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$\mathcal{I}_2$-LACUNARY STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF SETS

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Abstract. In this paper, we introduce the concepts of the Wijsman $\mathcal{I}_2$-statistical convergence, Wijsman $\mathcal{I}_2$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2$-lacunary convergence of double sequences of sets and investigate the relationship between them.

1. Introduction, Definitions and Notations

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [8] and Schoenberg [20]. This concept was extended to the double sequences by Mursaleen and Edely [12].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [11] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers $\mathbb{N}$. Nuray and Ruckle [14] indepedently introduced the same with another name generalized statistical convergence. Das et al. [5] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. Also, Das et al. [6] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence.

The concept of convergence of number sequences has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see, [2, 4, 26, 27]). Nuray and Rhoades [13] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [25] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its

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relation with Wijsman statistical convergence which was defined by Nuray and Rhoades.

Kişi and Nuray [10] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence. Ulusu and Dündar [24] studied the concepts of Wijsman $\mathcal{I}$-statistical convergence, Wijsman $\mathcal{I}$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}$-lacunary convergence of sequences of sets. The concepts of convergence, statistical convergence and ideal convergence of double sequences of sets were studied by Nuray et. al [15–18].

Now, we recall the basic definitions and concepts (See [1–4, 6, 7, 9, 11, 15–19, 21–23]).

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Definition 1.1. Let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X. A sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A),$$

for each $x \in X$. In this case, we write $W - \lim A_k = A$.

Throughout the paper, we let (X, ρ) be a metric space and A, A_{kj} be any non-empty closed subsets of X.

Definition 1.2. A double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A) \quad or \quad \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A),$$

for each $x \in X$. In this case, we write $W_2 - \lim A_{kj} = A$.

Definition 1.3. A double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n \to \infty} \frac{1}{mn} \Big| \big\{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \big\} \Big| = 0,$$

that is,

$$|d(x, A_{kj}) - d(x, A)| < \varepsilon, \quad \text{a.a. (k,j)}.$$

In this case, we write $st_2 - \lim_W A_k = A$.

If a double sequence of sets $\{A_{kj}\}$ is Wijsman statistically convergent to the set A, then $\{A_{kj}\}$ need not be convergent. Also, it is not necessary be bounded.

Example 1.1. Let $X = \mathbb{R}^2$ and a double sequence $\{A_{kj}\}$ be following sequence:

$$A_{kj} = \begin{cases} \big\{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 = kj \big\} & , \text{ if k and j is a square integer} \\ \{(2, 2)\} & , \text{ otherwise.} \end{cases}$$

This double sequence is Wijsman statistically convergent to the set $A = \big\{(2, 2)\big\}$ but it is neither Wijsman convergent nor bounded.

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad and \quad j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad r, u \to \infty.$$

We use following notations in the sequel:

$$k_{ru} = k_r j_u,\ \ h_{ru} = h_r \bar{h}_u,\ \ I_{ru} = \{(k,j) : k_{r-1} < k \leq k_r \ \ and \ \ j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \ \ and \ \ q_u = \frac{j_u}{j_{u-1}}.$$

Definition 1.4. Let θ be a double lacunary sequence. A double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)| = 0.$$

In this case, we write $A_{kj} \stackrel{[W_2 N_\theta]}{\longrightarrow} A$.

Definition 1.5. A double sequence $\{A_{kj}\}$ is Wijsman lacunary statistically convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \Big| \big\{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\big\} \Big| = 0.$$

In this case, we write $st_2 - \lim_{W_\theta} A_{kj} = A$.

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout the paper we take $\mathcal{I}_2$ as an admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

Let $\mathcal{I}_2^0 = \big\{A \subset \mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i,j) \notin A)\big\}$. Then, $\mathcal{I}_2^0$ is an strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

A family of sets $F \subseteq 2^{\mathbb{N}}$ is called a filter if and only if
(i) $\emptyset \notin F$, (ii) For each $A, B \in F$ we have $A \cap B \in F$, (iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$.

$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$ if and only if

$$F(\mathcal{I}) = \big\{M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N} \backslash A)\big\}$$

is a filter in $\mathbb{N}$.

Throughout the paper, we let $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal.

Definition 1.6. A double sequence $\{A_{kj}\}$ is $\mathcal{I}_{W_2}$-convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$,

$$\big\{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\big\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_{W_2} - \lim_{k,j\to\infty} A_{kj} = A$.

2. MAIN RESULTS

In this section, we define the concepts of Wijsman $\mathcal{I}_2$-statistical convergence, Wijsman $\mathcal{I}_2$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2$-lacunary convergence of double sequences of sets and investigate the relationship between them.

Definition 2.1. A double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A or $S(\mathcal{I}_{W_2})$-convergent to A if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}\Big|\big\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\big\}\Big| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\,(S\,(\mathcal{I}_{W_2}))$.

The set of Wijsman $\mathcal{I}_2$-statistical convergent double sequences will be denoted by $\big\{S\,(\mathcal{I}_{W_2})\big\}$.

Definition 2.2. Let θ be a double lacunary sequence. A double sequence $\{A_{kj}\}$ is said to be Wijsman $\mathcal{I}_2$-lacunary convergent to A or $N_\theta(\mathcal{I}_{W_2})$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \Big|\frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} d(x,A_{kj}) - d(x,A)\Big| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\,(N_\theta\,(\mathcal{I}_{W_2}))$.

Definition 2.3. Let θ be a double lacunary sequence. A double sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta[\mathcal{I}_{W_2}]$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\,(N_\theta\,[\mathcal{I}_{W_2}])$.

The set of Wijsman strongly $\mathcal{I}_2$-lacunary convergent double sequences will be denoted by $\big\{N_\theta\,[\mathcal{I}_{W_2}]\big\}$.

Definition 2.4. Let θ be a double lacunary sequence. A double sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-lacunary statistical convergent to A or $S_\theta\,(\mathcal{I}_{W_2})$-convergent to A if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u}\Big|\big\{(k,j) \in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\big\}\Big| \geq \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\,(S_\theta\,(\mathcal{I}_{W_2}))$.

The set of Wijsman $\mathcal{I}_2$-lacunary statistical convergent double sequences will be denoted by $\big\{S_\theta\,(\mathcal{I}_{W_2})\big\}$.

Theorem 2.1. *Let θ be a double lacunary sequence. Then,*

$$A_{kj} \to A(N_\theta[\mathcal{I}_{W_2}]) \Rightarrow A_{kj} \to A(S_\theta(\mathcal{I}_{W_2})).$$

Proof. Let $A_{kj} \to A(N_\theta[\mathcal{I}_{W_2}])$ and $\varepsilon > 0$. Then, for each $x \in X$ we can write

$$\begin{aligned} \sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| &\geq \sum_{\substack{(k,j)\in I_{ru} \\ |d(x,A_{kj})-d(x,A)|\geq\varepsilon}} |d(x,A_{kj}) - d(x,A)| \\ &\geq \varepsilon.\left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| \end{aligned}$$

and so

$$\frac{1}{\varepsilon\cdot h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \frac{1}{h_r\bar{h}_u}\left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right|.$$

Hence, for each $x \in X$ and for any $\delta > 0$,

$$\begin{aligned} &\left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u}\left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\qquad \subseteq \left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\cdot\delta\right\} \in \mathcal{I}_2. \end{aligned}$$

This proof is completed. □

Definition 2.5. A double sequence $\{A_{kj}\}$ is said to be bounded if

$$\sup_{k,j} d(x,A_{kj}) < \infty,$$

for each $x \in X$.

The set of all bounded double sequences of sets will be denoted by L^2_∞.

Theorem 2.2. *Let θ be a double lacunary sequence. Then, $\{A_{kj}\} \in L^2_\infty$ and*

$$A_{kj} \to A(S_\theta(\mathcal{I}_{W_2})) \Rightarrow A_{kj} \to A(N_\theta[\mathcal{I}_{W_2}]).$$

Proof. Suppose that $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$ and $A_{kj} \in L^2_\infty$. Then, for each $x \in X$ there exists an $M > 0$ such that

$$|d(x,A_{kj}) - d(x,A)| \leq M$$

for all $k, j \in \mathbb{N}$. Given $\varepsilon > 0$, for each $x \in X$ we have

$$\begin{aligned} &\frac{1}{h_r\bar{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \\ &\qquad = \frac{1}{h_r\bar{h}_u}\sum_{\substack{(k,j)\in I_{ru} \\ |d(x,A_{kj})-d(x,A)|\geq\frac{\varepsilon}{2}}} |d(x,A_{kj}) - d(x,A)| \\ &\qquad\quad + \frac{1}{h_r\bar{h}_u}\sum_{\substack{(k,j)\in I_{ru} \\ |d(x,A_{kj})-d(x,A)|<\frac{\varepsilon}{2}}} |d(x,A_{kj}) - d(x,A)| \\ &\qquad \leq \frac{M}{h_r\bar{h}_u}\left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\varepsilon}{2}. \end{aligned}$$

Hence, for each $x \in X$ we get

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}$$

$$\subseteq \left\{(r,u) : \frac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{\varepsilon}{2M}\right\} \in \mathcal{I}_2.$$

This proof is completed. □

We have the following Corollary by Theorem 2.1 and Theorem 2.2.

Corollary 2.1. *Let θ be a double lacunary sequence. Then,*

$$\left\{S_\theta(\mathcal{I}_{W_2})\right\} \cap L^2_\infty = \left\{N_\theta[\mathcal{I}_{W_2}]\right\} \cap L^2_\infty.$$

Theorem 2.3. *Let θ be a double lacunary sequence. If $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then $A_{kj} \to A(S(\mathcal{I}_{W_2}))$ implies $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$.*

Proof. Assume that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then there exist $\lambda, \mu > 0$ such that

$$q_r \geq 1+\lambda \ \ \textit{and} \ \ q_u \geq 1+\mu$$

for sufficiently large r, u which implies that

$$\frac{h_r\overline{h}_u}{k_{ru}} \geq \frac{\lambda\mu}{(1+\lambda)(1+\mu)}.$$

If $A_{kj} \to A(S(\mathcal{I}_{W_2})$, then for every $\varepsilon > 0$, for each $x \in X$ and for sufficiently large r, u, we have

$$\frac{1}{k_{ru}}\left|\left\{k \leq k_r, j \leq j_u : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right|$$

$$\geq \frac{1}{k_{ru}}\left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right|$$

$$\geq \frac{\lambda\mu}{(1+\lambda)(1+\mu)}.\left(\frac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right|\right).$$

Hence, for each $x \in X$ and for any $\delta > 0$ we have

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$\subseteq \left\{(r,u) : \frac{1}{k_{ru}}\left|\left\{k \leq k_r, j \leq j_u : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| \geq \tfrac{\delta\lambda\mu}{(1+\lambda)(1+\mu)}\right\} \in \mathcal{I}_2.$$

Therefore, $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$. □

Theorem 2.4. *Let θ be a double lacunary sequence. If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$ then, $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$ implies $A_{kj} \to A(S(\mathcal{I}_{W_2}))$.*

Proof. If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then there is an $M, N > 0$ such that $q_r < M$ and $q_u < N$, for all r, u. Suppose that $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$ and let

$$U_{ru} = U(r,u,x) := \left|\left\{(k,j)\in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right|.$$

Since $A_{kj} \to A(S_\theta(\mathcal{I}_{W_2}))$, it follows that for every $\varepsilon > 0, \delta > 0$ and for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \overline{h}_u}\left|\left\{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right| \geq \delta\right\}$$

$$= \left\{(r,u) \in \mathbb{N} \times \mathbb{N} : \frac{U_{ru}}{h_r \overline{h}_u} \geq \delta\right\} \in \mathcal{I}_2.$$

Hence, we can choose a positive integers $r_0, u_0 \in \mathbb{N}$ such that

$$\frac{U_{ru}}{h_r \overline{h}_u} < \delta, \text{ for all } r > r_0, u > u_0.$$

Now, let

$$K := \max\left\{U_{ru} : 1 \leq r \leq r_0, 1 \leq u \leq u_0\right\}$$

and let t and v be any integers satisfying $k_{r-1} < t \leq k_r$ and $j_{u-1} < v \leq j_u$. Then, for each $x \in X$ we have

$$\frac{1}{tv}\left|\left\{k \leq t, j \leq v : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right|$$

$$\leq \frac{1}{k_{r-1} j_{u-1}}\left|\left\{k \leq k_r, j \leq j_u : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{k_{r-1} j_{u-1}}\left(U_{11} + U_{12} + U_{21} + U_{22} + \cdots + U_{r_0 u_0} + \cdots + U_{ru}\right)$$

$$\leq \frac{K}{k_{r-1} j_{u-1}} \cdot r_0 u_0 + \frac{1}{k_{r-1} j_{u-1}}\left(h_{r_0} \overline{h}_{u_0+1} \frac{U_{r_0, u_0+1}}{h_{r_0} \overline{h}_{u_0+1}} + h_{r_0+1} \overline{h}_{u_0} \frac{U_{r_0+1, u_0}}{h_{r_0+1} \overline{h}_{u_0}} + \cdots + h_r \overline{h}_u \frac{U_{ru}}{h_r \overline{h}_u}\right)$$

$$\leq \frac{r_0 u_0 \cdot K}{k_{r-1} j_{u-1}} + \frac{1}{k_{r-1} j_{u-1}}\left(\sup_{\substack{r > r_0 \\ u > u_0}} \frac{U_{ru}}{h_r \overline{h}_u}\right)\left(h_{r_0} \overline{h}_{u_0+1} + h_{r_0+1} \overline{h}_{u_0} + \cdots + h_r \overline{h}_u\right)$$

$$\leq \frac{r_0 u_0 \cdot K}{k_{r-1} j_{u-1}} + \varepsilon \cdot \frac{(k_r - k_{r_0})(j_u - j_{u_0})}{k_{r-1} j_{u-1}}$$

$$\leq \frac{r_0 u_0 \cdot K}{k_{r-1} j_{u-1}} + \varepsilon \cdot q_r \cdot q_u \leq \frac{r_0 u_0 \cdot K}{k_{r-1} j_{u-1}} + \varepsilon \cdot M \cdot N.$$

Since $k_{r-1} j_{u-1} \to \infty$ as $t, v \to \infty$, it follows that

$$\frac{1}{tv}\left|\left\{k \leq t, j \leq v : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right| \to 0$$

and consequently for any $\delta_1 > 0$, the set

$$\left\{(t,v) \in \mathbb{N} \times \mathbb{N} : \frac{1}{tv}\left|\left\{k \leq t, j \leq v : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\right\}\right| \geq \delta_1\right\} \in \mathcal{I}_2.$$

This shows that $A_{kj} \to A(S(\mathcal{I}_{W_2}))$. □

Theorem 2.5. *Let θ be a double lacunary sequence. If*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty \quad \textit{and} \quad 1 < \liminf_u q_u \leq \limsup_u q_u < \infty,$$

then $\left\{S_\theta(\mathcal{I}_{W_2})\right\} = \left\{S(\mathcal{I}_{W_2})\right\}$.

Proof. This follows from Theorem 2.3 and Theorem 2.4. □

Theorem 2.6. *Let $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal satisfying property $(AP2)$ and $\theta \in \mathcal{F}(\mathcal{I}_2)$. If $\{A_{kj}\} \in \left\{S(\mathcal{I}_{W_2})\right\} \cap \left\{S_\theta(\mathcal{I}_{W_2})\right\}$, then*

$$S(\mathcal{I}_{W_2}) - \lim A_{kj} = S_\theta(\mathcal{I}_{W_2}) - \lim A_{kj}.$$

Proof. Assume that $S(\mathcal{I}_{W_2}) - \lim A_{kj} = A$ and $S_\theta(\mathcal{I}_{W_2}) - \lim A_{kj} = B$ and $A \neq B$. Let

$$0 < \varepsilon < \frac{1}{2}|d(x,A) - d(x,B)|$$

for each $x \in X$. Since $\mathcal{I}_2$ satisfies the property $(AP2)$, there exists $M \in \mathcal{F}(\mathcal{I}_2)$ (*i.e.*, $\mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that for each $x \in X$ and for $(m,n) \in M$,

$$\lim_{m,n\to\infty} \frac{1}{mn}\left|\left\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\}\right| = 0.$$

Let

$$P = \{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,A) \geq \varepsilon|\}$$

and

$$R = \{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,B)| \geq \varepsilon\}.$$

Then, $mn = |P \cup R| \leq |P| + |R|$. This implies that

$$1 \leq \frac{|P|}{mn} + \frac{|R|}{mn}.$$

Since

$$\frac{|R|}{mn} \leq 1 \quad \textit{and} \quad \lim_{m,n\to\infty} \frac{|P|}{mn} = 0,$$

so we must have

$$\lim_{m,n\to\infty} \frac{|R|}{mn} = 1.$$

Let $M^* = M \cap \theta \in \mathcal{F}(\mathcal{I}_2)$. Then, for each $x \in X$ and for $(k_l, j_t) \in M^*$ the $k_l j_t$th term of the statistical limit expression

$$\frac{1}{mn}\left|\{k \leq m, j \leq n : |d(x,A_{kj}) - d(x,B)| \geq \varepsilon\}\right|$$

is

$$\frac{1}{k_l j_t}\left|\left\{(k,j) \in \bigcup_{r,u=1,1}^{l,t} I_{ru} : |d(x,A_{kj}) - d(x,B)| \geq \varepsilon\right\}\right|$$

$$= \frac{1}{\sum\limits_{r,u=1,1}^{l,t} h_r \overline{h}_u} \sum_{r,u=1,1}^{l,t} v_{ru} h_r \overline{h}_u, \tag{2.1}$$

where

$$v_{ru} = \frac{1}{h_r \overline{h}_u}\left|\left\{(k,j) \in I_{ru} : |d(x,A_{kj}) - d(x,B)| \geq \varepsilon\right\}\right| \xrightarrow{\mathcal{I}_2} 0$$

because $A_{kj} \to B(S_\theta(\mathcal{I}_{W_2}))$. Since θ is a double lacunary sequence, (2.1) is a regular weighted mean transform of v_{ru}'s and therefore it is also $\mathcal{I}_2$-convergent to 0 as $l, t \to \infty$, and so it has a subsequence which is convergent to 0 since $\mathcal{I}_2$ satisfies property $(AP2)$. But since this is a subsequence of

$$\left\{ \frac{1}{mn} | \{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon \} | \right\}_{(m,n) \in M},$$

we infer that

$$\left\{ \frac{1}{mn} | \{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x, B)| \geq \varepsilon \} | \right\}_{(m,n) \in M}$$

is not convergent to 1. This is a contradiction. Hence, the proof is completed. □

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