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On strongly $\mathcal{I}$-lacunary Cauchy sequences of sets

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Abstract. In this study, we examinate the ideas of Wijsman strongly lacunary Cauchy, Wijsman strongly $\mathcal{I}$-lacunary Cauchy and Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequences of sets and investigate the relationship between them.

Key words. Lacunary sequences, $\mathcal{I}$-convergence, $\mathcal{I}$-Cauchy, sequences of sets.

1 Introduction

Let $\mathbb{N}$ be the set of all natural numbers and $\mathbb{R}$ be the set of all real numbers. Fast [6] and Schoenberg [14] independently presented the idea of statistical convergence. The notion of $\mathcal{I}$-convergence in a metric space was examinated by Kostyrko et al. [9]. Then it was further worked by many others. Nuray and Ruckle [11] presented the generalized statistical convergence. Nabiev et al. [10] investigated the ideas of $\mathcal{I}$-Cauchy and $\mathcal{I}^*$- Cauchy sequence. Also, Yamancı and Gürdal [20] gave the notion lacunary $\mathcal{I}$-convergence and lacunary $\mathcal{I}$-Cauchy in the topology in random n-normed spaces and gave a lot of useful properties. Debnath [5] studied the idea of lacunary $\mathcal{I}$-convergence in intuitionistic fuzzy normed linear spaces as a version of the idea of $\mathcal{I}$- convergence. Tripathy et al. [13] presented the notion of $\mathcal{I}$-lacunary convergent sequences.

The notion of convergence of sequences of numbers has been held out by some authors to convergence of sequences of sets. One of these studies which is considered in this work is the notion of Wijsman convergence (see [2–4]). Nuray and Rhoades [12] extended the idea of convergence of set sequences to statistical convergence, and gave some basic theorems. Ulusu and Nuray [15] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence, which was presented by Nuray and Rhoades. Also, Ulusu and Nuray [16] introduced the notion of Wijsman strongly lacunary summability of sequences of sets. Recently, Kişi and Nuray [7] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence. Kişi et al. [8] defined Wijsman $\mathcal{I}$-asymptotically lacunary statistical equivalence of sequences of sets. Sever et al. [17] investigated the ideas of Wijsman $\mathcal{I}$-convergence, Wijsman strongly $\mathcal{I}$-lacunary convergence, Wijsman strongly $\mathcal{I}^*$-lacunary convergence and Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequences of sets.

In this study, we examinate the idea of Wijsman strongly lacunary Cauchy, Wijsman strongly $\mathcal{I}$-lacunary Cauchy and Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequences of sets and investigate the relationship between them.

2 Definitions and notations

We recall the notion of lacunary sequence, ideal, and some fundamental definitions and notations (see [1, 2, 7–9, 15, 17–19]).

In this paper, we get (X, ρ) be a metric space and $B, B_k \subseteq X$ be any non-empty closed subsets.

For any point $x \in X$ we describe the distance from x to B by $d(x, B) = \inf_{a \in B} \rho(x, B)$.

The sequence $\{B_k\}$ is Wijsman convergent to B ($W - \lim B_k = B$) if $\lim_{k\to\infty} d(x, B_k) = d(x, B)$, for each $x \in X$.

By a lacunary sequence we define a increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. In this paper, the intervals defined by θ will be indicated by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be shortened by q_r.

We get $\theta = \{k_r\}$ be a lacunary sequence during this work.

Let $X \neq \emptyset$. A class $\mathcal{I} \subset 2^X$ is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$

(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,

(iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.

The $\mathcal{I}$ is a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. $\emptyset \neq \mathcal{F} \subset X$ is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,

(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,

(iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1 ([9]) *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, the set*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists H \in \mathcal{I})(M = X \backslash H)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

In this paper we get $\mathcal{I} \subset 2^{\mathbb{N}}$ as an admissible ideal. We give the following definitions:

Definition 2.1 *A sequence $(x_n) \in X$ is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ we have $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x) \geq \varepsilon\} \in \mathcal{I}$.*

Definition 2.2 *A sequence $(x_n) \in X$ is called an $\mathcal{I}$-Cauchy sequence, if for every $\varepsilon > 0$ there exists $k = k(\varepsilon) \in \mathbb{N}$ such that $A(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, x_k) \geq \varepsilon\} \in \mathcal{I}$.*

Definition 2.3 *A sequence $(x_n) \in X$ is called an $\mathcal{I}^*$-Cauchy sequence in X if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subset \mathbb{N}$ such that*

$$\lim_{k,p\to\infty} \rho(x_{m_k}, x_{m_p}) = 0.$$

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ satisfies the property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. (hence $B_j \in \mathcal{I}$ for each $j \in \mathbb{N}$).

The sequence $\{B_k\}$ is Wijsman $\mathcal{I}$-convergent to B ($\mathcal{I}_W - \lim B_k = B$ or $B_k \to B(\mathcal{I}_W)$), if for each $\varepsilon > 0$ and for each $x \in X$ the set $A(x, \varepsilon) = \{k \in \mathbb{N} : |d(x, B_k) - d(x, B)| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

Let (X, ρ) be a separable metric space. The sequence $\{B_n\}$ is Wijsman $\mathcal{I}$-Cauchy sequence if for each $\varepsilon > 0$ and for each $x \in X$, there exists $k = k(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \{n \in \mathbb{N} : |d(x, B_n) - d(x, B_k)| \geq \varepsilon\} \in \mathcal{I}.$$

The sequence $\{B_k\}$ is Wijsman strongly lacunary convergent to B $\big(B_k \to B\big([WN]_\theta\big)$ or $B_k \stackrel{[WN]_\theta}{\longrightarrow} B\big)$ if for each $x \in X$,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, B_k) - d(x, B)| = 0.$$

The sequence $\{B_k\}$ is said to be Wijsman strongly $\mathcal{I}$-lacunary convergent to B or $N_\theta(\mathcal{I}_W)$-convergent to B $\big(B_k \to B\big(N_\theta[\mathcal{I}_W]\big)\big)$ if for each $\varepsilon > 0$ and for each $x \in X$,

$$A(\varepsilon, x) = \{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |d(x, B_k) - d(x, B)| \geq \varepsilon\}$$

belongs to $\mathcal{I}$.

$N_\theta[\mathcal{I}_W]$ denotes the set of Wijsman strongly $\mathcal{I}$-lacunary convergent sequences.

Let (X, ρ) be a separable metric space. We can give the following definitions:

Definition 2.4 *The sequence $\{B_k\}$ is Wijsman strongly $\mathcal{I}^*$-lacunary convergent to B ($B_k \to B\,(N_\theta[\mathcal{I}_W^*])$), if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ and there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |d(x, B_{m_k}) - d(x, B)| = 0$$

for every $k \geq k_0$.

Definition 2.5 *The sequence $\{B_k\}$ is Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence if for each $\varepsilon > 0$ and $x \in X$, there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that*

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |d(x, B_k) - d(x, B_{k_0})| \geq \varepsilon \right\} \in \mathcal{I}.$$

A lacunary sequence $\theta' = (k'(r))$ is said to be a lacunary refinement of the lacunary sequence $\theta = (k_r)$ if $(k_r) \subset (k'(r))$.

Lemma 2.2 ([10], Lemma 4*) Let $\{S_m\}_{m=1}^{\infty}$ be a countable collection of subsets of $\mathbb{N}$ such that $S_m \in F(\mathcal{I})$ for each m, where $\mathcal{F}(\mathcal{I})$ is a filter associate with an admissible ideal $\mathcal{I}$ with the property (AP). Then there exists a set $S \subset \mathbb{N}$ such that $S \in \mathcal{F}(\mathcal{I})$ and the set $S \backslash S_m$ is finite for all m.*

3 Main results

First, we give a theorem related to lacunary refinement sequence.

Theorem 3.1 *If θ' is a lacunary refinement of a lacunary sequence θ and $\{B_k\} \in \big(N_{\theta'}[\mathcal{I}_W]\big)$, then $\{B_k\} \in \big(N_\theta[\mathcal{I}_W]\big)$.*

Proof. Assume that for each I_r of θ includes the points $(k'_{r,t})_{t=1}^{\eta(r)}$ of θ' such that

$$k_{r-1} < k'_{r,1} < k'_{r,2} < \ldots < k'_{r,\eta(r)} = k_r,$$

where $I'_{r,t} = (k'_{r,t-1}, k'_{r,t}]$. Since $k_r \subseteq (k'(r))$, so $r, \eta(r) \geq 1$.

Let $\left(I_j^*\right)_{j=1}^{\infty}$ be the sequence of intervals $\left(I'_{r,t}\right)$ ordered by increasing right end points. Because $\{B_k\} \in \left(N_{\theta'}[\mathcal{I}_W]\right)$, for each $\varepsilon > 0$ and $x \in X$,

$$\left\{ j \in \mathbb{N} : \frac{1}{h_j^*} \sum_{I_j^* \subset I_r} |d(x, B_k) - d(x, B)| \geq \varepsilon \right\} \in \mathcal{I}.$$

Also, since $h_r = k_r - k_{r-1}$, $h'_{r,t} = k'_{r,t} - k'_{r,t-1}$.

For each $\varepsilon > 0$ and $x \in X$, the following inclusion holds

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, B_k) - d(x, B)| \geq \varepsilon \right\}$$
$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left\{ j \in \mathbb{N} : \frac{1}{h_j^*} \sum_{\substack{I_j^* \subset I_r \\ k \in I_j^*}} |d(x, B_k) - d(x, B)| \geq \varepsilon \right\} \right\}.$$

Hence $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, B_k) - d(x, B)| \geq \varepsilon \right\} \in \mathcal{I}$ and so $\{B_k\} \in \left(N_\theta[\mathcal{I}_W]\right)$. ■

Definition 3.1 *The sequence $\{B_k\}$ is Wijsman strongly lacunary Cauchy if for each $\varepsilon > 0$ and $x \in X$, there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that*

$$\frac{1}{h_r} \sum_{k,p \in I_r} |d(x, B_k) - d(x, B_p)| < \varepsilon$$

for every $k, p \geq k_0$.

Now, we give following theorem without proof.

Theorem 3.2 *Let (X, ρ) be a separable metric space. If $\{B_k\}$ is Wijsman strongly lacunary Cauchy then $\{B_k\}$ is Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence of sets.*

We give the definition of Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence of sets.

Definition 3.2 *Let (X, ρ) be a separable metric space. The sequence $\{B_k\}$ is Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence if for each $\varepsilon > 0$ and $x \in X$, there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ and there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that*

$$\frac{1}{h_r} \sum_{k,p \in I_r} |d(x, B_{m_k}) - d(x, B_{m_p})| < \varepsilon$$

for every $k, p \geq k_0$.

Theorem 3.3 *Let (X, ρ) be a separable metric space. If the sequence $\{B_k\}$ is a Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence then $\{B_k\}$ is a Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence of sets.*

Proof. Suppose that $\{B_k\}$ is a Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence. Then, for each $\varepsilon > 0$ and $x \in X$, there exists $M' \in \mathcal{F}(\mathcal{I})$ and $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k,p \in I_r} |d(x, B_{m_k}) - d(x, B_{m_p})| < \varepsilon$$

for every $k, p \geq k_0$. Now, fix $p = m_{k_0+1}$. Then for each $\varepsilon > 0$ and $x \in X$, we get

$$\frac{1}{h_r} \sum_{k \in I_r} |d(x, B_{m_k}) - d(x, B_p)| < \varepsilon$$

for every $k \geq k_0$.

Let $K = \mathbb{N} \backslash M$. It is obvious that $K \in \mathcal{I}$ and

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k,p \in I_r} |d(x, B_k) - d(x, B_p)| \geq \varepsilon \right\} \subset K \cup \{m_1 < m_2 < \ldots < m_{k_0}\} \in \mathcal{I}.$$

Thus, for each $\varepsilon > 0$, we may choose $p \in \mathbb{N}$ such that $A(\varepsilon, x) \in \mathcal{I}$, that is, $\{B_k\}$ is a Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence of sets. ■

Theorem 3.4 *Let (X, ρ) be a separable metric space. If the sequence $\{B_k\}$ is a Wijsman strongly $\mathcal{I}^*$-lacunary convergence then $\{B_k\}$ is a Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence of sets.*

Proof. By assumption, there exists a set

$$M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$$

such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in \mathcal{F}(\mathcal{I})$ and there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, B_{m_k}) - d(x, B)| = 0 \quad (x \in X),$$

for every $k \geq k_0$. Since, for each $\varepsilon > 0$ and $x \in X$

$$\begin{aligned} \frac{1}{h_r} \sum_{k,p \in I_r} |d(x, B_{m_k}) - d(x, B_{m_p})| &\leq \frac{1}{h_r} \sum_{k \in I_r} |d(x, B_{m_k}) - d(x, B)| \\ &+ \frac{1}{h_r} \sum_{p \in I_r} |d(x, B_{m_p}) - d(x, B)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

for every $k, p > k_0$. Hence, we get

$$\frac{1}{h_r} \sum_{k,p \in I_r} |d(x, B_{m_k}) - d(x, B_{m_p})| < \varepsilon$$

for every $k, p \geq k_0$. This shows that $\{B_k\}$ is a Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence of sets. Then, by Theorem 3.3 $\{B_k\}$ is a Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence of sets. ■

Theorem 3.5 *Let (X, ρ) be a separable metric space. If $\mathcal{I} \subset 2^{\mathbb{N}}$ satisfies the property (AP), the ideals Wijsman strongly $\mathcal{I}$-lacunary Cauchy and Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence of sets coincide in X.*

Proof. If $\{B_k\}$ in X is Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence, it is Wijsman strongly $\mathcal{I}$-lacunary Cauchy sequence of sets by Theorem 3.3 where $\mathcal{I}$ need not have the property (AP).

Now, we get a sequence $\{B_k\}$ in X is a Wijsman strongly $\mathcal{I}$-lacunary Cauchy. We show that it is Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy. If $\{B_k\}$ in X be a Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy, for each $\varepsilon > 0$ and $x \in X$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, B_k) - d(x, B_N)| \geq \varepsilon\} \in \mathcal{I}.$$

Let $S_i = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, B_k) - d(x, B_{n_i})| < \frac{1}{i}\right\};\quad (i = 1, 2, \ldots)$, where $n_i = N(1 \backslash i)$. It is obvious that $S_i \in \mathcal{F}(\mathcal{I})$ $(i = 1, 2, \ldots)$. Because of the property (AP), by Lemma 2.2 there exists a set $S \subset \mathbb{N}$ such that $S \in \mathcal{F}(\mathcal{I})$, and $S \backslash S_i$ is finite for all i. We prove that

$$\lim_{\substack{k,n \to \infty \\ k,n \in P}} \frac{1}{h_r} \sum_{k,n \in I_r} |d(x, B_k) - d(x, B_n)| = 0 \text{ for each } x \text{ in } X.$$

In order to prove, let $\varepsilon > 0$ and $j \in \mathbb{N}$ such that $j > 2/\varepsilon$. If $k, n \in S$ then $S \backslash S_j$ is a finite set. Hence, there exists $m = m(j)$ such that $k, n \in S_j$ for all $k, n > m(j)$. Therefore, for each x in X,

$$\frac{1}{h_r} \sum_{k \in I_r} |d(x, B_k) - d(x, B_{n_j})| < \frac{1}{j} \text{ and } \frac{1}{h_r} \sum_{n \in I_r} |d(x, B_n) - d(x, B_{n_j})| < \frac{1}{j}$$

for all $k, n > m(j)$. Thus we get

$$\begin{aligned} \frac{1}{h_r} \sum_{k,n \in I_r} |d(x, B_k) - d(x, B_n)| &\leq \frac{1}{h_r} \sum_{k \in I_r} |d(x, B_k) - d(x, B_{n_j})| \\ &+ \frac{1}{h_r} \sum_{n \in I_r} |d(x, B_n) - d(x, B_{n_j})| \\ &< \frac{1}{j} + \frac{1}{j} = \frac{2}{j} \\ &< \varepsilon \end{aligned}$$

for all $k, n > m(j)$ and each x in X. Hence, for any $\varepsilon > 0$ there exists $m = m(\varepsilon)$ such that for $k, n > m(\varepsilon)$ and $k, n \in S \in \mathcal{F}(\mathcal{I})$

$$\frac{1}{h_r} \sum_{k,n \in I_r} |d(x, B_k) - d(x, B_n)| < \varepsilon, \text{ for each } x \text{ in } X.$$

Therefore, $\{B_k\}$ in X is Wijsman strongly $\mathcal{I}^*$-lacunary Cauchy sequence of sets. ■

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