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On strongly \mathcal{I} and \mathcal{I}^* -lacunary convergence of sequences of sets

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Abstract. In this paper we study the concepts of Wijsman strongly lacunary convergence, Wijsman strongly \mathcal{I} -lacunary convergence, Wijsman strongly \mathcal{I}^* -lacunary convergence and Wijsman strongly \mathcal{I} -lacunary Cauchy sequences of sets and investigate the relationship between them.

Keywords: Lacunary sequence, \mathcal{I} -convergence, Sequence of sets, Wijsman convergence

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INTRODUCTION

The concept of \mathcal{I} -convergence in a metric space, which is a generalized form of statistical convergence, was introduced by Kostyrko et al. [10]. Later it was further studied by many others [5, 18, 19]. Recently, Das et al. [5] introduced new notions, namely \mathcal{I} -statistical convergence and \mathcal{I} -lacunary statistical convergence by using ideal. Also, Yamancı and Gürdal [18] introduced the notion lacunary \mathcal{I} -convergence and lacunary \mathcal{I} -Cauchy in the topology induced by random n -normed spaces and prove some important results. Debnath [6] studied the notion of lacunary ideal convergence in intuitionistic fuzzy normed linear spaces as a variant of the notion of ideal convergence. Tripathy et al. [13] introduced the concepts of \mathcal{I} -lacunary convergent sequences.

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence [2, 4, 8, 11, 14, 16]. Nuray and Rhoades [11] extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems. Ulusu and Nuray [14] defined the Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Also, Ulusu and Nuray [15] introduced the concept of Wijsman strongly lacunary summability of sequences of sets. Recently, Kişi and Nuray [8] introduced a new convergence notion, for sequences of sets, which is called Wijsman \mathcal{I} -convergence. Kişi et al. [9] defined Wijsman \mathcal{I} -asymptotically lacunary statistical equivalence of sequences of sets.

In this paper we study the concepts of Wijsman strongly lacunary convergence, Wijsman strongly \mathcal{I} -lacunary convergence, Wijsman strongly \mathcal{I}^* -lacunary convergence and Wijsman strongly \mathcal{I} -lacunary Cauchy sequences of sets and investigate the relationship between them.

DEFINITIONS AND NOTATIONS

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X , we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Definition 1. [2] Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \rightarrow \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r .

Definition 2. [10] A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

- (i) $\emptyset \in \mathcal{I}$,
- (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$,
- (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Definition 3. [10] A family of sets $F \subseteq 2^{\mathbb{N}}$ is a filter if and only if

- (i) $\emptyset \notin F$,
- (ii) For each $A, B \in F$ we have $A \cap B \in F$,
- (iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$.

Proposition 1. [10] \mathcal{I} is a non-trivial ideal in \mathbb{N} , then the set

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \setminus A)\}$$

is a filter in \mathbb{N} , called the filter associated with \mathcal{I} .

Definition 4. [10] An admissible ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is said to satisfy the property (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\}$ belonging to \mathcal{I} there exists a countable family of sets $\{B_1, B_2, \dots\}$ such that $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Definition 5. [10] Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of \mathbb{N} . A sequence (x_k) of elements of \mathbb{R} is said to be \mathcal{I} -convergent to $L \in \mathbb{R}$ if for each $\varepsilon > 0$ the set

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$$

belongs to \mathcal{I} .

Definition 6. [8] Let (X, ρ) be a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of \mathbb{N} . For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman \mathcal{I} -convergent to A , if for each $\varepsilon > 0$ and for each $x \in X$ the set

$$A(x, \varepsilon) = \{k \in \mathbb{N} : |d(x, A_k) - d(x, A)| \geq \varepsilon\}$$

belongs to \mathcal{I} . In this case we write $\mathcal{I}_W - \lim A_k = A$ or $A_k \rightarrow A(\mathcal{I}_W)$.

Definition 7. [7] For any lacunary sequence $\theta = \{k_r\}$, a sequence (x_k) of elements of \mathbb{R} is said to be strongly lacunary convergent to $L \in \mathbb{R}$ if

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| = 0.$$

Definition 8. [14] Let (X, ρ) be a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| = 0.$$

In this case we write $A_k \rightarrow A([WN]_{\theta})$ or $A_k \xrightarrow{[WN]_{\theta}} A$.

Definition 9. [9] Let (X, ρ) be a metric space, θ be lacunary sequence and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of \mathbb{N} . For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly \mathcal{I} -lacunary convergent to A or $N_{\theta}(\mathcal{I}_W)$ -convergent to A if for each $\varepsilon > 0$ and for each $x \in X$, the set

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\}$$

belongs to \mathcal{I} . In this case, we write $A_k \rightarrow A(N_{\theta}[\mathcal{I}_W])$.

Definition 10. [8] Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal in \mathbb{N} and (X, d) be a separable metric space. For any non-empty closed subsets A_n in X , we say that the sequence $\{A_n\}$ is Wijsman \mathcal{I} -Cauchy sequence if for each $\varepsilon > 0$ and for each $x \in X$, there exists a number $N = N(\varepsilon)$ such that the set

$$A(x, \varepsilon) = \{n \in \mathbb{N} : |d(x, A_n) - d(x, A_N)| \geq \varepsilon\}$$

belongs to \mathcal{I} .

MAIN RESULTS

Throughout the paper we take (X, ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal and A, A_k be non-empty closed subsets of X .

In this section, we investigate the relationship between the concepts of Wijsman strongly lacunary convergence, Wijsman strongly \mathcal{I} -lacunary convergence, Wijsman strongly \mathcal{I}^* -lacunary convergence and Wijsman strongly \mathcal{I} -lacunary Cauchy sequences of sets.

Theorem 1. If $\{A_k\}$ is Wijsman strongly lacunary convergent to A , then it is Wijsman strongly \mathcal{I} -lacunary convergent to A .

Proof. Let $\{A_k\}$ is Wijsman strongly lacunary convergent to A . For each $\varepsilon > 0$ and $x \in X$ there exists $k_0 = k_0(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| < \varepsilon,$$

for all $k \geq k_0$. Then

$$A(x, \varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\} \subset \{1, 2, \dots, k_0 - 1\}.$$

Since \mathcal{I} is admissible ideal we have $\{1, 2, \dots, k_0 - 1\} \in \mathcal{I}$ and so $A(x, \varepsilon) \in \mathcal{I}$. This completes the proof. \square

Definition 11. Let (X, ρ) be a separable metric space. We say that the sequence $\{A_k\}$ is Wijsman \mathcal{I}^* -lacunary convergent to A if and only if there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in F(\mathcal{I})$ for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} d(x, A_{m_k}) = d(x, A).$$

In this case, we write $A_k \rightarrow A(N_{\theta}(\mathcal{I}_W^*))$.

Definition 12. Let (X, ρ) be a separable metric space. We say that the sequence $\{A_k\}$ is Wijsman strongly \mathcal{I}^* -lacunary convergent to A if and only if there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in F(\mathcal{I})$ for each $x \in X$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{m_k}) - d(x, A)| = 0.$$

In this case, we write $A_k \rightarrow A(N_{\theta}[\mathcal{I}_W^*])$.

Theorem 2. Let (X, ρ) be a separable metric space. $\{A_k\}$ is Wijsman strongly \mathcal{I}^* -lacunary convergent to A implies $\{A_k\}$ is Wijsman strongly \mathcal{I} -lacunary convergent to A .

Proof. Suppose that $\{A_k\}$ is Wijsman strongly \mathcal{I}^* -lacunary convergent to A . Then, there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that $M' = \{r \in \mathbb{N} : m_k \in I_r\} \in F(\mathcal{I})$ for each $x \in X$

$$\frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{m_k}) - d(x, A)| < \varepsilon,$$

for each $\varepsilon > 0$ and for all $k > k_0 = k_0(\varepsilon) \in \mathbb{N}$. Hence, for each $\varepsilon > 0$ and $x \in X$ we have

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{m_k}) - d(x, A)| \geq \varepsilon \right\} \subset H \cup \{1, 2, \dots, k_0 - 1\},$$

for $\mathbb{N} \setminus M' = H \in \mathcal{I}$. Since \mathcal{I} is an admissible ideal we have

$$H \cup \{m_1, m_2, \dots, m_{k_0-1}\} \in \mathcal{I}$$

and so $A(\varepsilon, x) \in \mathcal{I}$. Hence, this completes the proof. \square

Theorem 3. Let (X, ρ) be a separable metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be an admissible ideal with property (AP). $\{A_k\}$ is Wijsman strongly \mathcal{I} -lacunary convergent to A implies $\{A_k\}$ is Wijsman strongly \mathcal{I}^* -lacunary convergent to A .

Proof. Suppose that $\{A_k\}$ is Wijsman strongly \mathcal{I} -lacunary convergent to A . Then for each $\varepsilon > 0$, $x \in X$

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}.$$

Put

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq 1 \right\},$$

$$A_p = \left\{ r \in \mathbb{N} : \frac{1}{p} \leq \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| < \frac{1}{p-1} \right\}$$

for $p \geq 2$ and $p \in \mathbb{N}$. It is clear that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathcal{I}$ for each $i \in \mathbb{N}$. By property (AP) there is a countable family of sets $\{B_1, B_2, \dots\}$ such that $A_j \Delta B_j$ is a finite set for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. We prove that for each $x \in X$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{m_k}) - d(x, A)| = 0,$$

for $M = \mathbb{N} \setminus B \in F(\mathcal{I})$. Let $\delta > 0$ be given. Choose $q \in \mathbb{N}$ such that $\frac{1}{q} < \delta$. Then for each $x \in X$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \delta \right\} \subset \bigcup_{j=1}^{q-1} A_j.$$

Since $A_j \Delta B_j$ is a finite set for $j \in \{1, 2, \dots, q-1\}$, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{q-1} A_j \right) \cap \{k \in \mathbb{N} : k \geq k_0\} = \left(\bigcup_{j=1}^{q-1} B_j \right) \cap \{k \in \mathbb{N} : k \geq k_0\}.$$

If $k \geq k_0$ and $k \notin B$, then

$$k \notin \bigcup_{j=1}^{q-1} B_j \text{ and so } k \notin \bigcup_{j=1}^{q-1} A_j.$$

Thus, for each $x \in X$ we have

$$\frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| < \frac{1}{q} < \delta.$$

This implies that

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_{m_k}) - d(x, A)| = 0.$$

Hence, for each $x \in X$ we have $A_k \rightarrow A(N_\theta[\mathcal{I}_W^*])$. This completes the proof. \square

Definition 13. Let (X, ρ) be a separable metric space. We say that the sequence $\{A_k\}$ is Wijsman strongly \mathcal{I} -lacunary Cauchy sequence if for each $\varepsilon > 0$ and $x \in X$, there exists a number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A_N)| \geq \varepsilon \right\} \in \mathcal{I}.$$

Theorem 4. Let (X, d) be a separable metric space. If $\{A_k\}$ is Wijsman strongly \mathcal{I} -lacunary convergent then $\{A_k\}$ is Wijsman strongly \mathcal{I} -lacunary Cauchy sequence.

Proof. Let $\{A_k\}$ is Wijsman strongly \mathcal{I} -lacunary convergent to A . Then for each $\varepsilon > 0$ and $x \in X$, we have

$$A\left(\frac{\varepsilon}{2}, x\right) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| \geq \frac{\varepsilon}{2} \right\} \in \mathcal{I}.$$

Since \mathcal{I} is an admissible ideal, the set

$$A^c\left(\frac{\varepsilon}{2}, x\right) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| < \frac{\varepsilon}{2} \right\}$$

is non-empty and belongs to $F(\mathcal{I})$. So, we can choose positive integer r such that $r \notin A(\frac{\varepsilon}{2}, x)$, we have

$$\frac{1}{h_r} \sum_{k_0 \in I_r} |d(x, A_{k_0}) - d(x, A)| < \frac{\varepsilon}{2}.$$

Now, we define the set

$$B(\varepsilon, x) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k, k_0 \in I_r} |d(x, A_k) - d(x, A_{k_0})| \geq \varepsilon \right\}.$$

We show that $B(\varepsilon, x) \subset A(\frac{\varepsilon}{2}, x)$. Let $r \in B(\varepsilon, x)$ then we have

$$\begin{aligned} \varepsilon &\leq \frac{1}{h_r} \sum_{k, k_0 \in I_r} |d(x, A_k) - d(x, A_{k_0})| \leq \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| + \frac{1}{h_r} \sum_{k_0 \in I_r} |d(x, A_{k_0}) - d(x, A)| \\ &< \frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| + \frac{\varepsilon}{2}. \end{aligned}$$

This implies that

$$\frac{1}{h_r} \sum_{k \in I_r} |d(x, A_k) - d(x, A)| > \frac{\varepsilon}{2}$$

and therefore $r \in A(\frac{\varepsilon}{2}, x)$. Hence, we have $B(\varepsilon, x) \subset A(\frac{\varepsilon}{2}, x)$. This shows that $\{A_k\}$ is Wijsman strongly \mathcal{I} -lacunary Cauchy. \square

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