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A CHARACTERIZATION OF HOLOMORPHIC BIVARIATE FUNCTIONS OF BOUNDED INDEX

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ABSTRACT. The following notion of bounded index for complex entire functions was presented by Lepson. function $f(z)$ is of bounded index if there exists an integer N independent of z, such that

$$\max_{\{l:0\leq l\leq N\}}\left\{\frac{\left|f^{(l)}(z)\right|}{l!}\right\}\geq\frac{\left|f^{(n)}(z)\right|}{n!}\quad\text{for all } n.$$

The main goal of this paper is extend this notion to holomorphic bivariate function. To that end, we obtain the following definition. A holomorphic bivariate function is of bounded index, if there exist two integers M and N such that M and N are the least integers such that

$$\max_{\{(k,l):0,0\leq k,l\leq M,N\}}\left\{\frac{\left|f^{(k,l)}(z,w)\right|}{k!\,l!}\right\}\geq\frac{\left|f^{(m,n)}(z,w)\right|}{m!\,n!}\quad\text{for all } m \text{ and } n.$$

Using this notion we present necessary and sufficient conditions that ensure that a holomorphic bivariate function is of bounded index.

1. Introduction and preliminary results

An entire function $f(z)$ is of bounded index if there exists an integer N independent of z, such that

$$\max_{\{l:0\leq l\leq N\}}\left\{\frac{\left|f^{(l)}(z)\right|}{l!}\right\}\geq\frac{\left|f^{(n)}(z)\right|}{n!}\quad\text{for all } n.$$

The least such integer N is called the index of $f(z)$. The main goal of this paper is to extend this notion to multidimensional space. To accomplish this we begin with the presentation of the following notion. Thus, if $f(z,w)$ is a holomorphic function in the bicylinder

$$\{|z-a|<r_1,\ |w-b|<r_2\}$$

then at all point of the bicylinder

$$f(z,w)=\sum_{k,l=0,0}^{\infty,\infty}c_{k,l}(z-a)^k(w-b)^l$$

where

$$c_{k,l}=\frac{1}{k!\,l!}\left[\frac{\partial^{k+l}f(z,w)}{\partial w^k\partial z^l}\right]_{z=a;w=b}=\frac{1}{k!\,l!}f^{(k,l)}(a,b).$$

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Using this notion we present the following notion bounded index for holomorphic bivariate function A holomorphic bivariate function $f(z,w)$ is of bounded index if there exist integers M and N independent of z and w, respectively such that

$$\max_{\{(k,l):0,0\le k,l\le M,N\}}\left\{\frac{\left|f^{(k,l)}(z,w)\right|}{k!\,l!}\right\}\ge\frac{\left|f^{(m,n)}(z,w)\right|}{m!\,n!}\quad\text{for all } m \text{ and } n.$$

We shall say the bivariate holomorphic function f is of bounded index (M,N), if M and N are the smallest integers such that the above inequality holds. Using this notion we present necessary and sufficient conditions that ensure that f is of bounded index.

Let r, s be two positive real number and h and $\bar{h}$ be two positive integers and let z_0 and w_0 be two complex numbers then for any holomorphic bivariate entire function $f(z,w)$ with $m=0,1,2,\dots,h$ and $n=0,1,2,\dots,\bar{h}$ define

$$R_{m,n}(r,s,h,\bar{h},z_0,w_0)=\max\left\{\frac{\left|f^{(k,l)}(z,w)\right|}{k!\,l!}\ :\ |z-z_0|\le\frac{mr}{h},\ |w-w_0|\le\frac{ns}{\bar{h}},\ k,l=0,1,\dots\right\}.$$

2. Main results

Lemma 2.1. *If $f(z,w)$ is a holomorphic bivariate entire of index (M,N) and if r, s, h, and $\bar{h}$ are such that*

$$\frac{r}{h}\le\frac{1}{4(M+1)}\quad\textit{and}\quad\frac{s}{\bar{h}}\le\frac{1}{4(N+1)}$$

then

(1)

$$R_{\alpha,\beta}(r,s,h,\bar{h},z_0,w_0)\le 2R_{\alpha-1,\beta-1}(r,s,h,\bar{h},z_0,w_0)$$

for any complex numbers z_0 and w_0 and all $\alpha\in[1,h]$ and all $\beta\in[1,\bar{h}]$ and

(2)

$$\max_{k,l\le M,N;|z-z_0|=r,|w-w_0|=s}\left\{\frac{\left|f^{(k,l)}(z,w)\right|}{k!\,l!}\right\}\le 2\max_{k,l\le M,N}\left\{\frac{\left|f^{(k,l)}(z_0,w_0)\right|}{k!\,l!}\right\}.$$

Proof. Let us establish (1), suppose there exist integers $\alpha\in[1,h]$ and $\beta\in[1,\bar{h}]$ and complex numbers z_0 and w_0 such that

$$R_{\alpha,\beta}(r,s,h,\bar{h},z_0,w_0)>2R_{\alpha-1,\beta-1}(r,s,h,\bar{h},z_0,w_0).$$

Now

$$R_{\alpha,\beta}(r,s,h,\bar{h},z_0,w_0)=\frac{\left|f^{(k_\alpha,l_\beta)}(z_\alpha,w_\beta)\right|}{k_\alpha!\,l_\beta!}$$

for some complex numbers z_α and w_β with $|z_\alpha-z_0|=\frac{\alpha r}{h}$ and $|w_\beta-w_0|=\frac{\beta s}{\bar{h}}$ and some integers k_α and l_β with $k_\alpha\in[0,h]$ and $l_\beta\in[0,\bar{h}]$. Let us choose $z_\alpha^{'}$ and $w_\beta^{'}$ as follow:

$$z_\alpha^{'}=z_0+\frac{\alpha-1}{\alpha}(z_\alpha-z_0)$$

and

$$w_\beta^{'}=w_0+\frac{\beta-1}{\beta}(w_\beta-w_0).$$

Thus

$$\frac{\left|f^{(k_\alpha,l_\beta)}(z_\alpha^{'},w_\beta^{'})\right|}{k_\alpha!\,l_\beta!}\leq R_{\alpha-1,\beta-1}(r,s,h,\bar{h},z_0,w_0),$$

because,

$$\left|z_\alpha^{'}-z_0\right|=\frac{(\alpha-1)r}{h}\quad\text{and}\quad\left|w_\beta^{'}-w_0\right|=\frac{(\beta-1)s}{\bar{h}}.$$

Thus

$$\frac{\left|f^{(k_\alpha,l_\beta)}(z_\alpha,w_\beta)\right|}{k_\alpha!\,l_\beta!}-\frac{\left|f^{(k_\alpha,l_\beta)}(z_\alpha^{'},w_\beta^{'})\right|}{k_\alpha!\,l_\beta!}\geq R_{\alpha,\beta}(r,s,h,\bar{h},z_0,w_0)-R_{\alpha-1,\beta-1}(r,s,h,\bar{h},z_0,w_0).$$

In addition, there exist $\delta=z_\alpha^{'}+d(z_\alpha-z_\alpha^{'})$ and $\rho=w_\beta^{'}+\bar{d}(w_\beta-w_\beta^{'})$ for some $d,\bar{d}\in(0,1)$ such that

$$\begin{aligned}\frac{\left|f^{(k_\alpha+1,l_\beta+1)}(\delta,\rho)\right|}{(k_\alpha+1)!\,(l_\beta+1)!}&=\frac{1}{(k_\alpha+1)!\,(l_\beta+1)!}\frac{\left|f^{(k_\alpha,l_\beta)}(z_\alpha,w_\beta)\right|-\left|f^{(k_\alpha,l_\beta)}(z_\alpha^{'},w_\beta^{'})\right|}{|z_\alpha-z_\alpha^{'}|\,|w_\alpha-w_\alpha^{'}|}\\&\geq\frac{1}{(k_\alpha+1)(l_\beta+1)}\left[\frac{R_{\alpha,\beta}(r,s,h,\bar{h},z_0,w_0)-R_{\alpha-1,\beta-1}(r,s,h,\bar{h},z_0,w_0).}{|z_\alpha-z_\alpha^{'}|\,|w_\alpha-w_\alpha^{'}|}\right]\\&\geq\frac{1}{(N+1)(M+1)}\frac{\frac{1}{2}R_{\alpha,\beta}(r,s,h,\bar{h},z_0,w_0)}{\frac{1}{2(M+1)}\frac{r}{h}\frac{1}{2(N+1)}\frac{s}{\bar{h}}}\\&\geq 2R_{\alpha,\beta}(r,s,h,\bar{h},z_0,w_0).\end{aligned}$$

Since

$$|\delta-z_0|<\frac{\alpha r}{h}\quad\text{and}\quad|\rho-w_0|=\frac{\beta s}{\bar{h}},$$

we have a contradiction, thus

$$R_{\alpha,\beta}(r,s,h,\bar{h},z_0,w_0)\leq 2R_{\alpha-1,\beta-1}(r,s,h,\bar{h},z_0,w_0).$$

The establishment of (2) one should observe that the following clearly from part (1)

$$R_{h,\bar{h}}(r,s,h,\bar{h},z_0,w_0)\leq 2^{h+\bar{h}-2}R_{0,0}(r,s,h,\bar{h},z_0,w_0).$$

and since

$$\begin{aligned}R_{h,\bar{h}}(r,s,h,\bar{h},z_0,w_0)&=\max_{k,l\leq M,N;|z-z_0|=r,|w-w_0|=s}\left\{\frac{\left|f^{(k,l)}(z,w)\right|}{k!\,l!}\right\}\\&\leq 2\max_{k,l\leq M,N}\left\{\frac{\left|f^{(k,l)}(z_0,w_0)\right|}{k!\,l!}\right\}\\&=R_{h,\bar{h}}(r,s,h,\bar{h},z_0,w_0).\end{aligned}$$

□

Theorem 2.1. *A holomorphic bivariate entire function $f(z,w)$ is of bounded index if and only if for each ordered pair (r,s) with $r>0$ and $s>0$ there exist integers $N=N(r)$ and $M=M(s)$ and constants $\bar{N}=\bar{N}(r)>0$ and $\bar{M}=\bar{M}(s)>0$ such that for complex number z and w there exist integers $k=k(z)$ and $l=l(w)$ with $k\in[0,N]$ and $l\in[0,M]$ and*

$$\max_{|\delta-z|=r;|\rho-w|=s}\left\{\left|f^{(k,l)}(\delta,\rho)\right|\right\}\leq\bar{N}\bar{M}\left|f^{(k,l)}(z,w)\right|.$$

Proof. For the first part let us establish that the holomorphic bivariate entire function $f(z,w)$ is of bounded index. Let $r>0$, $s>0$, and let z and w be complex numbers. Let us also define $M_{l,k}(f,z,w,r,s)$ for $k,l=0,1,2,\dots$ by

$$M_{k,l}(f,z,w,r,s)=\max_{|\delta-z|=r;|\rho-w|=s}\left\{\left|f^{(k,l)}(\delta,\rho)\right|\right\}.$$

Without loss of generality we may assume $r=s=2$ thus there exist integers $N=N(2)$ and $M=M(2)$; and constants $\bar{N}=\bar{N}(r)>0$ and $\bar{M}=\bar{M}(s)>0$ such that for complex number z and w there exist integers $k=k(z)\le N$ and $l=l(w)\le M$ with

$$M_{k,l}(f,z,w,2,2)=\bar{N}\bar{M}\left|f^{(k,l)}(z,w)\right|.$$

Also there exist integers $n>0$ and $m>0$ such that

$$\frac{N!\,\bar{N}}{2^n}<1 \quad\text{and}\quad \frac{M!\,\bar{M}}{2^m}<1.$$

Let us now show that the index of $f(z,w)$ does not exceed $(n+N,m+M)$ in each term. Let $\bar{n}\ge n+N$, $\bar{m}\ge m+M$ and consider complex numbers z_0 and w_0. Now there exist integers $k_0=k(z_0)\le N$ and $l_0=l(w_0)\le M$ such that

$$M_{k_0,l_0}(f,z_0,w_0,2,2)=\bar{N}\bar{M}\left|f^{(k_0,l_0)}(z_0,w_0)\right|.$$

By a generalization of the Cauchy inequality we have the following, for an holomorphic bivariate entire function $g(z,w)$

$$\left|g^{(k,l)}(z,w)\right|\le k!\,l!\,R^kS^l\max_{|\delta-z|=R;|\rho-w|=S}\{|g(\delta,\rho)|\}$$

for $k,l=0,1,2,\dots$, and any $R>0$ and $S>0$. Thus, for $g(z,w)=f^{(k_0,l_0)}(z,w)$ and $R=S=2$,

$$\begin{aligned}\frac{\left|f^{(k_0+k,l_0+l)}(z,w)\right|}{k!\,l!}&\le 2^{-(k+l)}\max_{|\delta-z|=2;|\rho-w|=2}\left\{\left|f^{(k_0,l_0)}(\delta,\rho)\right|\right\}\\&=2^{-(k+l)}M_{k_0,l_0}(f,z,w,2,2).\end{aligned}$$

Thus

$$\begin{aligned}\frac{\left|f^{(m,n)}(z_0,w_0)\right|}{m!\,n!}&\le\frac{\left|f^{(k_0+m-k_0,l_0+n-l_0)}(z_0,w_0)\right|}{(m+k_0)!\,(n+l_0)}\le\frac{M_{k_0,l_0}(f,z,w,2,2)}{2^{m+n-k_0-l_0}}\\&\le\frac{2^{k_0+l_0}\bar{M}\bar{N}\left|f^{(k_0,l_0)}(z_0,w_0)\right|}{2^{m+n}}\le\frac{2^{M+N}\bar{M}\bar{N}\left|f^{(k_0,l_0)}(z_0,w_0)\right|}{2^{m+n}}\\&\le\frac{\bar{M}\bar{N}\left|f^{(k_0,l_0)}(z_0,w_0)\right|}{2^{\bar{m}+\bar{n}}}\le\frac{\left|f^{(k_0,l_0)}(z_0,w_0)\right|}{N!\,M!}\\&\le\frac{\left|f^{(k_0,l_0)}(z_0,w_0)\right|}{k_0!\,l_0!}.\end{aligned}$$

Thus the index of $f(z,w)$ at (z_0,w_0) does not exceed $(\bar{m}+M,\bar{n}+N)$ and since (z_0,w_0) was arbitrary, the index of f is bounded. Now suppose $f(z,w)$ is of bounded index (K,L). Thus for $r>0$ and $s>0$ let's choose $M=M(r)=K$, $N=N(s)=L$ and $\bar{M}=\bar{M}(r)=2^{\Delta+\bar{\Delta}}K!\,L!$ for some positive integers Δ and $\bar{\Delta}$ such that

$$\frac{rs}{\Delta\bar{\Delta}}\le\frac{1}{16(L+1)(K+1)}.$$

For complex numbers z_0 and w_0 let $k = k(z_0)$ and $l = l(w_0)$ be the index of f at (z_0, w_0). Thus $k \leq M = K$ and $l \leq N = L$. Thus by part (2) of Lemma 2.1

$$\max_{\alpha\beta \leq K,L;|z-z_0|=r,|w-w_0|=s} \left\{ \frac{\left|f^{(\alpha,\beta)}(z,w)\right|}{\alpha!\,\beta!} \right\} \leq 2^{\Delta+\bar{\Delta}} \max_{\alpha,\beta\leq K,L} \left\{ \frac{\left|f^{(\alpha,\beta)}(z_0,w_0)\right|}{\alpha!\,\beta!} \right\}$$
$$= 2^{\Delta+\bar{\Delta}} \left\{ \frac{\left|f^{(k,l)}(z_0,w_0)\right|}{k!\,l!} \right\}.$$

Therefore,

$$M_{k,l}(f,z_0,w_0,r,s) = \max_{|z-z_0|=r,|w-w_0|=s} \left\{ \left|f^{(k,l)}(z,w)\right| \right\}$$
$$\leq M!\,N! \max_{\alpha\beta\leq N,M;|z-z_0|=r,|w-w_0|=s} \left\{ \frac{\left|f^{(\alpha,\beta)}(z,w)\right|}{\alpha!\,\beta!} \right\}$$
$$\leq M!\,N!\,2^{\Delta+\bar{\Delta}} \left\{ \frac{\left|f^{(k,l)}(z_0,w_0)\right|}{k!\,l!} \right\} \leq \bar{N}\bar{M} \left|f^{(k,l)}(z_0,w_0)\right|.$$

Thus for each positive pair (r,s) there exist integers $N = N(r)$ and $M = M(s)$ and constants $\bar{N} = \bar{N}(r)$, $\bar{M} = \bar{M}(s)$ such that for each pair of complex numbers (z,w) there exist $l = l(z_0) \leq N$ and $k = k(s) \leq M$ with

$$M_{k,l}(f,z_0,w_0,r,s) = \bar{N}\bar{M} \left|f^{(k,l)}(z_0,w_0)\right|.$$

□

Theorem 2.2. *If the holomorphic bivariate $f(z,w)$ is of bounded index, then $g(z,w) = f(az+b, cw+d)$ is of bounded index for any complex numbers a,b,c and d.*

P r o o f. Without loss of generality we can assume $a \neq 0$ and $c \neq 0$ otherwise we have a constant function. We can also assume $b = d = 0$. Note the index of $f(z+b, w+d)$ at (z_0, w_0) is the same as the index of $f(z,w)$ at (z_0+b, w_0+d). Since $f(z,w)$ is of bounded index by Theorem 2.1 each ordered pair (r,s) with $r > 0$ and $s > 0$ there exist integers $N = N(r)$ and $M = M(s)$ and constants $\bar{N} = \bar{N}(r) > 0$ and $\bar{M} = \bar{M}(s) > 0$ such that for complex number z and w there exist integers $k = k(z) \leq N$ and $l = l(w) \leq M$ with

$$M_{k,l}(f,z,w,r,s) = \bar{N}\bar{M} \left|f^{(k,l)}(z_0,w_0)\right|.$$

Now, for $r = |a|\,r_0$ and $s = |c|\,s_0$ with $z = az_0$ and $w = cw_0$. Thus we obtain the following

$$M_{k,l}(g,z_0,w_0,r_0,s_0) = \max_{|\delta-z_0|=r_0,|\rho-w_0|=s_0} \left\{ \left|g^{(k,l)}(\delta,\rho)\right| \right\}$$
$$= \max_{|\delta-z_0|=r_0,|\rho-w_0|=s_0} \left\{ \left|a^k c^l f^{(k,l)}(\delta,\rho)\right| \right\}$$
$$= |a|^k\,|c|^l \max_{\left|\bar{\delta}-az_0\right|=|a|r_0,|\bar{\rho}-cw_0|=|c|s_0} \left\{ \left|f^{(k,l)}(\bar{\delta},\bar{\rho})\right| \right\}$$
$$= |a|^k\,|c|^l \max_{\left|\bar{\delta}-z\right|=r,|\bar{\rho}-w|=s} \left\{ \left|f^{(k,l)}(\bar{\delta},\bar{\rho})\right| \right\}$$
$$= |a|^k\,|c|^l\,M_{k,l}(f,z,w,r,s) = |a|^k\,|c|^l\,\bar{N}\bar{M} \left|f^{(k,l)}(z,w)\right|$$
$$= \bar{N}\bar{M} \left|a^k c^l f^{(k,l)}(z,w)\right| = \bar{N}\bar{M} \left|g^{(k,l)}(z,w)\right|.$$

Thus for each positive pair (r_0, s_0) there exist integers $\bar{K} = \bar{K}(r_0) = N(|a|\, r_0)$ and $\bar{L} = \bar{L}(s_0) = M(|c|\, s_0)$ and constants $\Gamma = \Gamma(r_0) = \bar{N}(|a|\, r_0)$ and $\bar{\Gamma} = \bar{\Gamma}(s_0) = \bar{M}(|c|\, s_0)$ such that for each complex numbers z_0 and w_0 there exist integers $m = m(z_0) = k(az_0) \leq \bar{K}$ and $n = n(w_0) = k(cw_0) \leq \bar{L}$ with

$$M_{m,n}(g, z_0, w_0, r_0, s_0) \leq \bar{N}\bar{M} \left| g^{(k,l)}(z_0, w_0) \right| \leq \Gamma\bar{\Gamma} \left| g^{(m,n)}(z_0, w_0) \right| .$$

Thus by Theorem 2.1 $g(z, w)$ is of bounded index. □

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