Afyon Kocatepe Üniversitesi Fen ve Mühendislik Bilimleri Dergisi

Afyon Kocatepe University Journal of Science and Engineering

AKÜ FEMÜBİD **20** (2020) 061301 (968-974) AKU J. Sci. Eng. **20** (2020) 061301 (968-974)

DOI: 10.35414/akufemubid.819410

Araştırma Makalesi / Research Article

Basic Properties of Statistical Epi-Convergence

Şükrü TORTOP[1*]

[1] *Afyon Kocatepe Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Afyonkarahisar.*

[*] *Sorumlu yazar e-posta: stortop@aku.edu.tr. ORCID ID: https://orcid.org/0000-0001-5342-7612*

Geliş Tarihi: 01.11.2020 Kabul Tarihi: 21.12.2020

Keywords
Epi-convergence; Statistical Convergence; Epigraph; Sequence of Functions

Abstract

In this paper, we give some basic properties in order to use statistical epi-convergence more efficiently in future studies. Such situations are studied: Uniform statistical convergence of sequence of functions, statistical epi-limit of compound of sequence of functions, statistical epi-limit of the sum of sequence of functions, the property of epi-limit function if the sequence of functions are lower semi-continuous and the convexity of epi-limit function if each function in the sequence is convex.

İstatistiksel Epi-Yakınsaklık ile İlgili Temel Özellikler

Anahtar kelimeler
Epi-Yakınsaklık; İstatistiksel Yakınsaklık; Epigraf; Fonksiyon Dizileri

Öz

Bu çalışmada, istatistiksel epi-yakınsaklığın sonraki çalışmalarda daha verimli kullanılabilmesi için bazı temel özelliklere yer verildi. Bir fonksiyon dizisinin düzgün istatistiksel yakınsaklık durumu, fonksiyon dizilerinin bileşkesinin istatistiksel epi-limiti, fonksiyon dizilerinin toplamının istatistiksel epi-limiti, fonksiyon dizisinin alttan yarı sürekli olması halinde epi-limit fonksiyonunun özelliği ve fonksiyon dizisindeki her bir fonksiyonun konveks olması halinde epi-limit fonksiyonunun konveksliği gibi durumlar çalışıldı.

1. Introduction

Wijsman (1964, 1966) studied epi-convergence in 1964 for the first time. Epi-convergence was called infimal convergence at that time. After Wijsman, Mosco (1969) used epi-convergence on variational inequalities, Joly (1973) on topological structures, Salinetti and Wets (1977) on equisemicontinuous convex functions, Attouch (1977) on convex functions, McLinden and Bergstrom (1981) on conservation of epi-convergence on convex functions. Moreover, epi-convergence was called Γ-convergence by Maso (1993). Wets (1980) called it epi-convergence in 1980, firstly. Epi-convergence offers solutions for stochastic optimization problems, variational problems and partial differential equations.

Zygmund (1979) studied statistical convergence in 1935 for the first time. Then it is investigated by other mathematicians including Fast (1951), Steinhaus (1951) and Schoenberg (1959). The definitions of pointwise and uniform statistical convergence of real-valued functions were given by Gökhan and Güngör (2002, 2005) and by Duman and Orhan (2004) independently. Statistical limit inferior and superior were studied by Fridy and Orhan (1997). Statistical limit points and cluster points were defined by Fridy (1993). Furthermore statistical lower and upper limits of closed sets were defined and characterized by Talo et al. (2016).

2. Preliminaries

In this part, fundamental definitions and theorems will be given. First of all, let (X, d) be a metric space and f, (f_n) are functions defined on X with $n \in \mathbb{N}$. If it is not mentioned explicitly the symbol d stands for the metric on X.

Let $K \subseteq \mathbb{N}$ and if the limit

$$\delta(K) = lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in K\}|$$

exists then it is called asymptotic density of K. $|\{k \leq n: k \in K\}|$ tells the number of elements of K less than or equal to n (Anastassiou and Duman 2011).

If $\delta(K_1) = \delta(K_2) = 1$ then,

$$\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 1.$$

If $\delta(K_1) = \delta(K_2) = 0$, then,

$$\delta(K_1 \cap K_2) = \delta(K_1 \cup K_2) = 0.$$

Let (x_n) be a sequence of real numbers. If $\forall \varepsilon > 0$, $\exists x_0$ such that

$$\lim_k \frac{1}{k} |\{n \leq k: |x_n - x_0| \geq \varepsilon\}| = 0,$$

then (x_n) is statistically convergent to x_0.

Let n be a positive integer and $x = (x_n)$ be a sequence of real numbers. Define the sets B_x and A_x as

$$B_x \coloneqq \{b \in \mathbb{R}: \delta(\{n: x_n > b\}) \neq 0\},$$

$$A_x := \{a \in \mathbb{R}: \delta(\{n: x_n < a\}) \neq 0\}.$$

Then statistical limit inferior and superior of $x = (x_n)$ is given by

$$st - liminf\ x := \begin{cases} inf A_x & if \quad A_x \neq \emptyset, \\ +\infty & if \quad A_x = \emptyset \end{cases}$$

$$st - limsup\ x := \begin{cases} sup B_x & if \quad B_x \neq \emptyset, \\ -\infty & if \quad B_x = \emptyset. \end{cases}$$

For every $\varepsilon > 0$, a sequence of functions (f_n) is uniformly statistically convergent to f on a set S if,

$$\lim_k \frac{1}{k} |\{n \leq k: |f_n(x) - f(x)| \geq \varepsilon \ \ for\ all\ \ x \in S\}| = 0.$$

For a sequence of functions $f_n: X \to \mathbb{R}$, if it is statistically alpha convergent to a function f, then it is uniformly statistically convergent to f (Caserta and Kočinac 2012).

Let $\sigma \in X$ and (x_n) is a sequence. If there exists a set $K = \{n_1 < n_2 < n_3 <\ldots\}$ with $\delta(K) \neq 0$ satisfying $x_{n_k} \to \sigma$ while $k \to \infty$, then σ is a statistical limit point of (x_n). Let Λ_x denote the set of all statistical limit points of (x_n).

Let $\mu \in X$ and (x_n) is a sequence of real numbers. If for any $\varepsilon > 0$, μ is a statistical cluster point of (x_n), then the following statement holds

$$\delta(\{n \in \mathbb{N}: d(x_n, \mu) < \varepsilon\}) \neq 0.$$

Γ_x will denote the set of all statistical cluster points of (x_n).

Let $\gamma \in X$ and (x_n) is a sequence of real numbers. If there exists a set $K = \{n_1 < n_2 < n_3 <\ldots\}$ satisfying $x_{n_k} \to \gamma$ while $k \to \infty$, then γ is a limit point of (x_n). The set of all limit points of (x_n) will be denoted by L_x.

Obviously we have $\Lambda_x \subseteq \Gamma_x \subseteq L_x$.

Following definitions are statistical inner and outer limits on the concept of set convergence which is fundamental to define statistical epi-limit using sets. In this paper, we deal with Painleve$'$-Kuratowski (1958) convergence and actually its statistical version will be studied here which is defined by Talo et al. (2016). Now we start with the following collections of subsets of $\mathbb{N}$.

$$\mathcal{S}^{\#} := \{N \subset \mathbb{N}: \delta(N) \neq 0\},$$

$$\mathcal{S} := \{N \subset \mathbb{N}: \delta(N) = 1\}.$$

Let (X, d) be a metric space. Statistical outer and inner limit of (A_n) are defined in the following equalities:

$$st - \liminf_n A_n := \{x | \forall V \in \mathcal{N}(x), \exists N \in \mathcal{S}, \forall n \in N: A_n \cap V \neq \emptyset\}$$

$$= \{x | \exists N \in \mathcal{S}, \forall n \in N, \exists y_n \in A_n: \lim_n y_n = x\}.$$

$$st - \limsup_n A_n := \{x | \forall V \in \mathcal{N}(x), \exists N \in \mathcal{S}^{\#}, \forall n \in N: A_n \cap V \neq \emptyset\}$$

$$= \{x | \exists N \in \mathcal{S}^{\#}, \forall n \in N, \exists y_n \in A_n: x \in \Gamma_y\}.$$

Let f be a function defined on X, the epigraph of f is the set $epif := \{(x,\alpha) \in X \times \mathbb{R} | \alpha \geq f(x)\}$ and its level set is defined by

$$lev_{\leq\alpha} f := \{x \in X | f(x) \leq \alpha\}.$$

Let $f_n: X \to \overline{\mathbb{R}}$ be a sequence consisting of lower semicontinuous functions and (X,d) a metric space. The lower statistical epi-limit, $e_{st} - \text{liminf}_n f_n$ is defined by the help of the sequence of sets:

$$epi(e_{st} - \underset{n}{liminf}\ f_n) = st - \underset{n}{limsup}\ (epif_n).$$

Similarly, the upper statistical epi-limit $e_{st} - \text{limsup}_n f_n$ is defined by:

$$epi(e_{st} - \underset{n}{limsup}\ f_n) = st - \underset{n}{liminf}(epif_n).$$

If we have the following equality, it is called statistical epi-convergence:

$$f = st - \underset{n}{lim}\ f_n = e_{st} - \underset{n}{limsup}\ f_n$$
$$= e_{st} - \underset{n}{liminf}\ f_n.$$

Following definition is a sequential characterication of epi-convergence.

For each $x \in X$ the sequence $f_n: X \to \overline{\mathbb{R}}$ is epi-convergent to f, if and only if the following conditions

(i) for all $x_n \in X$ whenever (x_n) is convergent to x, we have $f(x) \leq \text{liminf}_n\ f_n(x_n)$,

(ii) there exists a sequence (x_n) convergent to x such that $f(x) = \lim_n f_n(x_n)$

both hold.

Let $\mathcal{G}(f)$ be the set of all lower semicontinuous functions denoted by h on X satisfying $h(y) \leq f(y)$ for every $y \in X$. For every function $f: X \to \overline{\mathbb{R}}$, the lower semicontinuous envelope $sc^{-}f$ of f is defined by

$$(sc^{-}f)(x) = \sup_{g \in \mathcal{G}(f)} g(x)$$

for every $x \in X$.

Let $f: X \to \overline{\mathbb{R}}$ be a function. Then

$$(sc^{-}f)(x) = \sup_{V \in \mathcal{N}(x)} \inf_{y \in V} f(y)$$

for every $x \in X$ where $\mathcal{N}(x)$ is the neighbourhood of x.

More information about epi-convergence and statistical convergence we advise to look at papers in the reference part (Di Maio and Kočinac 2008, Rockafellar and Wets 2009, Šalaʹt 1980).

3. Main Result

Theorem 3.1 *Let $f_n: X \to \overline{\mathbb{R}}$ be a sequence of functions. If (f_n) is uniformly statistically convergent to f, then (f_n) is statistically epi-convergent to $sc^{-}f$.*

Proof: Assume that (f_n) is uniformly statistically convergent to f. Then, for every $\varepsilon > 0$, there exists $K \in \mathcal{S}$ such that for all $n \in K$ and for all $y \in X$ we have $|f_n(y) - f(y)| < \varepsilon$. Hence,

$$f(y) - \varepsilon < f_n(y) < f(y) + \varepsilon.$$

Since uniform statistical convergence is independent of y, the following equality is valid for an open set $U \in X$ and all $n \in K$.

$$\underset{y\in U}{inf} f(y) - \varepsilon < \underset{y\in U}{inf} f_n(y) < \underset{y\in U}{inf} f(y) + \varepsilon.$$

Then we have

$$st - \underset{n}{liminf} \underset{y\in U}{inf} f_n(y) = \underset{y\in U}{inf} f(y),$$

hence for every $x \in X$

$$\underset{U\in\mathcal{N}(x)}{sup}\ st - \underset{n}{liminf} \underset{y\in U}{inf} f_n(y) = \underset{U\in\mathcal{N}(x)}{sup} \underset{y\in U}{inf} f(y),$$

which implies that (f_n) is statistically epi-convergent to $sc^{-}f$.

In statistical pointwise convergence, ε is dependent on every point $x \in X$ hence it gives us an idea about why statistical pointwise convergence and statistical epi-convergence do not coincide in general.

Remark 3.2 Let *each function $f_n: X \to \overline{\mathbb{R}}$ be lower semicontinuous. If (f_n) statistically uniformly converges to f, then f is lower semicontinuous and (f_n) statistically epi-converges to f.*

Theorem 3.3 *Let $f_n: X \to \overline{\mathbb{R}}$ be a sequence of functions and $g: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a continuous and increasing function. Then*

$$e_{st} - \liminf_n (g \ o \ f_n) = g \ o \ (e_{st} - \liminf_n f_n),$$

(1)

$$e_{st} - \limsup_n (g \ o \ f_n) = g \ o \ (e_{st} - \limsup_n f_n).$$

(2)

Proof: As we know, g is a continuous and increasing function, then we have

$$g(\inf S) = \inf g(S) \quad \text{and} \quad g(\sup S = \sup g(S)$$

for each subset S of $\overline{\mathbb{R}}$. Since

$$(e_{st} - \liminf_n f_n)(x) = \sup_{U \in \mathcal{N}(x)} \sup_{N \in \mathcal{S}} \inf_{n \in N} \inf_{y \in U} f_n(y).$$

Hence the equation can be rewritten as

$$\sup_{U \in \mathcal{N}(x)} \sup_{N \in \mathcal{S}} \inf_{n \in N} \inf_{y \in U} g(f_n(y))$$
$$= g(\sup_{U \in \mathcal{N}(x)} \sup_{N \in \mathcal{S}} \inf_{n \in N} \inf_{y \in U} f_n(y)).$$

It gives the proof of (1). The proof of (2) is analogous to the previous one.

Theorem 3.4 *If $f_n: X \to \overline{\mathbb{R}}$ and $g_n: X \to \overline{\mathbb{R}}$ be sequences of functions and their sum is well defined, then the following inequalities are valid.*

$$e_{st} - \liminf_n (f_n + g_n) \geq e_{st} - \liminf_n f_n + e_{st} - \liminf_n g_n, \qquad (3)$$

$$e_{st} - \limsup_n (f_n + g_n) \geq e_{st} - \limsup_n f_n + e_{st} - \liminf_n g_n. \qquad (4)$$

Proof: First, we apply some additional restrictions for (f_n) and (g_n). $\exists \alpha \in \mathbb{R}$ such that $f_n \leq \alpha$ and $g_n \leq \alpha$ on X for every $n \in \mathbb{N}$. In our operations, all sums have become well defined. Let $U \in X$ be an open set. For every U,

$$\inf_{y \in U} (f_n + g_n)(y) \geq \inf_{y \in U} f_n(y) + \inf_{y \in U} g_n(y).$$

Hence, by using properties of statistical upper and lower limits, we get

$$st - \limsup_n \inf_{y \in U} (f_n + g_n)(y) \geq$$

$$st - \limsup_n \inf_{y \in U} f_n(y) + st - \liminf_n \inf_{y \in U} g_n(y). \qquad (5)$$

Now fix $x \in X$. If

$$e_{st} - \limsup_n f_n(x) + e_{st} - \liminf_n g_n(x) = -\infty$$

then we are done. Otherwise, for each $\varepsilon > 0$ there exists $V, W \in \mathcal{N}(x)$ such that

$$(e_{st} - \limsup_n f_n)(x) - \varepsilon < st - \limsup_n \inf_{y \in U} f_n(y),$$

(6)

$$(e_{st} - \liminf_n g_n)(x) - \varepsilon < st - \liminf_n \inf_{y \in U} g_n(y).$$

(7)

Let $U = V \cap W$. Since $U \in \mathcal{N}(x)$ and

$$\inf_{y \in V} f_n(y) \leq \inf_{y \in U} f_n(y) \ , \inf_{y \in W} g_n(y) \leq \inf_{y \in U} g_n(y).$$

By using definition of statistical upper epi-limit, (5), (6) and (7) we obtain

$$\left(e_{st} - \limsup_n (f_n + g_n)\right)(x)$$
$$\geq st - \limsup_n \inf_{y \in U} (f_n + g_n)(y)$$
$$\geq (e_{st} - \limsup_n f_n)(x) +$$
$$\left(e_{st} - \liminf_n g_n\right)(x) - 2\varepsilon.$$

ε was arbitrary, hence the proof is completed.

Now we deal with the general case. Assume that the sequences (f_n) and (g_n) are not restricted from above. Let us define a function $h_a: \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ as

$h_a(t) = \min\{t, a\}$ for every $a \in \mathbb{R}$. For every $n \in \mathbb{N}$ we know

$h_a \circ f_n \leq a$ and $h_a \circ g_n \leq a$

on X from previous part of the proof we get

$$e_{st} - \underset{n}{limsup}\big((h_a \circ f_n) + (h_a \circ g_n)\big)$$

$$\geq e_{st} - \underset{n}{limsup}(h_a \circ f_n) + e_{st} - \underset{n}{liminf}(h_a \circ g_n).$$

Theorem 3.3 implies that

$$e_{st} - \underset{n}{limsup}(f_n + g_n)$$

$$\geq e_{st} - \underset{n}{limsup}((h_a \circ f_n) + (h_a \circ g_n))$$

$$\geq h_a \circ (e_{st} - \underset{n}{limsup} f_n) + h_a \circ (e_{st} - \underset{n}{limsup}\, g_n).$$

When taking $a \to \infty$, the proof for the case of unboundedness is completed.

Even if (f_n) and (g_n) are statistically epi-convergent, the inequalities (3) and (4) can be strict. This situation can be seen in the following example.

Example 3.5 *Let (f_n) and (g_n) be real valued functions defined on $\mathbb{R}$ as,*

$$f_n(x) = \begin{cases} -2 & \text{if} \quad \text{n is even square,} \\ sin(nx) & \text{if} \quad \text{otherwise.} \end{cases}$$

$$g_n(x) = \begin{cases} -2 & \text{if} \quad \text{n is odd square,} \\ -sin(nx) & \text{if} \quad \text{otherwise.} \end{cases}$$

Then (f_n) and (g_n) are statistically epi-convergent to $h(x) = -1$ while $(f_n + g_n)$ is statistically epi-convergent to $h(x) = 0$.

A sequence $f_n: X \to \overline{\mathbb{R}}$ is statistically α-convergent to f if for every $x \in X$ and every sequence (x_n) in X converging to x, the sequence $f_n(x_n)$ statistically converges to $f(x)$. By Theorem 3.5 by (Caserta and Kočinac 2012), we know statistical α-convergence implies statistical uniform convergence. Also, we prooved in Theorem 3.1 that statistical epi-convergence is implied by statistical uniform convergence. Hence we will use it in the following Corollary.

Corollary 3.6 *Assume that $f_n: X \to \overline{\mathbb{R}}$ and $g_n: X \to \overline{\mathbb{R}}$ are sequences of functions. If (g_n) is statistically α-convergent to a function g provided that (g_n) and g are finite, then the following equalities hold.*

$$e_{st} - \underset{n}{liminf}(f_n + g_n) = e_{st} - \underset{n}{liminf} f_n + g,$$

(8)

$$e_{st} - \underset{n}{limsup}(f_n + g_n) = e_{st} - \underset{n}{limsup} f_n + g.$$

(9)

Proof: We shall proove only (9), the other one being analogous. First of all, we know that if $g_n \overset{st-\alpha}{\to} g$ then g is continuous and $g_n \overset{st-u}{\to} g$. Hence by Theorem 3.1 we have

$$g_n \overset{e_{st}}{\to} g. \tag{10}$$

From now on, we continue by using Theorem 3.4 and we get

$$e_{st} - \underset{n}{liminf}(f_n + g_n) \geq e_{st} - \underset{n}{liminf} f_n + g. \tag{11}$$

On the other hand, $(-g_n)$ is statistically epi-convergent to $-g$ in X and by Theorem 3.4 we have

$$e_{st} - \underset{n}{limsup} f_n = e_{st} - \underset{n}{limsup}(f_n + g_n - g_n)$$
$$\geq e_{st} - \underset{n}{limsup}(f_n + g_n) - g.$$

Hence,

$$e_{st} - \underset{n}{liminf} f_n + g \geq e_{st} - \underset{n}{liminf}(f_n + g_n).$$

(12)

Equality (9) follows from (11) and (12).

Corollary 3.7 *Let $f_n: X \to \overline{\mathbb{R}}$ be a sequence of functions. Suppose that $g: X \to \overline{\mathbb{R}}$ is a continuous function. Then the following equalities hold.*

$$e_{st} - \underset{n}{liminf}(f_n + g) = e_{st} - \underset{n}{liminf} f_n + g,$$

(13)

$$e_{st} - \underset{n}{limsup}(f_n + g) = e_{st} - \underset{n}{limsup} f_n + g.$$

(14)

Proof: The sequence (g_n) is statistically alpha convergent to g, since g is a continuous function. Then the result follows by using Corollary 3.6.

Continuity of g is essential in Corollary 3.7, as the following example shows.

Example 3.8 *Let (f_n) and g be real valued functions defined on $\mathbb{R}$ as,*

$$f_n(x) = \begin{cases} 2nxe^{-2n^2x^2} & \text{if} \quad \text{n is square,} \\ nxe^{-2n^2x^2} & \text{if} \quad \text{otherwise} \end{cases}$$

$$g(x) = \begin{cases} 1 & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

The function g is lower semicontinuous and each of f_n is continuous. (f_n) statistically epi-converges to $-\frac{1}{2}e^{-12}$ while $(f_n + g)$ statistically epi-converges to $1 - \frac{1}{2}e^{-12}$ at the point 0 where g is not continuous.

Corollary 3.9 *Let (f_n) and (g_n) be functions from X to $\overline{\mathbb{R}}$. Suppose that (f_n) is statistically epi-convergent and statistically pointwise convergent to f and (g_n) is statistically epi-convergent and statistically pointwise convergent to g. Then $(f_n + g_n)$ is statistically epi-convergent and statistically pointwise convergent to $(f + g)$, provided that the functions $(f_n + g_n)$ and $(f + g)$ are well defined on X.*

Proof: By Theorem 3.4 we have

$$\begin{aligned} f + g = e_{st} - \underset{n}{liminf} f_n + e_{st} - \underset{n}{liminf} g_n \\ \leq e_{st} - \underset{n}{liminf}(f_n + g_n) \\ \leq e_{st} - \underset{n}{limsup}(f_n + g_n) \end{aligned}$$

$$\leq st - \underset{n}{limsup}(f_n + g_n) = f + g.$$

Theorem 3.10 *For any sequence (f_n) of convex functions on X, the function $e_{st} - limsup_n f_n$ is convex.*

Proof: Since each f_n is convex function on X, each of $epif_n$ is convex set. Let $x, y \in st - \mathrm{liminf}_n(epif_n)$, then $\exists\, x_n \in epif_n$ such that $x_n \overset{st}{\rightarrow} x$, $\forall n \in N$ with $N \in \mathcal{S}$. Similarly there exists a sequence $y_n \in epif_n$ such that for all $n \in K$ with $K \in \mathcal{S}$, $y_n \overset{st}{\rightarrow} y$. Let $W = N \cap K$ that is $\delta(W) = 1$. For arbitrary $\lambda \in [0,1]$, define $z_n^\lambda := (1-\lambda)x_n + \lambda y_n$ and $z^\lambda := (1-\lambda)x + \lambda y$, then we have $z_n^\lambda \in epif_n$ and $z_n \overset{st}{\rightarrow} z$ for all $n \in W$, hence $z^\lambda \in st - \mathrm{liminf}_n(epif_n)$ and proves the convexity of this set. Consequently, $e_{st} - \mathrm{limsup}_n f_n$ is convex.

Following example shows that $e_{st} - \mathrm{liminf}_n f_n$ function need not be convex.

Example 3.11 *Let $f_n: \mathbb{R} \rightarrow \mathbb{R}$ be defined as $f_n = (x + (-1)^n)^2$. Indeed, $f = e_{st} - liminf_n f_n$ function is*

$$f(x) = \begin{cases} (x+1)^2 & if \quad x \leq 0, \\ (x-1)^2 & if \quad x > 0 \end{cases}$$

which is not convex.

References

Anastassiou, A. G. and Duman, O., 2011. Towards Intelligent Modeling: Statistical Approximation Theory, vol.14, Berlin.

Attouch, H., 1977. Convergence de fonctions convexes, de sous-differentiels et semi-groupes. *Comptes Rendus de lAcademie des Sciences de Paris*, **284,** 539-542.

Caserta, A. and Kočinac, Lj. D. R., 2012. On statistical exhaustiveness. *Applied Mathematics Letters*, **25,** 1447-1451.

Di Maio, G. and Kočinac, Lj. D. R., 2008. Statistical convergence in topology. *Topology and its Applications*, **156,** 28-45.

Duman, O. and Orhan, C., 2004. μ -statistically convergent function sequences. *Czechoslovak Mathematical Journal*, **54** (129)(2), 413-422.

Fast, H., 1951. Sur la convergence statistique. *Colloquium Mathematicum*, **2,** 241–244.

Fridy, J. A., 1993. Statistical limit points. *Proceedings of the American Mathematical Society***, 118** (4), 1182–1192.

Fridy, J. A. and Orhan, C., 1997. Statistical limit superior and limit inferior. *Proceedings of the American Mathematical Society*, **125,** 3625–3631.

Gökhan, A. and Güngör, M., 2002. On pointwise statistical convergence. *Indian Journal of Pure and Applied Mathematics*, **33** (9), 1379-1384.

Güngör, M. and Gökhan, A., 2005. On uniform statistical convergence. *International Journal of Pure and Applied Mathematics*, **19** (1), 17–24.

Joly, J.-L., 1973. Une famille de topologies sur lensemble des fonctions convexes pour lesquelles la polarite est bicontinue. *Journal de Mathematiques Pures et Appliquees*, **52,** 421–441.

Kuratowski, C., 1958. Topologie, vol.I, PWN, Warszawa.

Maso, G. D., 1993. An introduction to Γ-convergence, vol.8, Boston.

McLinden, L. and Bergstrom, R., 1981. Preservation of convergence of sets and functions in finite dimensions. *Transactions of the American Mathematical Society*, **268,** 127–142.

Mosco, U., 1969. Convergence of convex sets and of solutions of variational inequalities. *Advances in Mathematics*, **3,** 510–585.

Niven, I. and Zuckerman, H. S., 1980. An Introduction to the Theory of Numbers, New York.

Rockafellar, R.T. and Wets, R.J-B., 2009. Variational Analysis, Berlin.

$\check{S}$ala't, T., 1980. On statistically convergent sequences of real numbers. *Mathematica Slovaca*, **30,** 139-150.

Salinetti, G. and Wets, R.J-B., 1977. On the relation between two types of convergence for convex functions. *Journal of Mathematical Analysis and Applications*, **60,** 211–226.

Schoenberg, I.J.:, 1959. The integrability of certain functions and related summability methods. *American Mathematical Monthly*, **66,** 361-375.

Steinhaus, H., 1951. Sur la convergence ordinaire et la convergence asymptotique. *Colloquium Mathematicum*, **2,** 73-74.

Talo, Ö., Sever, Y. and Başar, F., 2016. On statistically convergent sequences of closed sets. *Filomat*, **30** (6), 1497-1509.

Wets, R.J-B., 1980. Convergence of convex functions, variational inequalities and convex optimization problems, New York.

Wijsman, R. A., 1964. Convergence of sequences of convex sets, cones and functions. *Bulletin of American Mathematical Society*, **70,** 186-188.

Wijsman, R. A., 1966. Convergence of sequences of convex sets, cones and functions II. *Transactions of the American Mathematical Society*, **123,** 32-45.

Zygmund, A., 1979. Trigonometric Series, Cambridge University Press, Cambridge, UK.