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Araştırma Makalesi / Research Article

Sums of Element Orders in Symmetric Groups

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Abstract

In literature, there are many papers on the sums of element orders of finite groups. In this study we deal with the cases in symmetric groups. Our main aim is to investigate the sums of element orders in symmetric groups and to give some properties of the sum of element orders in symmetric group. Moreover, we derive the formula for such sums.

Simetrik Gruplarda Eleman Mertebelerinin Toplamları

Anahtar kelimeler
Sonlu grup; Simetrik gruplar; Eleman mertebelerinin toplamı; Eleman mertebesi.

Özet

Literatürde sonlu grupların eleman mertebelerinin toplamı üzerine birçok çalışma yer almaktadır. Bu çalışmada simetrik gruplardaki durumlar üzerinde durulacaktır. Amacımız simetrik gruplarda eleman mertebelerinin toplamını incelemek ve simetrik gruplarda eleman mertebelerinin toplamlarının bazı özelliklerini vermektir. Ayrıca, bu toplamlar için bir formül üretmektir.

1. Introduction

Sums of element orders in finite groups is an interesting subject, which was studied in varies papers (see Amiri (2009), Amiri and Amiri (2011), Herzog et al. (2018)). Our main starting point is given by the papers H. Amiri et al. (2009), H. Amiri and S.M.J. Amiri (2011) which studied on the sums of element orders in finite groups. Given a finite group G, we denote the sum of element orders in G by $\psi(G)$. Historically, the most enlightening in this area is due H. Amiri, S.M.J. Amiri and I.M. Isaacs, who introduced the function $\psi(G)$ on G for a finite group G. Amiri et al. (2009) and proved that $\psi(G) < \psi(C_n)$, where C_n denotes a cycle group of order n. In Herzog et al. (2018), M. Herzog, P. Longobardi and M. Maj studied to find an exact upper bound for the sums of element orders in non-cyclic finite groups. Let S_n denote the symmetric group of degree n. In this note we will focus on the study of $\psi(S_n)$. Our goal is to derive an explicit formula for the sum of element orders in S_n.

2. Preliminaries

This section contains necessary definitions and preliminary results.

Notice that an arbitrary permutation $\sigma \in S_n$ can be written as a product of disjoint cycles. Suppose that σ has cycles of length $p_1, p_2, \dots, p_r$, where $p_1 \geq p_2 \geq \cdots \geq p_r$, $\sum_{i=1}^{r} p_i = n$ and 1's in this list are included for fixed points. The sequence $p = (p_1, p_2, \dots, p_r)$ is said to be the cycle type of σ. For instance, if $\sigma \in S_9$ and $\sigma = (1345)(278)$, then σ

has cycle type (4,3,1,1). If σ is a k-cycle in S_n, where $k \leq n$, then the cycle type of σ is $(k, 1, \ldots, 1)$, and the number of 1's in the sequence is $n-k$. The order of a permutation expressed as a product of disjoint cycles is the least common multiple of the lengths of the cycles, namely,

$$o(\sigma) = \operatorname{lcm}(p_1, p_2, \ldots, p_r). \tag{2.1}$$

Let n be a positive integer, a sequence of positive integers $p = (p_1, p_2, \ldots, p_r)$ such that $p_1 \geq p_2 \geq \cdots \geq p_r$ and $\sum_{i=1}^{r} p_i = n$ is called a partition of n. It is well-known that there is a bijection between the set of all partitions of n and the set of the conjugacy classes of S_n.

Lemma 2.1. Any two elements of S_n with the same cycle type are in the same conjugacy class.

Lemma 2.2. Let G be a group. Then G is the disjoint union of its conjugacy classes.

Let s be the number of distinct conjugacy classes of G. We suppose that the numbers of elements in the conjugacy classes are $n_1, n_2, \ldots, n_s$. These integers satisfy the class equation

$$|G| = n_1 + n_2 + \cdots + n_s.$$

The number of partitions of a positive number n is equal to the number of conjugacy classes of S_n.

Lemma 2.3. In S_n, let $p = (p_1, p_2, \ldots, p_r)$ be a partition of n such that for $1 \leq i \leq n$, k_i of the parts are i. Then, the number of permutations having cycle type $p = (p_1, p_2, \ldots, p_r)$ in S_n is calculated by the following formula

$$A_p = \prod_{i=1}^{n} \frac{n!}{(k_i)! i^{k_i}}.$$

This lemma will be an important ingredient in the proof of our main result. For more details we refer to (Gorenstein 1968, Herstein 1958, Herzog et al. 2018).

3. Main Results

This section is devoted to the description of the sum of element orders in symmetric group S_n. An explicit formula for $\psi(S_n)$ will be given by the following theorem.

Theorem 3.1. In S_n, let $p = (p_1, p_2, \ldots, p_r)$ be a partition of n and A_p denotes the number of permutations which have cycle type p. Then

$$\psi(S_n) = A_p . \operatorname{lcm}(p_1, p_2, \ldots, p_r).$$

Proof. The function $\psi(S_n)$ is defined as

$$\psi(S_n) = \sum_{\sigma \in S_n} o(\sigma),$$

where $o(\sigma)$ denotes the order of $\sigma \in S_n$. The number of all permutations with cycle type $p = (p_1, p_2, \ldots, p_r)$ is calculated by (2.1). Hence, the sum of orders of permutations with cycle type $p = (p_1, p_2, \ldots, p_r)$ is $A_p . \operatorname{lcm}(p_1, p_2, \ldots, p_r)$. Considering for each partition p of n, we obtain $\psi(S_n)$ which is the sum of orders of all permutations in S_n, that is, we get

$$\psi(S_n) = A_p . \operatorname{lcm}(p_1, p_2, \ldots, p_r).$$

Therefore, the proof of theorem completes.

□

Now, we can see some information on partitions of n, the sizes of conjugacy classes and the element orders of S_n for the cases $n = 3$ and $n = 4$. Moreover, we see how the formula is applied to the cycle sizes.

For $n = 3$,

Table 1. The case of $n = 3$

Partition	Elements with the cycle type	Size of conjugacy class	Element order
1+1+1	(1)	$A_{(1,1,1)} = 1$	$\operatorname{lcm}(1,1,1) = 1$
2+1	(12), (13), (23)	$A_{(2,1)} = 3$	$\operatorname{lcm}(2,1) = 2$
3	(123), (132)	$A_{(3)} = 1$	$\operatorname{lcm}(3) = 3$

$$\begin{aligned}\psi(S_3) &= A_{(1,1,1)} . \operatorname{lcm}(1,1,1) + A_{(2,1)} . \operatorname{lcm}(2,1) \\ &\quad + A_{(3)} . \operatorname{lcm}(3) \\ &= 1.1 + 3.2 + 2.3 \\ &= 13.\end{aligned}$$

For $n = 4$,

Table 2. The case of $n = 4$

Partition	Elements with the cycle type	Size of conjugacy class	Element order
1+1+1+1	(1)	$A_{(1,1,1,1)} = 1$	$\mathrm{lcm}(1,1,1,1) = 1$
2+1+1	(12),(13),(14), (23),(24),(34)	$A_{(2,1,1)} = 6$	$\mathrm{lcm}(2,1,1) = 2$
2+2	(12)(34), (13)(24), (14)(23)	$A_{(2,2)} = 3$	$\mathrm{lcm}(2,2) = 2$
3+1	(123), (132), (234), (243), (124), (142), (134), (143)	$A_{(3,1)} = 8$	$\mathrm{lcm}(3,1) = 3$
4	(1234), (1243), (1324), (1342), (1423), (1432)	$A_{(4)} = 6$	$\mathrm{lcm}(4) = 4$

$$\begin{aligned}\psi(S_4) &= A_{(1,1,1,1)}.\mathrm{lcm}(1,1,1,1)\\ &+A_{(2,1,1)}.\mathrm{lcm}(2,1,1) + A_{(2,2)}.\mathrm{lcm}(2,2)\\ &+A_{(3,1)}.\mathrm{lcm}(3,1) + A_{(4)}.\mathrm{lcm}(4)\\ &= 1.1 + 6.2 + 3.2 + 8.3 + 6.4\\ &= 67.\end{aligned}$$

Proposition 3.2. Let S_n be symmetric group of degree n for $n > 3$. Then,

$$\psi(S_n) < \frac{|S_n|^2}{2}.$$

Proof. Since S_n for $n > 3$ is non-cyclic and 2 is the smallest prime divisor of $|S_n|$, this implies that $o(x) \leq \frac{|S_n|}{2}$ for each $x \in S_n$. But $o(1) = 1$, so $\psi(S_n) \leq (|S_n| - 1)\left(\frac{|S_n|}{2}\right) + 1 < \frac{|S_n|^2}{2}$.

□

Recall that let n be a positive number, the Euler function $\varphi(n)$ is the number of integers k such that $1 \leq k < n$ and $(k, n) = 1$. We can calculate the Euler function $\varphi(n)$ by the following formula

$$\varphi(n) = n. \prod_{p|n, p \text{ prime}} (1 - \frac{1}{p}). \qquad (3.2)$$

Theorem 3.3. Let S_n be symmetric group of degree n. Then,

(i) For $n > 3$, $\psi(S_n) < \frac{1}{2}|S_n|\, \varphi(|S_n|)$.

(ii) For $n \leq 3$, $\psi(S_n) > \frac{1}{2}|S_n|\, \varphi(|S_n|)$.

Proof.

(i) By Proposition 3.2,

$$\psi(S_n) < \frac{|S_n|^2}{2} < \frac{1}{2}|S_n|\, \varphi(|S_n|).$$

(ii) It is clear to see for the cases $n = 1,2,3$.

□

Lemma 3.4. Let $|S_n|$ be the order of symmetric group of degree n with the largest prime divisor p. Then,

$$\varphi(|S_n|) = \frac{1}{p}\, |S_n|.$$

Proof. Let $|S_n| = p_1^{m_1} p_2^{m_2} \ldots p_k^{m_k}$ where p_i's are prime, m_i's are positive integers and $p_1 = 2 < p_2 < \cdots < p_k = p$. Using (3.2), the Euler's function $\varphi(|S_n|)$ satisfies the following equality:

$$\begin{aligned}\varphi(|S_n|) &= |S_n|\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \ldots \left(1 - \frac{1}{p}\right)\\ &= |S_n| \frac{1}{2}\frac{2}{3} \ldots \frac{p-1}{p}\\ &= \frac{1}{p}|S_n|.\end{aligned}$$

□

Proposition 3.5. Let $C_{|S_n|}$ be the cyclic group of order $|S_n|$, where S_n is a symmetric group of degree n. Then,

$$\psi(C_{|S_n|}) > \frac{1}{p}|S_n|^2.$$

Proof. It is clear that $\psi(C_{|S_n|}) > |S_n|\varphi(|S_n|)$. By Lemma 3.4, we have

$$\psi\left(C_{|S_n|}\right) > |S_n|\frac{1}{p}|S_n| = \frac{1}{p}|S_n|^2,$$

as required.

□

Theorem 3.6. Let S_n be the symmetric group of degree n for $n > 3$. Then,

$$\psi(S_n) < \psi\left(C_{|S_n|}\right),$$

where $C_{|S_n|}$ denote the cyclic group of order $|S_n|$.

Proof. By Theorem 3.3 (i),

$$\psi(S_n) < \frac{|S_n|^2}{2} < |S_n|\varphi(|S_n|) < \psi\left(C_{|S_n|}\right).$$

□

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