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On strongly lacunary summability of sequences of sets

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Abstract

In this paper, we introduce the concept of Wijsman strongly lacunary summability for set sequences. Then, we discus its relation with Wijsman strongly Cesàro summability. Furthermore, we also give its relation with Wijsman almost convergence.

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1 Introduction

The concept of convergence of sequences of points has been extended by several authors to convergence of sequences of sets. The one of these such

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extensions considered in this paper is the concept of Wijsman convergence (see, [2,3,4,11,14,15,16]). Baronti and Papini studied the relationship between Wijsman convergence and other types of convergence for sequence of sets in [2]. In [11], Nuray and Rhoades extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems. In [14], Ulusu and Nuray defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wiijsman statistical convergence, which was defined by Nuray and Rhoades.

Freedman, et al. established the connection between the strongly Cesàro summable sequences space $|\sigma_1|$ and the strongly lacunary summable sequences space N_θ in their work [7] published in 1978.

This paper is inspired by the paper [7] by Freedman, Sember and Raphael entitled "Some Cesàro type summability spaces". We introduce the concept of Wijsman strongly lacunary summability for sequences of sets. Then, we discus its relation with Wijsman strongly Cesàro summability, which was defined by Nuray and Rhoades. Furthermore, we also give its relation with Wijsman almost convergence, which was again defined by Nuray and Rhoades.

Let us start with fundamental definitions from the literature;

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and let $q_r = \frac{k_r}{k_{r-1}}$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, A).$$

Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

The concepts of Wijsman strongly Cesàro summability was introduced by Nuray and Rhoades as follows: Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly Cesàro summable to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \to A[W\sigma_1]$ or $A_k \stackrel{[W\sigma_1]}{\to} A$.

Also the concept of Wijsman strongly almost convergence for sequences of sets was given by Nuray and Rhoades as follows: Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that $\{A_k\}$ is Wijsman strongly almost convergent to A if for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)| = 0$$

uniformly in i.

2 Main results

In this section, we will define Wijsman strongly lacunary summability of sequences of sets. Then, we will give its the relationship between the Wijsman strongly Cesàro summability and the Wijsman almost convergence.

Definition 2.1. *Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary summable to A if for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} d(x, A_k) = d(x, A).$$

In this case we write $A_k \to A(WN_\theta)$.

Definition 2.2. *Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly lacunary summable to A if for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \to A([WN]_\theta)$ or $A_k \overset{[WN]_\theta}{\to} A$.

The set of Wijsman strongly lacunary summable sequences will be denoted

$$[WN]_\theta = \left\{ \{A_k\} : \lim_r \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| = 0 \right\}.$$

Example 2.3. *Let $X = \mathbb{R}^2$ and we define a sequence $\{A_k\}$ as follows:*

$$A_k := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \frac{1}{k}\right\} & , \quad \text{if } k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r}\right], \\ \\ \{(0,0)\} & , \quad \text{otherwise.} \end{cases}$$

This sequence is Wijsman strongly lacunary summable to the set $A = \{(0,0)\}$ since

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, \{(0,0)\})| \leq \lim_{r \to \infty} \frac{1}{h_r} \cdot 2\sqrt{k_r - k_{r-1}} = 0.$$

i.e., $\{A_k\} \in [WN]_\theta$.

Definition 2.4. *Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets A, $A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly $p-$lacunary summable to A if for each p positive real number and for each $x \in X$,*

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)|^p = 0.$$

In this case we write $A_k \to A([WN]_\theta^p)$.

Lemma 2.5. *For any lacunary sequence $\theta = \{k_r\}$, $[W\sigma_1] \subseteq [WN]_\theta$ if and only if $\liminf_r q_r > 1$.*

Proof. Suppose first that $\liminf_r q_r > 1$, then there exists a $\lambda > 0$ such that
$q_r \geq 1 + \lambda$ for all $r \geq 1$, which implies that

$$\frac{k_r}{h_r} \leq \frac{1+\lambda}{\lambda} \qquad \text{and} \qquad \frac{k_{r-1}}{h_r} \leq \frac{1}{\lambda}.$$

If $\{A_k\} \in |W\sigma_1|$, then we can write

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}|d(x,A_k)-d(x,A)| \\
&\quad= \frac{1}{h_r}\sum_{i=1}^{k_r}|d(x,A_i)-d(x,A)| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}}|d(x,A_i)-d(x,A)| \\
&\quad= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{i=1}^{k_r}|d(x,A_i)-d(x,A)|\right) - \frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}|d(x,A_i)-d(x,A)|\right).
\end{aligned}$$

Since $\{A_k\} \in |W\sigma_1|$, the terms

$$\frac{1}{k_r}\sum_{i=1}^{k_r}|d(x,A_i)-d(x,A)| \qquad \text{and} \qquad \frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}}|d(x,A_i)-d(x,A)|$$

both convergent to 0, and it follows that

$$\frac{1}{h_r}\sum_{k\in I_r}|d(x,A_k)-d(x,A)| \to 0,$$

that is, $\{A_k\} \in [WN]_\theta$.

Conversely, suppose that $[W\sigma_1] \subseteq [WN]_\theta$ and $\liminf_r q_r = 1$. Since θ is lacunary sequence, we can select a subsequence $\{k_{r_j}\}$ of θ satisfying

$$\frac{k_{r_j}}{k_{r_j-1}} < 1+\frac{1}{j} \qquad \text{and} \qquad \frac{k_{r_j-1}}{k_{r_{j-1}}} > j, \qquad \text{where } r_j \geq r_{j-1}+2.$$

Now we define a sequence $\{A_k\}$ as follows:

$$A_k := \begin{cases} \left\{(x,y)\in\mathbb{R}^2,\ \ x^2+(y-1)^2=\frac{1}{k^4}\right\} & , \ \text{if } k \in I_{r_j}, \qquad j=1,2,\cdots, \\ \\ \{(0,0)\} & , \ \textit{otherwise}. \end{cases}$$

Then, for at least one $x \in X$, we have

$$\frac{1}{h_{r_j}}\sum_{k\in I_{r_j}}|d(x,A_k)-d(x,\{(0,0)\})| = T \qquad \text{for } j=1,2,\cdots, \qquad (T\in\mathbb{R}^+)$$

and

$$\frac{1}{h_r}\sum_{k\in I_r}|d(x,A_k)-d(x,\{(0,0)\})| = 0 \qquad \text{for } r \neq r_j.$$

It follows that $\{A_k\} \notin [WN_\theta]$ since

$$\frac{1}{h_r}\sum_{k\in I_r}|d(x,A_k)-d(x,\{(0,0)\})| \neq 0.$$

However, $\{A_k\}$ is Wijsman strongly Cesàro summable, since if n is any sufficiently large integer we can find the unique j for which $k_{r_j-1} < n < k_{r_{j+1}-1}$ and write

$$\frac{1}{n}\sum_{k=1}^{n}|d(x,A_k)-d(x,\{(0,0)\})| \leq \frac{k_{r_{j-1}}+h_{r_j}}{k_{r_j-1}} \leq \frac{1}{j}+\frac{1}{j}=\frac{2}{j}.$$

As $n\to\infty$ it follows that also $j\to\infty$. Hence $\{A_k\}\in|W\sigma_1|$. This contradicts to our assumption. Therefore, $\liminf_r q_r > 1$. $\square$

Lemma 2.6. *For any lacunary sequence $\theta=\{k_r\}$, $[WN]_\theta \subseteq [W\sigma_1]$ if and only if* $\limsup_r q_r < \infty$.

Proof. Suppose first that $\limsup_r q_r < \infty$, then there exists $M>0$ such that $q_r < M$ for all r. Let $\{A_k\}\in[WN_\theta]$ and $\varepsilon>0$. Then we can find $R>0$ and $K>0$ such that

$$\sup_{i\geq R}\tau_i < \varepsilon \qquad \text{and} \qquad \tau_i < K \qquad \text{for all } i=1,2,\cdots,$$

where

$$\tau_r = \frac{1}{h_r}\sum_{I_r}|d(x,A_k)-d(x,A)|.$$

If t is any integer with $k_{r-1} < t \leq k_r$, where $r>R$, we can write

$$\begin{aligned}
&\frac{1}{t}\sum_{i=1}^{t}|d(x,A_i)-d(x,A)| \\
&\leq \frac{1}{k_{r-1}}\sum_{i=1}^{k_r}|d(x,A_i)-d(x,A)| \\
&= \frac{1}{k_{r-1}}\Bigg(\sum_{I_1}|d(x,A_i)-d(x,A)| + \sum_{I_2}|d(x,A_i)-d(x,A)| + \cdots \\
&\qquad + \sum_{I_{r-1}}|d(x,A_i)-d(x,A)| + \sum_{I_r}|d(x,A_i)-d(x,A)|\Bigg)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{k_1}{k_{r-1}}.\tau_1 + \frac{k_2 - k_1}{k_{r-1}}.\tau_2 + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}}\tau_R \\
&\qquad + \frac{k_{R+1} - k_R}{k_{r-1}}\tau_{R+1} + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}\tau_r \\
&\leq \left(\sup_{i\geq 1}\tau_i\right)\frac{k_R}{k_{r-1}} + \left(\sup_{i\geq R}\tau_i\right)\frac{k_r - k_R}{k_{r-1}} \\
&\leq K \cdot \frac{k_R}{k_{r-1}} + \varepsilon \cdot M.
\end{aligned}$$

Since $k_{r-1} \to \infty$ as $t \to \infty$, it follows that

$$\frac{1}{t}\sum_{i=1}^{t} |d(x, A_i) - d(x, A)| \to 0$$

and, consequently, $\{A_k\} \in |W\sigma_1|$.

Conversely, we assume that $[WN_\theta] \subseteq [W\sigma_1]$ and $\limsup_r q_r = \infty$. To proceed we construct a sequence in $[WN_\theta]$ that is not Wijsman strongly Cesàro summable. First select a subsequence (k_{r_j}) of the lacunary sequence $\theta = \{k_r\}$ such that $q_{r_j} > j$, and then we define the sequence $\{A_k\}$ as follows:

$$A_k := \begin{cases} \{1\} & , \quad \text{if } k_{r_j-1} < k \leq 2k_{r_j-1}, \qquad j = 1, 2, \cdots, \\ \\ \{0\} & , \quad \text{otherwise.} \end{cases}$$

Then, we have

$$\tau_{r_j} = \frac{1}{h_{r_j}}\sum_{I_{r_j}} |d(x, A_k) - d(x, \{0\})| = \frac{k_{r_j-1}}{k_{r_j-1} - k_{r_{j-1}}} < \frac{1}{j-1},$$

and

$$\tau_r = \frac{1}{h_r}\sum_{I_r} |d(x, A_k) - d(x, \{0\})| = 0, \qquad \text{for } r \neq r_j.$$

It follows that $\{A_k\} \in [WN_\theta]$ since

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |d(x, A_k) - d(x, \{0\})| = 0.$$

Observe next that any sequence in $|W\sigma_1|$ consisting of only $\{0\}$'s and $\{1\}$'s has an associated Wijsman strongly limit A which is $\{0\}$ or $\{1\}$. For the sequence $\{A_k\}$ above, and $k = 1, 2, \cdots, k_{r_j}$,

$$\frac{1}{k_{r_j}}\sum_{k} |d(x, A_k) - d(x, \{1\})| \geq \frac{(k_{r_j} - 2k_{r_j-1})}{k_{r_j}} = 1 - \frac{2k_{r_j-1}}{k_{r_j}} > 1 - \frac{2}{j}$$

which converges to $\{1\}$, and, for $k = 1, 2, ...2k_{r_j-1}$,

$$\frac{1}{2k_{r_j-1}} \sum_{k} |d(x, A_k) - d(x, \{0\})| \geq \frac{k_{r_j-1}}{2k_{r_j-1}} = \frac{1}{2},$$

and it follows that $\{A_k\} \notin |W\sigma_1|$. $\square$

Combining Lemma 1 and Lemma 2 we have.

Theorem 2.7. *Let $\theta = \{k_r\}$ be a lacunary sequence, then $[WN]_\theta = [W\sigma_1]$ if and only if*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty.$$

Proof. This follows from Lemma 1 and Lemma 2. $\square$

Theorem 2.8. *Let $\{A_k\} \in [W\sigma_1] \cap [WN]_\theta$. If $A_k \stackrel{[W\sigma_1]}{\rightarrow} A$ and $A_k \stackrel{[WN]_\theta}{\rightarrow} B$, then $A = B$.*

Proof. Let $A_k \stackrel{[W\sigma_1]}{\rightarrow} A$, $A_k \stackrel{[WN]_\theta}{\rightarrow} B$, and suppose that $A \neq B$. We can write

$$\begin{aligned} \upsilon_r + \tau_r &= \tfrac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| + \tfrac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, B)| \\ &\geq \tfrac{1}{h_r} \sum_{k \in I_r} |d(x, A) - d(x, B)| \\ &= |d(x, A) - d(x, B)| \end{aligned}$$

where

$$\upsilon_r = \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \qquad \text{and} \qquad \tau_r = \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, B)|.$$

Since $\{A_k\} \in [WN]_\theta$, $\tau_r \to 0$. Thus for sufficiently large r we must have

$$\upsilon_r > \frac{1}{2} |d(x, A) - d(x, B)|.$$

Observe that

$$\begin{aligned}
\frac{1}{k_r}\sum_{i=1}^{k_r}|d(x,A_i)-d(x,A)| \geq& \frac{1}{k_r}\sum_{I_r}|d(x,A_i)-d(x,A)| \\
=& \frac{k_r-k_{r-1}}{k_r}.\upsilon_r \\
=& \left(1-\frac{1}{q_r}\right).\upsilon_r \\
>& \frac{1}{2}\left(1-\frac{1}{q_r}\right).|d(x,A)-d(x,B)|
\end{aligned}$$

for sufficiently large r. Since $\{A_k\} \in [W\sigma_1]$, the left hand side of the inequality above convergent to 0, so we must have $q_r \to 1$. But this implies, by proof of Lemma 2, that

$$[WN]_\theta \subset [W\sigma_1].$$

That is, we have

$$A_k \overset{[WN]_\theta}{\longrightarrow} B \Rightarrow A_k \overset{[W\sigma_1]}{\longrightarrow} B,$$

and therefore

$$\frac{1}{t}\sum_{i=1}^{t}|d(x,A_i)-d(x,B)| \to 0.$$

Then we have

$$\frac{1}{t}\sum_{i=1}^{t}|d(x,A_i)-d(x,B)| + \frac{1}{t}\sum_{i=1}^{t}|d(x,A_i)-d(x,A)| \geq |d(x,A)-d(x,B)| > 0,$$

which yields a contradiction to our assumption, since both terms on the left hand side convergent to 0. That is, for each $x \in X$,

$$|d(x,A)-d(x,B)| = 0,$$

and therefore $A = B$. $\square$

Nuray and Rhoades (Nuray and Rhoades, 2012) introduced the notion of strongly almost convergence for sequence of sets as follows: For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly almost convergent to A if for each $x \in X$,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)| = 0$$

uniformly in i. In this case we write $A_k \to A([WAC])$.

The set of Wijsman strongly almost convergent sequences will be denoted

$$[WAC] := \left\{ \{A_k\} : \lim_n \frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)| = 0 \right\}.$$

Lemma 2.9. *For any lacunary sequence $\theta = \{k_r\}$, $[WAC] \subset [WN]_\theta$.*

Proof. If $A_k \in [WAC]$, then there exist $N > 0$ and a non-empty closed subset $A \subset X$ such that for each $\varepsilon > 0$ and for each $x \in X$,

$$\frac{1}{n} \sum_{k=1}^{n} |d(x, A_{k+i}) - d(x, A)| < \varepsilon, \qquad \text{for } n > N,$$

uniformly in i. We can choose $R > 0$ such that $r \geq R$ implies $h_r > N$ since θ is lacunary sequence and, consequently,

$$\tau_r = \frac{1}{h_r} \sum_{I_r} |d(x, A_k) - d(x, A)| < \varepsilon.$$

Thus $\{A_k\} \in [WN]_\theta$. To obtain a sequence in $[WN]_\theta$ and not in $[WAC]$, we define the sequence $\{A_k\}$ as follows:

$$A_k = \begin{cases} \{1\} \quad , & \text{if } \; k_{r-1} < k \leq k_{r-1} + \sqrt{h_r} \\ \\ \{0\} \quad , & \text{otherwise.} \end{cases}$$

Then there are arbitrarily long strings of consecutive $\{0\}$'s in the coordinates of $\{A_k\}$, as well as arbitrarily long strings of consecutive $\{1\}$'s, from which it follows that $\{A_k\}$ is not Wijsman strongly almost convergent. However, $\{A_k\} \in [WN]_\theta$ since

$$\lim_{r\to\infty} \tau_r = \lim_{r\to\infty} \frac{1}{h_r} \sum_{I_r} |d(x, A_k) - d(x, \{0\})| = \lim_{r\to\infty} \frac{\left[\left|\sqrt{h_r}\right|\right]}{h_r} = 0.$$

$\square$

Theorem 2.10. $[WAC] = \bigcap [WN]_\theta$.

Proof. We need show only that if $\{A_k\} \notin [WAC]$ there exists a lacunary θ such that $\{A_k\} \notin [WN]_\theta$. We may assume that $\{A_k\}$ is Wijsman strongly Cesàro Summable. (otherwise $\{A_k\} \notin [WN]_{(\theta=2^r)}$). Consequently, there exists a unique non-empty closed subset $A \subseteq X$ for which

$$\lim_{t\to\infty} \frac{1}{t} \sum_{k=1}^{t} |d(x, A_k) - d(x, A)| = 0.$$

Let $\{A_k\} \notin [WAC]$. Then there exists $\varepsilon > 0$ for each $x \in X$ such that for each N there exist $n > N$ and an integer m for which,

$$\frac{1}{n} \sum_{k=m+1}^{m+n} |d(x, A_k) - d(x, A)| \geq \varepsilon, \qquad m = 1, 2, \cdots .$$

Then we can select sequences (m_r) and (n_r) such that $n_r \to \infty$ and

$$\frac{1}{n_r} \sum_{k=m_r+1}^{m_r+n_r} |d(x, A_k) - d(x, A)| \geq \varepsilon.$$

Since $\{A_k\} \in [W\sigma_1]$ it follows that $m_r \to \infty$ also. We construct a lacunary sequence $\theta = \{k_r\}$ as follows:

$$\begin{array}{lll}
 & k_1 = m_1, & \\
k_2 = m_1 + n_1, & k_3 = m_{r_2}, & \text{where} \quad m_{r_2} > 2k_2 \\
k_4 = m_{r_2} + n_{r_2}, & k_5 = m_{r_3}, & \text{where} \quad m_{r_3} > 2k_4 \\
k_6 = m_{r_3} + n_{r_3}, & \vdots & \\
\vdots & k_{2i-1} = m_{r_i}, & \text{where} \quad m_{r_i} > 2k_{2i-2} \\
k_{2i} = m_{r_i + n_{r_i}}, & & \\
\vdots & &
\end{array}$$

Then clearly $\theta = \{k_r\}$ is lacunary and, for $r = 2j$,

$$\tau_r = \frac{1}{n_r} \sum_{k=m_{r_j}+1}^{m_{r_j}+n_{r_j}} |d(x, A_k) - d(x, A)| \geq \varepsilon.$$

It follows from Theorem 2 that $\{A_k\} \notin [WN]_\theta$. $\square$

We now consider the inclusion of $[WN]_{\theta'}$, by $[WN]_{\theta}$, where θ' is lacunary refinement of θ. Recall (Freedman, et al., 1978) that the lacunary sequence $\theta' = \{k'_r\}$ is called a lacunary refinement of the lacunary sequence $\theta = \{k_r\}$ if $\{k_r\} \subseteq \{k'_r\}$.

Lemma 2.11. *If θ', is a lacunary refinement of θ and if $A_k \notin [WN]_{\theta}$, then*
$A_k \notin [WN]_{\theta'}$.

Proof. Let $A_k \notin [WN]_{\theta}$. Then, for any non-empty closed subset $A \subseteq X$ there exists $\varepsilon > 0$ and a subsequence (k_{r_j}) of (k_r) such that

$$\tau_{r_j} = \frac{1}{h_{r_j}} \sum_{k=1}^{k_{r_j}} |d(x, A_k) - d(x, A)| \geq \varepsilon.$$

Writing

$$I_{r_j} = I'_{s+1} \cup I'_{s+2} \cup ... \cup I'_{s+p}$$

where

$$k_{r_j - 1} = k'_s < k'_{s+1} < ... < k'_{s+p} = k_{r_j}.$$

Then we have

$$\tau_{r_j} = \frac{\sum\limits_{I'_{s+1}} |d(x, A_k) - d(x, A)| + \sum\limits_{I'_{s+2}} |d(x, A_k) - d(x, A)| + ... + \sum\limits_{I'_{s+p}} |d(x, A_k) - d(x, A)|}{h'_{s+1} + h'_{s+2} + ... + h'_{s+p}}.$$

It follows from [Lemma 3.2, in (7)] that

$$\frac{1}{h'_{s+j}} \sum_{I'_{s+j}} |d(x, A_k) - d(x, A)| \geq \varepsilon$$

for some j and, consequently, $\{A_k\} \notin [WN]_{\theta'}$. $\square$

Theorem 2.12. $[WAC] = \bigcap \{[WN]_{\theta} : \lim_r q_r = 1\}$

Proof. If $\{A_k\} \notin [WAC]$, then there exists, by Theorem 3, a lacunary sequence θ such that $\{A_k\} \notin [WN]_{\theta}$. If we define

$$\theta' = (k'_r) = (k_r) \cup \left\{n^2 : [(n-1)^2, (n+1)^2] \cap (k_r) = \emptyset\right\},$$

then θ' is lacunary sequence, $\lim_r q'_r = 1$, and, by Lemma 4, $\{A_k\} \notin [WN]_{\theta'}$. $\square$

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