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$\mathcal{I}_2$-CONVERGENCE OF DOUBLE SEQUENCES OF FUNCTIONS

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Abstract. In this work, we discuss various kinds of ideal convergence for double sequences of functions with values in $\mathbb{R}$. We introduce the concepts of $\mathcal{I}_2$, $\mathcal{I}_2^*$-pointwise convergence and the concepts of $\mathcal{I}_2$, $\mathcal{I}_2^*$-pointwise Cauchy for double sequences of functions and show the relation between them.

1. Background and Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [12] and Schoenberg [30]. This concept was extended to the double sequences by Mursaleen and Edely [23]. A lot of development have been made in this area after the works of Šalát [29], Móricz [22] and Fridy [14, 15]. Furthermore, Gökhan et al. [17] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions. In general, statistically convergent sequences satisfy many of the properties of ordinary convergent sequences in metric spaces [12, 14, 15, 27]. Çakan and Altay [4] presented multidimensional analogues of the results presented by Fridy and Orhan [13].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [19] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subset of the set of natural numbers. Nuray and Ruckle [25] indepedently introduced the same with another name generalized statistical convergence. Kostyrko et al. [20] gave some of basic properties of $\mathcal{I}$-convergence and dealt with extremal $\mathcal{I}$-limit points. Das et al. [5] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Also Das and Malik [6] introduced the concept of $\mathcal{I}$-limit points, $\mathcal{I}$-cluster points and $\mathcal{I}$-limit superior and $\mathcal{I}$-limit inferior of double sequences. Balcerzak et al. [3] discussed various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in $\mathbb{R}$ or in a metric space. Gezer and Karakuş [16] investigated $\mathcal{I}$-pointwise and uniform convergence and $\mathcal{I}^*$-pointwise and uniform convergence of function sequences and then they examined the relation between them. Dündar and Altay [10] studied

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the concepts of $\mathcal{I}_2$-Cauchy and $\mathcal{I}_2^*$-Cauchy for double sequences in a linear metric space and investigated the relation between $\mathcal{I}_2$-convergence and $\mathcal{I}_2^*$-convergence of double sequences of functions defined between linear metric spaces. Also some results on $\mathcal{I}_2$-convergence may be found in [2, 7, 8, 9, 11, 18, 21, 24, 28, 31].

In this study, we discuss various kinds of ideal convergence for double sequences of functions with values in $\mathbb{R}$. We introduce the concepts of $\mathcal{I}_2$, $\mathcal{I}_2^*$-pointwise convergence and $\mathcal{I}_2$, $\mathcal{I}_2^*$-pointwise Cauchy sequences for double sequences of functions and show the relation between them.

2. Definitions and Notations

Now, we recall the concept of statistical, ideal convergence of sequences and basic concepts. (See [1, 5, 10, 12, 17, 19, 23, 26, 28]). Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers.

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{mn} - L| < \varepsilon$, whenever $m, n > N_\varepsilon$. In this case we write

$$\lim_{m,n\to\infty} x_{mn} = L.$$

A double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of real numbers is said to be bounded if there exists a positive real number M such that $|x_{mn}| < M$, for all $m, n \in \mathbb{N}$. That is

$$\|x\|_\infty = \sup_{m,n} |x_{mn}| < \infty.$$

Let $K \subset \mathbb{N} \times \mathbb{N}$. Let K_{mn} be the number of $(j, k) \in K$ such that $j \leq m$, $k \leq n$. That is,

$$K_{mn} = |\{(j, k) : j \leq m, k \leq n\}|,$$

where $|A|$ denotes the number of elements in A. If the double sequence $\left\{\frac{K_{mn}}{m.n}\right\}$ has a limit then we say that K has double natural density and is denoted by

$$d_2(K) = \lim_{m,n\to\infty} \frac{K_{mn}}{m.n}.$$

A double sequence $x = (x_{mn})$ of real numbers is said to be statistically convergent to $L \in \mathbb{R}$, if for any $\varepsilon > 0$ we have $d_2(A(\varepsilon)) = 0$, where $A(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - L| \geq \varepsilon\}$.

A double sequence of functions $\{f_{mn}\}$ is said to be pointwise convergent to f on a set $S \subset \mathbb{R}$, if for each point $x \in S$ and for each $\varepsilon > 0$, there exists a positive integer $N = N(x, \varepsilon)$ such that

$$|f_{mn}(x) - f(x)| < \varepsilon,$$

for all $m, n > N$. In this case we write

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x) \ or \ f_{mn} \to f,$$

on S.

A double sequence of functions $\{f_{ij}\}$ is said to be pointwise statistically convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(i, j), i \leq m \ and \ j \leq n : |f_{ij}(x) - f(x)| \geq \varepsilon\}| = 0,$$

for each (fixed) $x \in S$, i.e., for each (fixed) $x \in S$,

$$|f_{ij}(x) - f(x)| < \varepsilon, \ \ a.a.(i,j).$$

In this case we write

$$st - \lim_{i,j\to\infty} f_{ij}(x) = f(x) \ or \ f_{ij} \to_{st} f,$$

on S.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:
i) $\emptyset \in \mathcal{I}$, ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, iii) $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.
$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:
i) $\emptyset \notin \mathcal{F}$, ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, iii) $A \in \mathcal{F}$, $A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1. [19] *If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : (\exists A \in \mathcal{I})(M = X \backslash A)\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Throughout the paper we take $\mathcal{I}_2$ as a nontrivial admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belong to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

Let $\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : \big(\exists m(A) \in \mathbb{N}\big)\big(i, j \geq m(A) \Rightarrow (i,j) \notin A\big)\}$. Then $\mathcal{I}_2^0$ is a nontrivial strongly admissible ideal and clearly $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2$-convergent to $L \in X$, if for any $\varepsilon > 0$ we have

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, L) \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we say that x is $\mathcal{I}_2$ -convergent to $L \in X$ and we write

$$\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L.$$

If $\mathcal{I}_2$ is a strongly admissible ideal on $\mathbb{N} \times \mathbb{N}$, then usual convergence implies $\mathcal{I}_2$-convergence.

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2^*$-convergent to $L \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \backslash M \in \mathcal{I}_2$) such that

$$\lim_{m,n\to\infty} x_{mn} = L,$$

for $(m,n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L.$$

Let (X, ρ) be a linear metric space and $\mathcal{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a strongly admissible ideal. A double sequence $x = (x_{mn})$ of elements of X is said to be $\mathcal{I}_2$-Cauchy if for every $\varepsilon > 0$, there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : \rho(x_{mn}, x_{st}) \geq \varepsilon\} \in \mathcal{I}_2.$$

We say that an admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

Now, we begin with quoting the lemmas due to Das et al. [5] which are needed throughout the paper.

Lemma 2.2 ([5], Theorem 1)**.** *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If $\mathcal{I}^*{}_2 - \lim_{m,n\to\infty} x_{mn} = L$ then $\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$.*

Lemma 2.3 ([5], Theorem 3)**.** *If $\mathcal{I}_2$ is an admissible ideal of $\mathbb{N} \times \mathbb{N}$ having the property (AP2) and (X, ρ) is an arbitrary metric space, then for an arbitrary double sequence $x = (x_{mn})_{m,n\in\mathbb{N}}$ of elements of X, $\mathcal{I}_2 - \lim_{m,n\to\infty} x_{mn} = L$ implies $\mathcal{I}_2^* - \lim_{m,n\to\infty} x_{mn} = L$.*

3. $\mathcal{I}_2$ and $\mathcal{I}_2^*$-Convergence Of Double Sequences Of Functions

Throughout the paper we take convergent instead of pointwise convergent.

Definition 3.1. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to $\mathcal{I}_2$-convergent to f on a set $S \subset \mathbb{R}$, if for every $\varepsilon > 0$*

$$\{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each (fixed) $x \in S$. This can be written by the formula

$$(\forall x \in S)\ (\forall \varepsilon > 0)\ (\exists H \in \mathcal{I}_2)\ (\forall (m,n) \notin H)\ \ |f_{mn}(x) - f(x)| < \varepsilon.$$

This convergence can be showed by

$$f_{mn} \to_{\mathcal{I}_2} f.$$

The function f is called the double $\mathcal{I}_2$- limit (or Pringsheim $\mathcal{I}_2$-limit) function of the $\{f_{mn}\}$.

Theorem 3.2. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. $\mathcal{I}_2$-limit of any double sequence $\{f_{mn}\}$ of functions on $S \subset \mathbb{R}$ if exist is unique.*

Proof. Let a double sequence $\{f_{mn}\}$ of functions on $S \subset \mathbb{R}$. Assume that

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x_0) = f_1(x_0) \text{ and } \mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x_0) = f_2(x_0),$$

on S, where $f_1(x_0) \neq f_2(x_0)$, for a $x_0 \in S$. Since $f_1(x_0) \neq f_2(x_0)$, so we may suppose that $f_1(x_0) > f_2(x_0)$. Select

$$\varepsilon = \frac{f_1(x_0) - f_2(x_0)}{3},$$

so that the neighborhoods $(f_1(x_0) - \varepsilon, f_1(x_0) + \varepsilon)$ and $(f_2(x_0) - \varepsilon, f_2(x_0) + \varepsilon)$ of points $f_1(x_0)$ and $f_2(x_0)$, respectively, are disjoints. Since

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x_0) = f_1(x_0) \text{ and } \mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x_0) = f_2(x_0),$$

therefore we have

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x_0) - f_1(x_0)| \geq \varepsilon\} \in \mathcal{I}_2$$

and

$$B(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x_0) - f_2(x_0)| \geq \varepsilon\} \in \mathcal{I}_2.$$

This implies that the sets

$$A^c(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x_0) - f_1(x_0)| < \varepsilon\}$$

and

$$B^c(\varepsilon) \quad = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x_0) - f_2(x_0)| < \varepsilon\}$$

belongs to $\mathcal{F}(\mathcal{I}_2)$ and $A^c(\varepsilon) \cap B^c(\varepsilon)$ is a non empty set in $\mathcal{F}(\mathcal{I}_2)$.

Since $A^c(\varepsilon) \cap B^c(\varepsilon) \neq \emptyset$, we obtain a contradiction to the fact that the neighborhoods $(f_1(x_0)-\varepsilon, f_1(x_0)+\varepsilon)$ and $(f_2(x_0)-\varepsilon, f_2(x_0)+\varepsilon)$ of points $f_1(x_0)$ and $f_2(x_0)$, respectively, are disjoints. Hence, it is clear that $f_1(x_0) = f_2(x_0)$ and consequently we have

$$f_1(x) = f_2(x), \quad (i.e., f_1 = f_2),$$

for each $x \in S$. □

Theorem 3.3. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ be a double sequence of functions and f be a function on $S \subset \mathbb{R}$. Then*

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ implies } \mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in S$.

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{m,n\to\infty} f_{mn}(x) = f(x)$ for each $x \in S$, therefore there exists a positive integer $k_0 = k_0(\varepsilon, x)$ such that $|f_{mn}(x) - f(x)| < \varepsilon$, whenever $m, n \geq k_0$. This implies that the set

$$\begin{aligned} A(\varepsilon) &= \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\} \\ &\subset \big((\mathbb{N} \times \{1,2,...,(k_0-1)\}) \cup (\{1,2,...,(k_0-1)\} \times \mathbb{N})\big). \end{aligned}$$

Since $\mathcal{I}_2$ be a strongly admissible ideal, therefore

$$\big((\mathbb{N} \times \{1,2,...,(k_0-1)\}) \cup (\{1,2,...,(k_0-1)\} \times \mathbb{N})\big) \in \mathcal{I}_2.$$

Hence, it is clear that $A(\varepsilon) \in \mathcal{I}_2$ and consequently we have

$$f_{mn} \to_{\mathcal{I}_2} f.$$

□

Theorem 3.4. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ and $\{g_{mn}\}$ be double sequences of functions, f and g be functions on $S \subset \mathbb{R}$ and*

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ and } \mathcal{I}_2 - \lim_{m,n\to\infty} g_{mn}(x) = g(x),$$

for each $x \in S$. Then, for every $(m,n) \in K$ we have

(i) If $f_{mn}(x) \geq 0$ then $f(x) \geq 0$ and (ii) If $f_{mn}(x) \leq g_{mn}(x)$ then $f(x) \leq g(x)$, for each $x \in S$, where $K \subseteq \mathbb{N} \times \mathbb{N}$ and $K \in \mathcal{F}(\mathcal{I}_2)$.

Proof. (i) Suppose that $f(x) < 0$. Select $\varepsilon = -\frac{f(x)}{2}$ for each $x \in S$. Since $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$, so there exists the set M such that

$$M = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| < \varepsilon\} \in \mathcal{F}(\mathcal{I}_2).$$

Since $M, K \in \mathcal{F}(\mathcal{I}_2)$ then $M \cap K$ is a non empty set in $\mathcal{F}(\mathcal{I}_2)$. So, we can find out a pair (m_0, n_0) in K such that

$$|f_{m_0 n_0}(x) - f(x)| < \varepsilon.$$

Since $f(x) < 0$ and $\varepsilon = -\frac{f(x)}{2}$ for each $x \in S$, then we have

$$f_{m_0 n_0}(x) < 0.$$

This is a contradiction to the fact $f_{mn}(x) > 0$ for every $(m, n) \in K$. Hence we have $f(x) > 0$, for each $x \in S$.

(ii) Suppose that $f(x) > g(x)$. Select $\varepsilon = \frac{f(x)-g(x)}{3}$ for each $x \in S$, so that the neighborhoods $(f(x)-\varepsilon, f(x)+\varepsilon)$ and $(g(x)-\varepsilon, g(x)+\varepsilon)$ of $f(x)$ and $g(x)$, respectively, are disjoints. Since $\mathcal{I}_2-\lim_{m,n\to\infty} f_{mn}(x) = f(x)$ and $\mathcal{I}_2-\lim_{m,n\to\infty} g_{mn}(x) = g(x)$ and $\mathcal{F}(\mathcal{I}_2)$ is a filter on $\mathbb{N} \times \mathbb{N}$, therefore we have

$$A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| < \varepsilon\} \in \mathcal{F}(\mathcal{I}_2)$$

and

$$B = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |g_{mn}(x) - g(x)| < \varepsilon\} \in \mathcal{F}(\mathcal{I}_2).$$

This implies that $\emptyset \neq A \cap B \cap K \in \mathcal{F}(\mathcal{I}_2)$. There exists a pair (m_0, n_0) in K such that

$$|f_{m_0 n_0}(x) - f(x)| < \varepsilon \text{ and } |g_{m_0 n_0}(x) - g(x)| < \varepsilon.$$

Since $f(x) > g(x)$ and $\varepsilon = \frac{f(x)-g(x)}{3}$ for each $x \in S$, then we have

$$f_{m_0 n_0}(x) > g_{m_0 n_0}(x).$$

This is a contradiction to the fact $f_{mn}(x) \leq g_{mn}(x)$ for every $(m, n) \in K$. Thus we have

$$f(x) \leq g(x),$$

for each $x \in S$. □

Theorem 3.5. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$, $\{g_{mn}\}$ and $\{h_{mn}\}$ be double sequences of functions and k be a function on $S \subset \mathbb{R}$. If*
(i) $\{f_{mn}\} \leq \{g_{mn}\} \leq \{h_{mn}\}$, for every $(m, n) \in K$, where $\mathbb{N} \times \mathbb{N} \supseteq K \in \mathcal{F}(\mathcal{I}_2)$ and
(ii) $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = k(x)$ and $\mathcal{I}_2 - \lim_{m,n\to\infty} h_{mn}(x) = k(x)$,
then $\mathcal{I}_2 - \lim_{m,n\to\infty} g_{mn}(x) = k(x)$.

Proof. Let $\varepsilon > 0$ be given. By condition (ii) we have

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - k(x)| \geq \varepsilon\} \in \mathcal{I}_2$$

and

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |h_{mn}(x) - k(x)| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in S$. This implies that the sets

$$P = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - k(x)| < \varepsilon\}$$

and

$$R = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |h_{mn}(x) - k(x)| < \varepsilon\}$$

belongs to $\mathcal{F}(\mathcal{I}_2)$ for each $x \in S$. Let

$$Q = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |g_{mn}(x) - k(x)| < \varepsilon\},$$

for each $x \in S$. It is clear that, the set $P \cap R \cap K$ is contained in Q. Since $P \cap R \cap K \in \mathcal{F}(\mathcal{I}_2)$ and $P \cap R \cap K \subset Q$, then from the property of filter we have $Q \in \mathcal{F}(\mathcal{I}_2)$. Hence

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |g_{mn}(x) - k(x)| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in S$. This completes the proof. $\square$

Definition 3.6. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to be pointwise $\mathcal{I}_2^*$-convergent to f on $S \subset \mathbb{R}$, if and only if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e. $\mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that*

$$\lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for $(m,n) \in M$ and we write

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ or } f_{mn} \to_{\mathcal{I}_2^*} f.$$

Theorem 3.7. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, $\{f_{mn}\}$ be a double sequence of functions and f be a function on $S \subset \mathbb{R}$. Then*

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x) \text{ implies } \mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for each $x \in S$.

Proof. Since $\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$, so there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $\mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) such that

$$\lim_{\substack{m,n\to\infty \\ (m,n)\in M}} f_{mn}(x) = f(x).$$

Let $\varepsilon > 0$. Then there exists $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that $|f_{mn}(x) - f(x)| < \varepsilon$, for all $(m,n) \in M$ such that $m,n \geq k_0$ and for each $x \in S$. Then, we have

$$\begin{aligned} A(\varepsilon) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\} \\ &\subset H \cup \left[M \cap \left((\{1,2,3,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,3,...,(k_0-1)\})\right)\right], \end{aligned}$$

for each $x \in S$. Since $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a strongly admissible ideal, then

$$H \cup \left[M \cap \left((\{1,2,3,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,3,...,(k_0-1)\})\right)\right] \in \mathcal{I}_2$$

and therefore $A(\varepsilon) \in \mathcal{I}_2$. This implies that

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x).$$

$\square$

Theorem 3.8. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, (X, d_x) and (Y, d_y) are metric spaces, $f_{mn} : X \to Y$ is a double sequence of functions and $f : X \to Y$ is a function. If Y has no accumulation point, then $\mathcal{I}_2^*$ and $\mathcal{I}_2$-convergence coincide.*

Proof. By Theorem 3.7, we must show that if $\{f_{mn}\}$ double sequence of functions is $\mathcal{I}_2$-convergent, so it is $\mathcal{I}_2^*$-convergent. We suppose that

$$\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x),$$

for $x \in X$ and $f(x) \in Y$. Since Y has no accumulation point, so there exists a $\delta > 0$ such that

$$B_\delta(f(x)) = \{f_{mn}(x) : d_y(f_{mn}(x), f(x)) < \delta\} = \{f(x)\},$$

for each $x \in X$. Since $\mathcal{I}_2 - \lim_{m,n\to\infty} f_{mn}(x) = f(x)$ then we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : d_y(f_{mn}(x), f(x)) \geq \delta\} \in \mathcal{I}_2,$$

for each $x \in X$. This gives

$\{(m,n) \in \mathbb{N}\times\mathbb{N} : d_y(f_{mn}(x), f(x)) < \delta\} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : f_{mn}(x) = f(x)\} \in \mathcal{F}(\mathcal{I}_2)$.

Therefore, we have

$$\mathcal{I}_2^* - \lim_{m,n\to\infty} f_{mn}(x) = f(x).$$

□

4. $\mathcal{I}_2$-Cauchy Of Double Sequences Of Functions

Definition 4.1. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2$-Cauchy on $S \subset \mathbb{R}$, if for every $\varepsilon > 0$ there exist $s = s(\varepsilon), t = t(\varepsilon) \in \mathbb{N}$ such that*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f_{st}(x)| \geq \varepsilon\} \in \mathcal{I}_2,$$

for each $x \in S$.

Theorem 4.2. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent on $S \subset \mathbb{R}$ if and only if it is $\mathcal{I}_2$-Cauchy sequences.*

Proof. Suppose that $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent to f on S. Then

$$A\left(\frac{\varepsilon}{2}\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}_2,$$

for $\varepsilon > 0$ and for each $x \in S$. This implies that

$$A^c\left(\frac{\varepsilon}{2}\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f(x)| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}_2),$$

for each $x \in S$ and therefore $A^c(\frac{\varepsilon}{2})$ is non empty. So we can choose positive integers k, l such that $(k,l) \notin A\left(\frac{\varepsilon}{2}\right)$ and $|f_{kl}(x) - f(x)| < \frac{\varepsilon}{2}$. Now, we define the set

$$B(\varepsilon) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f_{kl}(x)| \geq \varepsilon\},$$

for each $x \in S$, such that we show that $B(\varepsilon) \subset A\left(\frac{\varepsilon}{2}\right)$. Let $(m,n) \in B(\varepsilon)$, then we have

$$\begin{aligned} \varepsilon \leq |f_{mn}(x) - f_{kl}(x)| &\leq |f_{mn}(x) - f(x)| + |f_{kl}(x) - f(x)| \\ &< |f_{mn}(x) - f(x)| + \frac{\varepsilon}{2}, \end{aligned}$$

for each $x \in S$. This implies that

$$\frac{\varepsilon}{2} < |f_{mn}(x) - f(x)|$$

and therefore $(m,n) \in A\left(\frac{\varepsilon}{2}\right)$. Hence, we have $B(\varepsilon) \subset A\left(\frac{\varepsilon}{2}\right)$. This shows that $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence.

Conversely, suppose that $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence. We prove that $\{f_{mn}\}$ is $\mathcal{I}_2$-convergent. Let (ε_{pq}) be a strictly decreasing sequence of numbers converging to zero. Since $\{f_{mn}\}$ is $\mathcal{I}_2$-Cauchy sequence, there exist two strictly increasing sequences (k_p) and (l_q) of positive integers such that

$$A(\varepsilon_{pq}) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f_{k_p l_q}(x)| \geq \varepsilon_{pq}\} \in \mathcal{I}_2, \text{ (p, q= 1, 2,...)},$$

for each $x \in S$. This implies that

$$\emptyset \neq \{(m,n) \in \mathbb{N}\times\mathbb{N} : |f_{mn}(x) - f_{k_p l_q}(x)| < \varepsilon_{pq}\} \in \mathcal{F}(\mathcal{I}_2), \text{ (p, q= 1, 2,...)} \quad (4.1)$$

for each $x \in S$. Let p, q, s and t be four positive integers such that $p \neq q$ and $s \neq t$. By (4.1), both the sets

$$C(\varepsilon_{pq}) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f_{k_p l_q}(x)| < \varepsilon_{pq}\}$$

and

$$D(\varepsilon_{st}) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f_{k_s l_t}(x)| < \varepsilon_{st}\}$$

are non empty sets in $\mathcal{F}(\mathcal{I}_2)$, for each $x \in S$. Since $\mathcal{F}(\mathcal{I}_2)$ is a filter on $\mathbb{N} \times \mathbb{N}$, therefore

$$\emptyset \neq C(\varepsilon_{pq}) \cap D(\varepsilon_{st}) \in \mathcal{F}(\mathcal{I}_2).$$

Thus, for each pair (p,q) and (s,t) of positive integers with $p \neq q$ and $s \neq t$, we can select a pair $(m_{(p,q),(s,t)}, n_{(p,q),(s,t)}) \in \mathbb{N} \times \mathbb{N}$ such that

$$|f_{m_{pqst}n_{pqst}}(x) - f_{k_p l_q}(x)| < \varepsilon_{pq} \text{ and } |f_{m_{pqst}n_{pqst}}(x) - f_{k_s l_t}(x)| < \varepsilon_{st},$$

for each $x \in S$. It follows that

$$\begin{aligned} |f_{k_p l_q}(x) - f_{k_s l_t}(x)| &\leq |f_{m_{pqst}n_{pqst}}(x) - f_{k_p l_q}(x)| + |f_{m_{pqst}n_{pqst}}(x) - f_{k_s l_t}(x)| \\ &\leq \varepsilon_{pq} + \varepsilon_{st} \to 0, \end{aligned}$$

as $p, q, s, t \to \infty$. This implies that $\{f_{k_p l_q}\}$ $(p, q = 1, 2, ...)$ is a Cauchy sequence and therefore it satisfies the Cauchy convergence criterion. Thus, the sequence $\{f_{k_p l_q}\}$ converges to a limit f (say). i.e.,

$$\lim_{p,q \to \infty} f_{k_p l_q}(x) = f(x),$$

for each $x \in S$. Also, we have $\varepsilon_{pq} \to 0$ as $p, q \to \infty$, so for each $\varepsilon > 0$ we can choose positive integers p_0, q_0 such that

$$\varepsilon_{p_0 q_0} < \frac{\varepsilon}{2} \text{ and } |f_{k_p l_p}(x) - f(x)| < \frac{\varepsilon}{2}, \text{ (for } p > p_0 \text{ and } q > q_0). \tag{4.2}$$

Now, we define the set

$$A(\varepsilon) = \{(m,n) \in \mathbb{N} \times \mathbb{N} : |f_{mn}(x) - f(x)| \geq \varepsilon\},$$

for each $x \in S$. We prove that $A(\varepsilon) \subset A(\varepsilon_{p_0 q_0})$. Let $(m,n) \in A(\varepsilon)$, then by second half of (4.2) we have

$$\begin{aligned} \varepsilon \leq |f_{mn}(x) - f(x)| &\leq |f_{mn}(x) - f_{k_{p_0} l_{q_0}}(x)| + |f_{k_{p_0} l_{q_0}}(x) - f(x)| \\ &< |f_{mn}(x) - f_{k_{p_0} l_{q_0}}(x)| + \frac{\varepsilon}{2}, \end{aligned}$$

for each $x \in S$. This implies that

$$\frac{\varepsilon}{2} < |f_{mn}(x) - f_{k_{p_0} l_{q_0}}(x)|$$

and therefore by first half of (4.2)

$$\varepsilon_{p_0 q_0} < |f_{mn}(x) - f_{k_{p_0} l_{q_0}}(x)|,$$

for each $x \in S$. Thus, we have $(m,n) \in A(\varepsilon_{p_0 q_0})$ and therefore $A(\varepsilon) \subset A(\varepsilon_{p_0 q_0})$. Since $A(\varepsilon_{p_0 q_0}) \in \mathcal{I}_2$ so $A(\varepsilon) \in \mathcal{I}_2$ by property of ideal. Hence, $\{f_{k_p l_q}\}$ is $\mathcal{I}_2$-convergent. □

Now we introduce the notion of $\mathcal{I}_2^*$-Cauchy sequence.

Definition 4.3. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. A double sequence of functions $\{f_{mn}\}$ is said to be $\mathcal{I}_2^*$-Cauchy sequence on $S \subset \mathbb{R}$, if there exists a set $M \in \mathcal{F}(\mathcal{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2$) and $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that for every $\varepsilon > 0$ and for $(m,n),(s,t) \in M$*

$$|f_{mn}(x) - f_{st}(x)| < \varepsilon.$$

whenever $m,n,s,t > k_0$. In this case we write

$$\lim_{m,n,s,t\to\infty} |f_{mn}(x) - f_{st}(x)| = 0.$$

Theorem 4.4. *Let $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal. If $\{f_{mn}\}$ is $\mathcal{I}_2^*$-Cauchy sequence, then it is $\mathcal{I}_2$-Cauchy sequence on $S \subset \mathbb{R}$.*

Proof. The proof is straightforward and so is omitted. □

References

[1] B. Altay, F. Başar, *Some new spaces of double sequences*, J. Math. Anal. Appl. **309**(1) (2005), 70-90.
[2] V. Baláz, J. Červeňanský, P. Kostyrko, T. Šalát, *I-convergence and I-continuity of real functions*, Acta Mathematica, Faculty of Natural Sciences, Constantine the Philosopher University, Nitra, **5** (2004), 43–50.
[3] M. Balcerzak, K. Dems, A. Komisarski, *Statistical convergence and ideal convergence for sequences of functions*, J. Math. Anal. Appl. **328** (2007), 715-729.
[4] C. Çakan, B. Altay, *Statistically boundedness and statistical core of double sequences*, J. Math. Anal. Appl. **317** (2006), 690-697.
[5] P. Das, P. Kostyrko, W. Wilczyński, P. Malik, *I and I^*-convergence of double sequences*, Math. Slovaca, **58** (2008), No. 5, 605-620.
[6] P. Das, P. Malik, *On extremal I-limit points of double sequences*, Tatra Mt. Math. Publ. **40** (2008), 91-102.
[7] K. Demirci, *I-limit superior and limit inferior*, Math. Commun. **6** (2001), 165-172.
[8] K. Dems, *On I-Cauchy sequence*, Real Anal. Exchange **30** (2004/2005) 123–128.
[9] E. Dündar, B. Altay *Multipliers for bounded $\mathcal{I}_2$-convergent of double sequences*, Math. Comput. Modelling, **55**(3-4) (2012), 1193–1198.
[10] E. Dündar, B. Altay, *On some properties of $\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences* , Gen. Math. Notes, **7**(1) (2011), 1–12.
[11] E. Dündar, B. Altay, *$\mathcal{I}_2$-convergence and $\mathcal{I}_2$-Cauchy of double sequences*, Acta Mathematica Scientia, **34B**(2) (2014), 343–353.
[12] H. Fast, *Sur la convergenc statistique*, Colloq. Math. **2** (1951), 241-244.
[13] J. A. Fridy, C. Orhan, *Statistical limit superior and inferior*, Proc. Amer. Math. Soc. **125** (1997), 3625–3631.
[14] J. A. Fridy, *On statistical convergence*, Analysis, **5** (1985), 301-313.
[15] J. A. Fridy, *Statistical limit points*, Proc. Amer. Math. Soc. **118** (1993), 1187–1192.
[16] F. Gezer, S. Karakuş, *$\mathcal{I}$ and $\mathcal{I}^*$ convergent function sequences*, Math. Commun. **10** (2005), 71-80.
[17] A. Gökhan, M. Güngör and M. Et, *Statistical convergence of double sequences of real-valued functions*, Int. Math. Forum, **2(8)** (2007), 365-374.
[18] M. Gürdal, A. Şahiner, *Extremal $\mathcal{I}_2$-limit points of double sequences*, Applied Mathematics E-Notes, **2** (2008), 131-137.
[19] P. Kostyrko, T. Salat and W. Wilczyński, *I-convergence*, Real Anal. Exchange, **26(2)** (2000), 669-686.
[20] P. Kostyrko, M. Macaj, T. Salat and M. Sleziak, *I-convergence and extremal I-limit points*, Math. Slovaca, **55** (2005), 443-464.
[21] V. Kumar, *On I and I^*-convergence of double sequences*, Math. Commun. **12** (2007), 171-181.
[22] F. Móricz, *Statistical convergence of multiple sequences*, Arch. Mat. **81**(1) (2003), 82–89.
[23] Mursaleen, O. H. H. Edely, *Statistical convergence of double sequences*, J. Math. Anal. Appl. **288** (2003), 223-231.

[24] A. Nabiev, S. Pehlivan and M. Gürdal, *On I-Cauchy sequence*, Taiwanese J. Math. **11(2)** (2007), 569-576.
[25] F. Nuray, W. H. Ruckle, *Generalized statistical convergence and convergence free spaces*, J. Math. Anal. Appl. **245** (2000), 513–527.
[26] A. Pringsheim, *Zur theorie der zweifach unendlichen Zahlenfolgen*, Math. Ann. **53** (1900), 289–321.
[27] D. Rath, B. C. Tripaty, *On statistically convergence and statistically Cauchy sequences*, Indian J. Pure Appl. Math. **25(4)** (1994), 381-386.
[28] T. Šalát, B. C. Tripathy and M. Ziman, *On I-convergence field*, Ital. J. Pure Appl. Math. **17** (2005),45-54.
[29] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30** (1980), 139-150.
[30] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361-375.
[31] B. Tripathy, B. C. Tripathy, *On I-convergent double sequences*, Soochow J. Math. **31** (2005), 549-560.

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