LE MATEMATICHE
Vol. LXX (2015) – Fasc. II, pp. 225–233
doi: 10.4418/2015.70.2.14

MULTIVALENCE OF BIVARIATE FUNCTIONS OF BOUNDED INDEX

FATİH NURAY - RICHARD F. PATTERSON

This paper examines the relationship between the concept of bounded index and the radius of univalence, respectively p-valence, of entire bivariate functions and their partial derivatives at arbitrary points in $\mathbb{C}^2$.

1. Introduction

If $f(z_1,z_2)$ is a bivariate entire function in the bicylinder

$$\{(z_1,z_2)\in\mathbb{C}^2: \quad |z_1-a_1|<r_1, |z_2-a_2|<r_2\}$$

then $f(z_1,z_2)$ have following Taylor expansion at point (a,b),

$$f(z_1,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}(z_1-a_1)^m(z_2-a_2)^n$$

where

$$a_{mn}=\frac{1}{m!n!}\left[\frac{\partial^{m+n}f(z_1,z_2)}{\partial z_1^m\partial z_2^n}\right]_{z_1=a_1;z_2=a_2}=\frac{1}{m!n!}f^{(m,n)}(a_1,a_2).$$

Similar to Gross[6] we presented in [9] the following notion of bounded index of bivariate entire function.

Entrato in redazione: 27 ottobre 2014

AMS 2010 Subject Classification: 30D10, 30D20.

Keywords: Entire function, bivariate function, function of bounded index.

The first author acknowledges the support of The Scientific and Technological Research Council of Turkey in the preparation of this work.

Definition 1.1. A entire bivariate function $f(z_1,z_2)$ is said to be of bounded index provided that there exist integers M and N independent of z_1 and z_2 such that

$$\max_{0\le k\le M;0\le l\le N}\left\{\frac{|f^{(k,l)}(z_1,z_2)|}{k!l!}\right\}\ge\frac{|f^{(m,n)}(z_1,z_2)|}{m!n!}$$

for all m and n.

We shall say that f is of index (M,N) if N and M are the smallest integers for which above inequality holds. A entire bivariate function which is not of bounded index is said to be of unbounded index. One should observe that a f bivariate entire function is of bounded index then there exist integers $M\ge 0$, $N\ge 0$ and some $C>0$,

$$\sum_{k=0}^{M}\sum_{l=0}^{N}\frac{|f^{(k,l)}(z_1,z_2)|}{k!l!}\ge C\frac{|f^{(m,n)}(z_1,z_2)|}{m!n!} \tag{1}$$

where $m=M+1,M+2,\dots$ and $n=N+1,N+2,\dots$. In addition if the last inequality holds then

$$\max_{0\le k\le M;0\le l\le N}\left\{\frac{|f^{(k,l)}(z_1,z_2)|}{k!l!}\right\}\ge\frac{1}{(M+1)(N+1)}\frac{|f^{(m,n)}(z_1,z_2)|}{m!n!}$$

where $m,n=0,1,2,3,\dots$.

Let $r=(r_1,r_2)$ be a 2-tuples of positive real numbers. If the power series is convergent in the polydisc (or bicylinder) $|z_1-a_1|<r_1$, $|z_2-a_2|<r_2$ then r is called associated biradius of convergence of the power series. The $(a_1,a_2)\in\mathbb{C}^2$ is called the center of the bicylinder. Let $D\subset\mathbb{C}^2$ be a domain, that is, an open and connected non-empty subset of $\mathbb{C}^2$. Let $a=(a_1,a_2)\in D$ and $f:D\to\mathbb{C}$. We say that f is analytic or holomorphic at a if for some $\varepsilon>0$,

$$B(a,\varepsilon)=\{z=(z_1,z_2)\in\mathbb{C}^2:\quad \|z-a\|<\varepsilon\}\subseteq D$$

and f is given at $B(a,\varepsilon)\subseteq D$ as a power series

$$f(z_1,z_2)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}(z_1-a_1)^m(z_2-a_2)^n$$

such that for $0\le r\le\varepsilon$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|a_{mn}|r^{m+n}<\infty.$$

If f is continuous and analytic in every variable then f itself is analytic. A function f analytic in a domain D is said to be univalent there if it does not take the same value twice: $f(z_1,z_2) \neq f(z_3,z_4)$ for all pairs of distinct points (z_1,z_2) and (z_3,z_4) in D. If $f(z_1,z_2)$ is univalent in D, then $f^{(1,1)}(z_1,z_2) \neq 0$ in D. An entire function is a function that is analytic at each point in the entire $\mathbb{C}^2$.

For an entire bivariate function $f(z_1,z_2)$ and complex numbers w_1 and w_2 let $r(w_1,w_2,f(z_1,z_2))$ denote the radius of univalence of $f(z_1+w_1,z_2+w_2)$. Let

$$\begin{aligned} R_{MN}(w_1,w_2) &= R_{MN}(w_1,w_2,f(z_1,z_2)) \\ &= \max_{0\leq i\leq M;0\leq j\leq N}\{r(w_1,w_2,f^{(i,j)}(z_1,z_2))\}. \end{aligned}$$

2. Main Results

Theorem 2.1. *If*

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} mn|a_{mn}| < 1$$

then

$$f(z_1,z_2) = z_1z_2 + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} a_{mn}z_1^m z_2^n$$

is univalent and starlike in $|z_1| < 1$, $|z_2| < 1$.

Proof. Suppose that $\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} mn|a_{mn}| < 1$ and that

$$f(z_1,z_2) = z_1z_2 + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} a_{mn}z_1^m z_2^n.$$

Then in $|z_1| < 1$, $|z_2| < 1$

$$\begin{aligned} &|z_1z_2f^{(1,1)}(z_1,z_2) - f(z_1,z_2)| - |f(z_1,z_2)| \\ &= |\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(mn-1)a_{mn}z_1^m z_2^n| - |z_1z_2 + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} a_{mn}z_1^m z_2^n| \\ &< \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(mn-1)|a_{mn}| - (1 - \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}|a_{mn}|) \\ &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} mn|a_{mn}| - 1 \leq 0. \end{aligned}$$

Hence it follows that in $|z_1| < 1$, $|z_2| < 1$

$$|z_1z_2\frac{f^{(1,1)}(z_1,z_2)}{f(z_1,z_2)} - 1| < 1.$$

This shows that $f(z_1,z_2)$ is univalent and starlike in $|z_1| < 1$, $|z_2| < 1$. □

Then we prove following results.

Theorem 2.2. *Let $f(z_1,z_2)$ be a entire bivariate function. Then $f^{(1,1)}(z_1,z_2)$ is bounded index if and only if there exist integers $M>0, N>0$ and a constant $\delta>0$ such that*

$$R_{MN}(w_1,w_2) \geq \delta \text{ for all } w_1,w_2 \in \mathbb{C}^2$$

that is, for any point in the $\mathbb{C}^2$, $f^{(1,1)}(z_1,z_2)$ or one $f^{(M,N)}(z_1,z_2)$ has radius of univalence of at least δ.

Proof. Let $f^{(1,1)}(z_1,z_2)$ be of bounded index. Then

$$f^{(1,1)}(\alpha z_1,\beta z_2)$$

is of bounded index for any $\alpha,\beta \in \mathbb{C}$. Let (M,N) be the index of $f^{(1,1)}(2z_1,2z_2)$ and for $w_1,w_2 \in \mathbb{C}^2$ let

$$f(z_1,z_2)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}(z_1-w_1)^m(z_2-w_2)^n.$$

Hence, there exists (k,l), $0\leq k\leq M$, $0\leq l\leq N$, such that

$$\begin{aligned}|a_{k+1,l+1}| &\geq \frac{1}{(k+1)(l+1)}(k+1+i)(l+1+j)2^{i+j}|a_{k+1+i,l+1+j}|\\ &\geq 2^{i+j}|a_{k+1+i,l+1+j}| \text{ for } i=1,2,\ldots; j=1,2,\ldots.\end{aligned}$$

Clearly,

$$\begin{aligned}&f^{(k,l)}(z_1,z_2)\\ &=\sum_{\substack{m=k\\ n=l}}^{\infty} m(m-1)\ldots(m-k+1)n(n-1)\ldots(n-l+1)a_{mn}(z_1-w_1)^{m-k}(z_2-w_2)^{n-l}\\ &=\sum_{i,j=0}^{\infty} b_{ij}(z_1-w_1)^i(z_2-w_2)^j.\end{aligned}$$

Therefore, for $i=1,2,\ldots; j=1,2,\ldots,$

$$\begin{aligned}&\frac{|b_{i+1,j+1}|}{|b_{11}|}\\ &=\frac{(k+1+i)(k+i)\ldots(i+2)(l+1+j)(l+j)\ldots(j+2)|a_{k+1+i,l+1+j}|}{(k+1)!(l+1)!|a_{k+1,l+1}|}\\ &\leq (i+1)^k(j+1)^l\frac{|a_{k+1+i,l+1+j}|}{|a_{k+1,l+1}|}\\ &\leq \frac{(i+1)^k(j+1)^l}{2^{i+j}}.\end{aligned}$$

Thus

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} mn|b_{mn}| \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(i+1)^k(j+1)^l}{2^{i+j}}|b_{11}|$$
$$\leq |b_{11}|\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{(i+1)^{M+1}(j+1)^{N+1}}{2^{i+j}}$$
$$= |b_{11}|B_1B_2$$

where B_1 and B_1 are constants independent of w_1 and w_2. By Theorem 1,

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} mn|c_{mn}| < 1$$

implies

$$g(z_1,z_2) = z_1.z_2 + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} c_{mn}z_1^m z_2^n$$

is univalent and starlike in $|z_1| < 1$, $|z_2| < 1$. Therefore

$$f^{(k,l)}(z_1,z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}(z_1-w_1)^i(z_2-w_2)^j$$

is univalent in $|z_1 - w_1| < \frac{1}{B_1}$ and $|z_2 - w_2| < \frac{1}{B_2}$.

Conversely, choose $M > 0$, $N > 0$ and $\delta > 0$ such that $R_{MN}(w_1,w_2) \geq \delta$ for all $w_1,w_2 \in \mathbb{C}^2$. Then given $w_1,w_2 \in \mathbb{C}^2$, there exist integers $k \leq M$ and $l \leq N$ such that

$$f^{(k,l)}(z_1,z_2) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b_{ij}(z_1-w_1)^i(z_2-w_2)^j$$

is univalent in $|z_1 - w_1| < \delta$, $|z_2 - w_2| < \delta$. Obviously $b_{11} \neq 0$ and therefore

$$g(z_1,z_2) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{b_{mn}}{b_{11}}\delta^{m+n-2}z_1^m z_2^n$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}z_1^m z_2^n$$

is univalent in $|z_1| < 1$, $|z_2| < 1$ with $c_{11} = 1$. We can write

$$|c_{mn}| < e^2mn$$

for $m = 2,3,\ldots; n = 2,3,\ldots$. Hence

$$|b_{11}| \geq \frac{|b_{mn}|}{e^2mn}\delta^{m+n-2}$$

for $m=2,3,\dots;n=2,3,\dots$. Now if

$$f(z_1,z_2)=\sum_{t=0}^{\infty}\sum_{s=0}^{\infty}a_{ts}(z_1-w_1)^t(z_2-w_2)^s \text{ then}$$

$$\begin{aligned}|a_{k+1,l+1}| &\geq \frac{|b_{11}|}{(k+1!)(l+1)!}\\ &\geq \frac{|b_{mn}|}{(k+1)!(l+1)!}\frac{\delta^{m+n-2}}{e^2mn}\\ &\geq |a_{k+m,l+n}|\frac{\delta^{m+n-2}}{e^2mn}\end{aligned}$$

for $m=1,2,\dots;n=1,2,\dots$. Hence, $f^{(k,l)}(\frac{z_1}{T_1},\frac{z_2}{T_2})$ is of index not exceeding M and N for T_1 and T_2 sufficiently large and thus $f^{(1,1)}(z_1,z_2)$ is of bounded index. □

Theorem 2.3. *Let $f(z_1,z_2)$ be an entire bivariate function. Then*

$$f^{(1,1)}(z_1,z_2)$$

is of bounded index if and only if there exists an integers $p>0$, such that

$$f^{(1,1)}(z_1,z_2)$$

is p-valent in any bicylinder of radius 1.

Proof. Let $f^{(1,1)}(z_1,z_2)$ be of bounded index (M,N), (b_1,b_2) be root of

$$f(z_1,z_2)=w.$$

Let

$$f(z_1,z_2)=w+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}(z_1-b_1)^m(z_2-b_2)^n.$$

Since

$$f^{(1,1)}(z_1,z_2)$$

is of bounded index (M,N), there exists integers k and l with $1\leq k\leq M+1$, $1\leq l\leq N+1$ such that

$$|a_{kl}|\geq|a_{ij}| \text{ for } i=1,2,\dots;j=1,2,\dots.$$

Let

$$P(z_1,z_2)=\sum_{m=1}^{k}\sum_{n=1}^{l}\frac{a_{mn}}{a_{kl}}z_1^m z_2^n.$$

Then

$$P(z_1,z_2)=\prod_{i=1}^{k}\prod_{j=1}^{l}(z_1-c_i)(z_2-c_j)$$

for some $(c_i,c_j)\in\mathbb{C}^2$. Let

$$\tau_1=\{4(M+1)\}^{(M+1)} \text{ and } \tau_2=\{4(N+1)\}^{(N+1)}.$$

Then there exist constants r_1 and r_2 with

$$\frac{1}{2k\tau_1}\le r_1\le\frac{1}{\tau_1},\quad \frac{1}{2l\tau_2}\le r_2\le\frac{1}{\tau_2}$$

such that for $|z_1|=r_1$, $|z_2|=r_2$ we have

$$|z_1-c_i|\ge\frac{1}{2k\tau_1} \text{ for } i=1,2,\dots,k$$

$$|z_2-c_j|\ge\frac{1}{2l\tau_2} \text{ for } j=1,2,\dots,l.$$

Thus

$$|P(z_1,z_2)|\ge\left(\frac{1}{4k\tau_1}\right)^k\left(\frac{1}{4l\tau_2}\right)^l$$

for all $|z_1|=r_1$, $|z_2|=r_2$. Now for $|z_1|=r_1$, $|z_2|=r_2$,

$$\begin{aligned}\left|\sum_{m=k+1}^{\infty}\sum_{n=l+1}^{\infty}\frac{a_{mn}}{a_{kl}}z_1^m z_2^n\right| &\le \sum_{m=k+1}^{\infty}\sum_{n=l+1}^{\infty}\left|\frac{a_{mn}}{a_{kl}}z_1^m z_2^n\right|\\ &\le\sum_{m=1}^{k+1}\sum_{n=l+1}^{\infty}|z_1|^m|z_2|^n\\ &\le\sum_{m=k+1}^{\infty}\sum_{n=l+1}^{\infty}\frac{1}{\tau_1^m\tau_2^n}\\ &<\frac{4}{\tau_1^{k+1}\tau_2^{l+1}}.\end{aligned}$$

Therefore

$$\begin{aligned}|P(z_1,z_2)| &\ge\left(\frac{1}{2k\tau_1}\right)^k\left(\frac{1}{2l\tau_2}\right)^l\\ &\ge\frac{4}{\tau_1^{k+1}\tau_2^{l+1}}\\ &>\left|\sum_{m=k+1}^{\infty}\sum_{n=l+1}^{\infty}\frac{a_{mn}}{a_{kl}}z_1^m z_2^n\right|\end{aligned}$$

for $|z_1| = r_1$, $|z_2| = r_2$. Similarly we can find bounds for

$$\sum_{m=1}^{\infty}\sum_{n=l+1}^{\infty}\frac{a_{mn}}{a_{kl}}z_1^m z_2^n$$

and

$$\sum_{m=k+1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{a_{kl}}z_1^m z_2^n$$

from above. By the multidimensional analogue of the classical Rouché principle [1, Theorem 2.5],

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{a_{kl}}z_1^m z_2^n$$

has the same numbers of zeros in $|z_1| < r_1$, $|z_2| < r_2$ as $P(z_1,z_2)$. Hence

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}z_1^m z_2^n$$

has at most kl zeros in $|z_1| < (2k\tau_1)^{-1}$ and $|z_2| < (2l\tau_2)^{-1}$ and $f(z_1,z_2) = w$ has at most kl solutions in $|z_1 - b_1| < (2k\tau_1)^{-1}$ and $|z_2 - b_2| < (2l\tau_2)^{-1}$. In general $f(z_1,z_2)$ is at most $(M+1)(N+1)$-valent in $|z_1 - \gamma_1| < \{4(M+1)\tau_1\}^{-1}$ and $|z_2 - \gamma_2| < \{4(N+1)\tau_2\}^{-1}$ for all $\gamma_1, \gamma_2 \in \mathbb{C}^2$. Thus there exist $p \geq (M+1)(N+1)$ such that $f(z_1,z_2)$ is p-valent in any bicylinder of radius 1.

Conversely, Let

$$f(z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{mn}(z_1 - w_1)^m(z_2 - w_2)^n$$

be kl-valent in $|z_1 - w_1| < 1$ and $|z_2 - w_2| < 1$. Without loss of generality we may assume that $a_{00} = 0$. Then by [10], we have, for $m = 1,2,\ldots; n = 1,2,\ldots$,

$$|a_{mn}| < A(k,l)\max_{1\leq u\leq k;1\leq v\leq l}\{|a_{uv}|\}m^{2k-1}n^{2l-1},$$

where $A(k,l)$ depends only k and l. Thus, in general

$$\frac{f^{(m,n)}}{m!n!} < A(k,l)m^{2k-1}n^{2l-1}\max_{1\leq u\leq k;1\leq v\leq l}\left\{\frac{|f^{(u,v)}(z_1,z_2)|}{u!v!}\right\}$$

for $m = 1,2,\ldots; n = 1,2,\ldots$. Hence $f^{(1,1)}(\frac{z_1}{\tau_1},\frac{z_2}{\tau_2})$ is of index (k,l) for τ_1 and τ_2 sufficiently large and therefore $f^{(1,1)}(z_1,z_2)$ is bounded index. $\square$

REFERENCES

[1] L. A. Aizenberg - A. P. Yuzhakov, *Integral representations and residues in multidimensional complex analysis*, Translation of Mathematical Monographs, AMS, 58 (1983).

[2] E. Borel, *Sur les zeros des functions entières*, Acta. Math. 20 (1987), 357–396.

[3] J. Clunie - F. R. Keogh, *On starlile and convex schlicht functions*, J. London Math. Soc. 35 (1960), 229–233.

[4] G. H. Fricke - S. M. Shah, *Entire Functions Satisfying a Linear Differential Equation*, Indagationes Mathematicae, 78 (1) (1975), 39–41.

[5] G. H. Fricke , *A note on multivalence of functions of bounded index*, Proc. Amer. Math. Soc. 40 (1) (1973), 140–142.

[6] F. Gross, *Entire Function of Bounded Index*, Proc. Amer. Math. Soc. 18 (1967), 974–980.

[7] W. K. Hayman, *Multivalent Functions*, Cambridge Tracts in Math. and Math. Phys. 48, Cambridge Univ. Press, Cambridge, 1958.

[8] B. Lepson, *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*, Lecture Notes, 1966 Summer Institute on Entire Functions, Univ. of California, La Jolla, Ca.

[9] R. F. Patterson - F. Nuray, *Holomorfic bivariate functions of bounded index*, (Under consideration)

[10] R. F. Patterson - F. Nuray, *A characterization of holomorphic bivariate functions of bounded index*, (Under consideration)

[11] S. M. Shah, *A note on the derivatives of integral functions*, Bull. Amer. Math. Soc. 53 (1947), 1156–1163.

[12] S. M. Shah, *Entire functions of bounded index*, Proc. Amer. Math. Soc. 19 (1968), 1017–1022.

[13] S. M. Shah, *The maximum term of an entire series III*, J. Math. Oxford Ser. 19 (1948), 220–223.

FATİH NURAY
Department of Mathematics
Afyon Kocatepe University
e-mail: fnuray@aku.edu.tr

RICHARD F. PATTERSON
Department of Mathematics and Statistics
University of North Florida
e-mail: rpatters@unf.edu