ can be chosen such that $h_r > P$ where $r \geq R$. Thereby

$$\frac{1}{h_r}\sum_{k\in I_r}\left|d_x(A_{\sigma^k(nr)}) - d_x(A)\right| < \varepsilon,$$

that is, $A_k \stackrel{[WQV_{\sigma\theta}]}{\longrightarrow} A$. The proof of theorem is completed. □

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