CREAT. MATH. INFORM. Online version at http://creative-mathematics.ubm.ro/
28 (2019), No. 2, 143 - 150 Print Edition: ISSN 1584 - 286X Online Edition: ISSN 1843 - 441X

Lacunary invariant statistical convergence of double sequences of sets

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ABSTRACT. In this paper, we introduce the concepts of Wijsman invariant convergence, Wijsman invariant statistical convergence, Wijsman lacunary invariant convergence, Wijsman lacunary invariant statistical convergence for double sequences of sets. Also, we investigate existence of some relations among these new convergence concepts for double sequences of sets.

1. INTRODUCTION

The concept of statistical convergence was firstly introduced by Fast [7] and then this concept, using the concepts of lacunary sequence, invariant mean and double sequence, has been extended by many others (see, [6, 8, 9, 12, 13, 18, 24, 25]).

The concept of convergence of real sequences has been extended by several authors to convergence of set sequences (see, [3–5, 14–17, 27–30]). Nuray and Rhoades [14] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Using the concept of lacunary sequence, the concepts of Wijsman lacunary statistical convergence for set sequences was introduced by Ulusu and Nuray [27] and investigated relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades [14]. Also, Ulusu and Nuray [28] defined the concept of Wijsman strongly lacunary summability for set sequences and examined relation with the concept of Wijsman strongly Cesàro summability. Then, the concepts of invariant convergence, invariant statistical convergence, lacunary invariant convergence and lacunary invariant statistical convergence for set sequences was introduced by Pancaroğlu and Nuray [17]. Recently, Nuray et. al [16] studied the concepts of Wijsman statistical convergence and Wijsman statistical Cauchy for double sequences of sets and investigate the relationship between this concepts. Also, Using the concept of double lacunary sequence, the concepts of Wijsman lacunary statistical convergence for double sequences of sets was introduced Nuray et al. [15].

In this study, we introduce the concepts of Wijsman invariant convergence, Wijsman invariant statistical convergence, Wijsman lacunary invariant convergence, Wijsman lacunary invariant statistical convergence for double sequences of sets. Also, we investigate existence of some relations among these new convergence concepts for double sequences of sets.

2. DEFINITIONS AND NOTATIONS

Now, we recall the basic definitions and concepts (See, [1–5, 10, 11, 15, 16, 18–23, 26]).

Received: 17.12.2018. In revised form: 08.04.2019. Accepted: 15.04.2019
2010 *Mathematics Subject Classification.* 40A05, 40A35.
Key words and phrases. *Statistical convergence, lacunary sequence, invariant convergence, double sequence of sets, Wijsman convergence.*
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Let X be any non-empty set and $\mathbb{N}$ be the set of natural numbers. The function $f : \mathbb{N} \to P(X)$ is defined by $f(k) = A_k \in P(X)$ for each $k \in \mathbb{N}$, where $P(X)$ is power set of X. The sequence $\{A_k\} = (A_1, A_2, \ldots)$, which is the range's elements of f, is said to be sequences of sets.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, the distance from x to A is defined by $d(x, A) = \inf_{a \in A} \rho(x, a)$.

Throughout this paper, (X, ρ) and A, A_k will be taken as a metric space and any non-empty closed subsets of X, respectively.

We say that the sequence $\{A_k\}$ is Wijsman convergent to A if for each $x \in X$,

$$\lim_{k \to \infty} d(x, A_k) = d(x, A).$$

A double sequence $x = (x_{kj})$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. It is denoted by

$$P - \lim_{k,j \to \infty} x_{kj} = L.$$

We say that a double sequence $\{A_{kj}\}$ is Wijsman convergent to A if for each $x \in X$,

$$P - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A).$$

We say that a double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A if for every $\varepsilon > 0$ and each $x \in X$,

$$P - \lim_{m,n \to \infty} \frac{1}{mn} \left| \left\{ k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \right| = 0.$$

A double sequence $\theta_2 = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \;\; h_r = k_r - k_{r-1} \to \infty \;\text{ as }\; r \to \infty \;\text{ and }\; j_0 = 0, \; \bar{h}_u = j_u - j_{u-1} \to \infty \;\text{ as }\; u \to \infty.$$

We use following notations in the sequel:

$$k_{ru} = k_r j_u, \;\; h_{ru} = h_r \bar{h}_u, \;\; I_{ru} = \left\{ (k, j) : k_{r-1} < k \leq k_r \;\text{ and }\; j_{u-1} < j \leq j_u \right\},$$

$$q_r = \frac{k_r}{k_{r-1}} \;\text{ and }\; q_u = \frac{j_u}{j_{u-1}}.$$

Throughout this paper, $\theta_2 = \{(k_r, j_s)\}$ will be taken as lacunary sequence.

A double sequence $\{A_{kj}\}$ is Wijsman lacunary convergent to A if for each $x \in X$,

$$P - \lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} d(x, A_{kj}) = d(x, A).$$

It is denoted by $A_{kj} \stackrel{W_2 N_\theta}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$P - \lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)| = 0.$$

It is denoted by $A_{kj} \stackrel{[W_2 N_\theta]}{\longrightarrow} A$.

A double sequence $\{A_{kj}\}$ is Wijsman lacunary statistically convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$P - \lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} \left| \left\{ (k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \right| = 0.$$

It is denoted by $st_2 - \lim_{W_\theta} A_{kj} = A$.

Let σ be a mapping of the positive integers into themselves. A continuous linear functional ϕ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

(1) $\phi(x) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all n,
(2) $\phi(e) = 1$, where $e = (1, 1, 1, ...)$, and
(3) $\phi(x_{\sigma(n)}) = \phi(x_n)$ for all $x \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, ϕ extends the limit functional on c, the space of convergent sequences, in the sense that $\phi(x) = \lim x$ for all $x \in c$. In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown that

$$V_\sigma = \left\{ x = (x_n) \in \ell_\infty : \lim_{m\to\infty} t_{mn}(x) = L, \text{ uniformly in } n, \ L = \sigma - \lim x \right\},$$

where

$$t_{mn}(x) = \frac{x_n + Tx_n + ... + T^m x_n}{m+1}.$$

A sequence $x = (x_n)$ is said to be σ-convergent to L if and only if all of its σ-means coincide with L.

In [26], Schaefer proved that a bounded sequence $x = (x_n)$ of real numbers is σ-convergent to L if and only if

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=1}^{p} x_{\sigma^k(m)} = L,$$

uniformly in m.

A double sequence $x = (x_{kj})$ of real numbers is said to be σ-convergent to L if

$$P - \lim_{p,q\to\infty} \frac{1}{pq} \sum_{k=1}^{p} \sum_{j=1}^{q} x_{\sigma^k(s),\sigma^j(t)} = L,$$

uniformly in s and t.

3. Main results

In this section, we introduce the concepts of Wijsman invariant convergence, Wijsman invariant statistical convergence, Wijsman lacunary invariant convergence, Wijsman lacunary invariant statistical convergence for double sequences of sets. Also, we investigate existence of some relations among these new convergence concepts for double sequences of sets.

Definition 3.1. A double sequence $\{A_{kj}\}$ is Wijsman invariant convergent to A if for each $x \in X$

$$P - \lim_{p,q\to\infty} \frac{1}{pq} \sum_{k,j=1,1}^{p,q} d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) = d(x, A),$$

uniformly in s and t. In this case, we write $A_{kj} \stackrel{W_2 V_\sigma}{\longrightarrow} A$.

Definition 3.2. A double sequence $\{A_{kj}\}$ is Wijsman strongly invariant convergent to A if for each $x \in X$

$$P - \lim_{p,q\to\infty} \frac{1}{pq} \sum_{k,j=1,1}^{p,q} \left| d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A) \right| = 0,$$

uniformly in s and t. In this case, we write $A_{kj} \stackrel{[W_2V_\sigma]}{\longrightarrow} A$.

Definition 3.3. A double sequence $\{A_{kj}\}$ is Wijsman invariant statistically convergent to A if for every $\varepsilon > 0$ and each $x \in X$

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \left| \left\{ k \le m, j \le n : \left| d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A) \right| \ge \varepsilon \right\} \right| = 0,$$

uniformly in s and t. In this case, we write $A_{kj} \stackrel{W_2S_\sigma}{\longrightarrow} A$.

Definition 3.4. A double sequence $\{A_{kj}\}$ is Wijsman lacunary invariant convergent to A if for each $x \in X$

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k,j\in I_{ru}} d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) = d(x,A),$$

uniformly in s and t. In this case, we write $A_{kj} \stackrel{W_2V_\sigma^\theta}{\longrightarrow} A$.

Definition 3.5. A double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary invariant convergent to A if for each $x \in X$

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k,j\in I_{ru}} \left| d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A) \right| = 0,$$

uniformly in s and t. In this case, we write $A_{kj} \stackrel{[W_2V_\sigma^\theta]}{\longrightarrow} A$.

Definition 3.6. A double sequence $\{A_{kj}\}$ is Wijsman lacunary invariant statistically convergent to A if for every $\varepsilon > 0$ and for each $x \in X$

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \left| \left\{ (k,j) \in I_{ru} : \left| d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A) \right| \ge \varepsilon \right\} \right| = 0,$$

uniformly in s and t. In this case, we write $A_{kj} \stackrel{W_2S_\sigma^\theta}{\longrightarrow} A$.

Theorem 3.1.

(i) $A_{kj} \stackrel{[W_2V_\sigma^\theta]}{\longrightarrow} A$ *implies* $A_{kj} \stackrel{W_2S_\sigma^\theta}{\longrightarrow} A$,

(ii) $d(x, A_{kj}) = O\big(d(x,A)\big)$ *and* $A_{kj} \stackrel{W_2S_\sigma^\theta}{\longrightarrow} A$ *implies* $A_{kj} \stackrel{[W_2V_\sigma^\theta]}{\longrightarrow} A$.

Proof. (i) : Let $A_{kj} \stackrel{[W_2V_\sigma^\theta]}{\longrightarrow} A$. For every $\varepsilon > 0$ and each $x \in X$, we can write

$$\begin{aligned} &\sum_{k,j\in I_{ru}} \left| d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A) \right| \\ &\qquad \ge \sum_{\substack{k,j\in I_{ru} \\ \left| d\left(x, A_{\sigma^k(s),\sigma^j(t)}\right) - d(x,A) \right| \ge \varepsilon}} \left| d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A) \right| \\ &\qquad \ge \varepsilon \cdot \left| \left\{ (k,j) \in I_{ru} : \left| d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A) \right| \ge \varepsilon \right\} \right| \end{aligned}$$

which yields the result.

(ii) : Suppose that $d(x, A_{kj}) = O\big(d(x, A)\big)$ and $A_{kj} \to A(W_2S_\sigma^\theta)$. Since $d(x, A_{kj}) = O\big(d(x, A)\big)$, then there exists $M > 0$ such that for each $x \in X$,

$$\left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \leq M$$

for all $k,\ j,\ s$ and t. Thus, for every $\varepsilon > 0$ and each $x \in X$, we can write

$$\begin{aligned}
&\frac{1}{h_r\overline{h}_u}\sum_{k\in I_{ru}}\left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \\
&= \frac{1}{h_r\overline{h}_u}\sum_{\substack{k,j\in I_{ru} \\ \left|d\left(x,A_{\sigma^k(s),\sigma^j(t)}\right)-d(x,A)\right|\geq\varepsilon}}\left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \\
&+\frac{1}{h_r\overline{h}_u}\sum_{\substack{k,j\in I_{ru} \\ \left|d\left(x,A_{\sigma^k(s),\sigma^j(t)}\right)-d(x,A)\right|<\varepsilon}}\left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \\
&\leq \frac{M}{h_r\overline{h}_u}\left|\left\{k \in I_{ru} : \left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \geq \varepsilon\right\}\right| + \varepsilon
\end{aligned}$$

from which the result follows. □

Theorem 3.2. *Suppose that for given $\varepsilon_1 > 0$ and every $\varepsilon > 0$, there exists m_0, n_0, s_0 and t_0 such that for each $x \in X$,*

$$\frac{1}{mn}\left|\left\{0 \leq k \leq m-1,\ 0 \leq j \leq n-1 : \left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \geq \varepsilon\right\}\right| < \varepsilon_1$$

for all $m \geq m_0, n > n_0, s > s_0$ and $t \geq t_0$, then $\{A_{kj}\} \in W_2S_\sigma$.

Proof. Let $\varepsilon_1 > 0$ be given. For every $\varepsilon > 0$, choose m_0', n_0', s_0 and t_0 such that for each $x \in X$,

$$\frac{1}{mn}\left|\left\{0 \leq k \leq m-1, 0 \leq j \leq n-1 : \left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \geq \varepsilon\right\}\right| < \frac{\varepsilon_1}{2} \qquad (3.1)$$

for all $m \geq m_0'$, $n \geq n_0'$, $s \geq s_0$ and $t \geq t_0$. It is enough to prove that there exist m_0'', n_0'' such that for each $x \in X$,

$$\frac{1}{mn}\left|\left\{0 \leq k \leq m-1, 0 \leq j \leq n-1 : \left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \geq \varepsilon\right\}\right| < \varepsilon_1 \qquad (3.2)$$

for $m \geq m_0'', n \geq n_0'', 0 \leq s \leq s_0$ and $0 \leq t \leq t_0$.

Since taking $m_0 = max\{m_0', m_0''\}$ and $n_0 = max\{n_0', n_0''\}$, (3.2) will hold for each $x \in X$, for $m \geq m_0, n \geq n_0$ and for all s and t, which gives the result.

Once s_0 and t_0 have been chosen $0 \leq s \leq s_0,\ 0 \leq t \leq t_0$, s_0 and t_0 are fixed. So, put

$$K = \left|\left\{0 \leq k \leq s_0 - 1, 0 \leq j \leq t_0 - 1 : \left|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x, A)\right| \geq \varepsilon\right\}\right|.$$

Now taking $0 \le s \le s_0,\ 0 \le t \le t_0$ and $m \ge s_0,\ n \ge t_0$, by (3.1) for each $x \in X$, we get

$$\frac{1}{mn}\Big|\big\{0 \le k \le m-1, 0 \le j \le n-1 : \big|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A)\big| \ge \varepsilon\big\}\Big|$$

$$\le \frac{1}{mn}\Big|\big\{0 \le k \le s_0-1, 0 \le j \le t_0-1 : \big|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A)\big| \ge \varepsilon\big\}\Big|$$

$$+\frac{1}{mn}\Big|\big\{s_0 \le k \le m-1, t_0 \le j \le n-1 : \big|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A)\big| \ge \varepsilon\big\}\Big|$$

$$\le \frac{K}{mn} + \frac{\varepsilon_1}{2}$$

and taking m, n sufficiently large, we can write

$$\frac{K}{mn} + \frac{\varepsilon_1}{2} < \varepsilon_1$$

which gives (3.2) and hence, the result follows. □

Theorem 3.3. *$A_{kj} \overset{W_2S_\sigma^\theta}{\longrightarrow} A$ if and only if $A_{kj} \overset{W_2S_\sigma}{\longrightarrow} A$, for every double lacunary sequence θ_2.*

Proof. Let $A_{kj} \overset{W_2S_\sigma^\theta}{\longrightarrow} A$. Then, for given $\varepsilon_1 > 0$ there exists r_0, u_0 such that for every $\varepsilon > 0$ and each $x \in X$,

$$\frac{1}{h_r\overline{h}_u}\Big|\big\{0 \le k \le h_{r-1}, 0 \le j \le \overline{h}_{u-1} : \big|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A)\big| \ge \varepsilon\big\}\Big| < \varepsilon_1$$

for $r \ge r_0, u > u_0$ and $s = k_{r-1}+1+v$, $v \ge 0$, $t = j_{r-1}+1+w$, $w \ge 0$. Let $m \ge h_r$ and $n \ge \overline{h}_u$. Write $m = \alpha h_r + y$ and $n = \beta\overline{h}_u + z$ where $0 \le y \le h_r$ and $0 \le z \le \overline{h}_u$, α and β are integers. Since $m \ge h_r$ and $n \ge \overline{h}_u$, we can write

$$\frac{1}{mn}\Big|\big\{0 \le k \le m-1, 0 \le j \le n-1 : \big|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A)\big| \ge \varepsilon\big\}\Big|$$

$$\le \frac{1}{mn}\Big|\big\{0 \le k \le (\alpha+1)h_r - 1, 0 \le j \le (\beta+1)\overline{h}_u - 1 : \big|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A)\big| \ge \varepsilon\big\}\Big|$$

$$= \frac{1}{mn}\sum_{f=0}^{\alpha}\sum_{g=0}^{\beta}\Big|\big\{fh_r \le k \le (f+1)h_r - 1, g\overline{h}_u \le j \le (g+1)\overline{h}_u - 1 : \big|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A)\big| \ge \varepsilon\big\}\Big|$$

$$\le \frac{1}{mn}(\alpha+1)(\beta+1)\cdot h_r\overline{h}_u\cdot\varepsilon_1$$

$$\le \frac{1}{mn}\cdot 4\cdot\alpha\cdot\beta\cdot h_r\overline{h}_u\cdot\varepsilon_1$$

and since for $\frac{1}{m}\cdot\alpha\cdot h_r \le 1$ and $\frac{1}{n}\cdot\beta\cdot\overline{h}_u \le 1$, we get

$$\frac{1}{mn}\Big|\big\{0 \le k \le m-1, 0 \le j \le n-1 : \big|d\big(x, A_{\sigma^k(s),\sigma^j(t)}\big) - d(x,A)\big| \ge \varepsilon\big\}\Big| \le 4\cdot\varepsilon_1.$$

Thus, by Theorem (3.2), $W_2S_\sigma^\theta \subset W_2S_\sigma$. It is easy to see that $W_2S_\sigma \subset W_2S_\sigma^\theta$. This completes the proof. □

By using the same techniques as in Theorem 3.3, we can prove the following theorem.

Theorem 3.4. *$A_{kj} \xrightarrow{W_2V_\sigma^\theta} A$ if and only if $A_{kj} \xrightarrow{W_2V_\sigma} A$, for every double lacunary sequence θ_2.*

43When $(\sigma(s), \sigma(t)) = (s+1, t+1)$, from Definitions 3.1-3.6 we have the definitions of Wijsman almost convergence, Wijsman strongly almost convergence, Wijsman almost statistically convergence, Wijsman lacunary almost convergence, Wijsman strongly lacunary almost convergence and Wijsman lacunary almost statistically convergence for double sequences of sets. So, similar inclusions to Theorems 3.1-3.4 hold between Wijsman strongly almost convergent double set sequences, Wijsman almost statistically convergent double set sequences, Wijsman strongly lacunary almost convergent double set sequences and Wijsman lacunary almost statistically convergent double set sequences, which have not appeared anywhere by this time.

Acknowledgements. The authors are grateful to referees for his careful reading of this paper and several valuable suggestions which improved the presentation of the paper.

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