CONTINUOUS DEPENDENCE OF SOLUTIONS ON THE COEFFICIENT THERMAL DIFFUSIVITY FOR PHASE FIELD EQUATIONS

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ABSTRACT

In this paper, investigate the continuous dependence on the coefficient thermal diffusivity.

Keyword: Phase field equation, continuous dependence.

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ÖZET

Bu çalışmada faz alan denklemlerinin çözümlerinin ısı iletkenlik katsayısına sürekli bağımlılığı incelenmiştir.

Anahtar Kelimeler: Faz alan denklemi, sürekli bağımlılık.

I. INTRODUCTION

We consider the problem

$$\tau\phi_t - \xi^2\Delta\phi + f(x,\phi) = 2u + h_1(x,t) \qquad (x,t)\in Q_T \tag{1}$$

$$u_t + \tfrac{l}{2}\phi_t = K\Delta u + h_2(x,t) \quad (x,t)\in Q_T \tag{2}$$

$$\phi\big|_\Gamma = \phi_\partial(x,t),\ u\big|_\Gamma = u_\partial(x,t) \qquad (x,t)\in\partial\Omega\times(0,T] \tag{3}$$

$$\phi(x,0) = \phi_0(x),\ u(x,0) = u_0(x) \quad x\in\Omega \tag{4}$$

where $Q_T = \Omega \times (0,T]$, $\Omega \subset R^n$ $(n \geq 1)$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$; $\Gamma = \partial\Omega \times (0,T]$, ξ, τ, l and K are positive constants characterizing the length scale, the relaxation time, the latent heat and the thermal diffusivity respectively. $\phi_0, u_0, \phi_\partial, u_\partial, h_1, h_2$ and $f(x,\phi)$ are given functions.

In [1], G.Çağınalp has considered, as a model describing the phase transitions with a seperation surface of finite thickness and proves a global existence theorem for the classical solution of problem such type. In [2], Brochet, Hilhorst ve Chen investigated problem (1)-(4) considering $v = u + \frac{l}{2}\phi$, $f(s) = \sum_{j=0}^{2p-1} b_j s^j$, $b_{2p-1} > 0, p \geq 2$, $h_i(x,t) = 0$, $(i = 1,2)$ and homogeneous Neumann boundary condition and they proved that the problem is well posed if $(\phi_0, u_0) \in (L_2(\Omega))^2$. In [3], Kalantarov has proved that the initial boundary value problem for system (1)-(2), under some conditions on $f(x,\phi)$ homogeneous boundary conditions, global uniqualy solvable in $C(R^+, X)$, $X = \overset{0}{H}{}^1(\Omega) \times \overset{0}{H}{}^1(\Omega)$ and the existence of global attractor.

II. CONTINUOUS DEPENDENCE

THEOREM:

If

$$|f(x,s_1) - f(x,s_2)| \leq c(1 + |s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2| \tag{5}$$

then the solution of problem (1)-(4) from $V(Q_T) \times V(Q_T)$ ([4],[5],[6]) depends continuously on the thermal diffusivity coefficient. Where,

$$V(Q_T) = W_2^1(Q_T) \cap \{v(x,t) : \Delta v \in L_2(Q_T)\}$$

and

$$p \in [1, \infty] \text{ if } n = 1,2, \; p \in \left[1, \frac{n}{n-2}\right] \text{ if } n \geq 3 .$$

PROOF:

Let $\{\phi_1, u_1\}$ and $\{\phi_2, u_2\}$ be the solutions from $V(Q_T)\times V(Q_T)$ of problem (1)-(4) for different coefficient K_1 and K_2 respectively. We define difference variables φ, u and K by

$\varphi = \phi_1 - \phi_2, u = u_1 - u_2$, and $K = K_1 - K_2$ $(K_1 > K_2)$.

Then $\{\varphi, u\}$ satifies the initial boundary value problem

$$\tau\varphi_t - \xi^2\Delta\varphi + f(x,\phi_1) - f(x,\phi_2) = 2u \qquad (x,t)\in Q_T \tag{6}$$

$$u_t + \tfrac{l}{2}\varphi_t = K_1\Delta u + K\Delta u_2 \qquad (x,t)\in Q_T \tag{7}$$

$$\varphi\big|_\Gamma = u\big|_\Gamma = 0 \tag{8}$$

$$\varphi(x,0) = u(x,0) = 0 \tag{9}$$

If we take the inner product in $L_2(\Omega)$ of (6) by $\varphi_t + \varphi$ and of (7) by $\dfrac{2\tau}{l^2}u_t + \dfrac{4}{l}u$ and we add the obtained equations, then by inequalities from making use of (5)

$$\left|\int_\Omega (f(x,\phi_1) - f(x,\phi_2))\varphi_t\,dx\right| \le c\|\varphi\|\|\varphi_t\| + cc_1(t)\|\varphi\|_{L_{\frac{2n}{n-2}}}\|\varphi_t\| \tag{10}$$

and

$$\left|\int_\Omega (f(x,\phi_1) - f(x,\phi_2))\varphi\,dx\right| \le c\|\varphi\|^2 + cc_1(t)\|\varphi\|_{L_{\frac{2n}{n-2}}}\|\varphi\| \tag{11}$$

we obtain

$$\tau\|\varphi_t\|^2 + \xi^2\|\nabla\varphi\|^2 + \frac{4K_1}{l}\|\nabla u\|^2 + \frac{2\tau}{l^2}\|u_t\|^2 +$$

$$\frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{2}{l}\|u\|^2 + \frac{\tau}{2}\|\varphi\|^2 + \frac{\tau K_1}{l^2}\|\nabla u\|^2\right] \le 2(u,\varphi) + \frac{4K}{l}(\Delta u_2, u) +$$

$$\frac{2\tau K}{l^2}(\Delta u_2, u_t) + \frac{\tau}{l}|(\varphi_t, u_t)| + c\|\varphi\|\|\varphi_t\| + cc_1(t)\|\varphi\|_{L_{\frac{2n}{n-2}}}\|\varphi_t\| + c\|\varphi\|^2 +$$

$$cc_1(t)\|\varphi\|_{L_{\frac{2n}{n-2}}}\|\varphi\| \tag{12}$$

where $\|\bullet\|$ denotes the norm on $L_2(\Omega)$ and $c_1(t)$ is depens on the given functions. Making use of Cauchy-Schwarz and ε-Young's inequalities, right hand side of (12) can be estimated. If we select the number $\varepsilon > 0$ sufficiently small and by the inequality

$$\|\varphi\|_{L_{\frac{2n}{n-2}}} \leq c_2 \|\nabla \varphi\|$$

(12) rewrite as follows:

$$\frac{\tau}{4}\|\varphi_t\|^2 + \frac{4K_1}{l}\|\nabla u\|^2 + \frac{3\tau}{4l^2}\|u_t\|^2 +$$

$$\frac{d}{dt}\left[\frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{2}{l}\|u\|^2 + \frac{\tau}{2}\|\varphi\|^2 + \frac{\tau K_1}{l^2}\|\nabla u\|^2\right] \leq a_1(t)\|\nabla\varphi\|^2 +$$

$$+\left(\frac{l+2}{l}\right)\|u\|^2 + a_2(t)\|\varphi\|^2 + \left(\frac{6l+4\tau}{3l^2}\right)K^2\|\Delta u_2\|^2 \qquad (13)$$

Where $a_1(t)$ and $a_2(t)$ are depends on given function and the parameters. If we denote

$$\tilde{c}(t) = \max\left\{\frac{2a_1(t)}{\xi^2}, \frac{l+2}{2}, \frac{2a_2(t)}{\tau}, 1\right\}$$

and

$$Y(t) = \frac{\xi^2}{2}\|\nabla\varphi\|^2 + \frac{2}{l}\|u\|^2 + \frac{\tau}{2}\|\varphi\|^2 + \frac{\tau K_1}{l^2}\|\nabla u\|^2$$

then from (13)

$$\begin{cases} \dfrac{dY(t)}{dt} \leq \tilde{c}(t)Y(t) + \left(\dfrac{6l+4\tau}{3l^2}\right)K^2\|\Delta u_2\|^2 \\ Y(0) = 0 \end{cases}$$

According to Gronwall's lemma, we have

$$Y(t) \leq \left(\frac{6l+4\tau}{3l^2}\right)e^{\int_0^t \tilde{c}(s)ds} K^2 \|\Delta u_2\|^2_{L_2(Q_t)}$$

Since $\{\phi_i, u_i\} \in V(Q_T) \times V(Q_T)$

$\|\Delta u_2\|_{L_2(Q_T)} \leq$

$$C(\|h_1\|^2_{L_2(Q_T)}, \|h_2\|^2_{L_2(Q_T)}, \|\tilde{u}\|^2_{W_2^1(0,T;W_2^1(\Omega))}, \|\tilde{\phi}\|^{p+1}_{W_{p+1}^1(0,T;W_2^1(\Omega))}, \|u_0\|^2_{H^1(\Omega)}, \|\phi_0\|^2_{H^1(\Omega)}, \xi, \tau, l, K_2)$$

$$= C_1(K_2)$$

$$Y(t) \le \left\{ \left(\frac{6l+4\tau}{3l^2} \right) C_1(K_2) e^{\int_0^t \tilde{c}(s)ds} \right\} K^2$$

Moreover,

$$\|u_1 - u_2\|^2_{W_2^1(\Omega)}(t) \le \left\{ \left(\frac{6l+4\tau}{3l^2} \right) C_1(K_2) e^{\int_0^t \tilde{c}(s)ds} \right\} (K_1 - K_2)^2$$

and

$$\|\phi_1 - \phi_2\|^2_{W_2^1(\Omega)}(t) \le \left\{ \left(\frac{6l+4\tau}{3l^2} \right) C_1(K_2) e^{\int_0^t \tilde{c}(s)ds} \right\} (K_1 - K_2)^2$$

Hence we have proved the theorem.

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