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e-CORE OF DOUBLE SEQUENCES

Y. SEVER[1,*] and Ö. TALO[2]

[1]Department of Mathematics, Faculty of Art and Sciences, Afyon Kocatepe University, 03200 Afyonkarahisar, Turkey
e-mails: yurdalsever@hotmail.com, ysever@aku.edu.tr

[2]Department of Mathematics, Faculty of Art and Sciences, Celal Bayar University, 45040 Manisa, Turkey
e-mails: ozertalo@hotmail.com, ozer.talo@cbu.edu.tr

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Abstract. Boos, Leiger and Zeller [1,2] defined the concept of e-convergence. In this paper we introduce the concepts of e-limit superior and inferior for real double sequences and prove some fundamental properties of e-limit superior and inferior. In addition to these results we define e-core for double sequences. Also, we show that that if A is a nonnegative $\mathcal{C}_{\mathrm{e}}$-regular matrix then the e-core of Ax is contained in e-core of x, provided that Ax exists.

1. Introduction

By Ω, we denote the set of all complex valued double sequences, i.e.,

$$\Omega = \left\{ x = (x_{mn}) : x_{mn} \in \mathbb{C} \text{ for all } m, n \in \mathbb{N} \right\},$$

which is a vector space with co-ordinatewise addition and scalar multiplication of double sequences, where $\mathbb{N}$ and $\mathbb{C}$ denote the set of positive integers and the complex field, respectively. Any vector subspace of Ω is called a double sequence space. The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u = \left\{ x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty \right\}$$

which is a Banach space with the norm $\|\cdot\|_\infty$. Consider the sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

* Corresponding author.

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$|x_{mn} - \ell| < \varepsilon$ for all $m, n > n_0$ then we say that the double sequence x is convergent in the Pringsheim's sense to the limit ℓ and write P-$\lim_{m,n} x_{mn} = \ell$. By $\mathcal{C}_{\rm p}$, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_{\rm p}$ but not in the space $\mathcal{M}_u$. So, we may mention the space $\mathcal{C}_{\rm bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{\rm bp} = \mathcal{C}_{\rm p} \cap \mathcal{M}_u$. Móricz [8] proved that $\mathcal{C}_{\rm bp}$ is a Banach space with the norm $\|\cdot\|_\infty$. By $\mathcal{C}_{\rm bp0}$, we denote the space of double sequences which are both convergent to zero in the Pringsheim's sense and bounded.

Boos, Legier and Zeller [1,2] introduced and investigated the notion of e-convergence of double sequences, which is essentially weaker than the Pringsheim convergence. Zeltser [16] characterized SM-methods (see [12,14]) mapping bounded or convergent sequences into e-, be- or c-convergent double sequences, as well as 4-dimensional matrices being conservative with respect to the one of these notions of convergence. A double sequence $x = (x_{kl}) \in \Omega$ is said to be e-convergent to a number a if

$$\forall \varepsilon > 0\ \exists l_0 \in \mathbb{N},\ \forall l \geqq l_0,\ \exists k_l \in \mathbb{N}\ \ni\ \forall k \geqq k_l \Rightarrow |x_{kl} - a| < \varepsilon.$$

The space of all double sequences converging in this way is denoted by $\mathcal{C}_{\rm e}$. More precisely,

$$\mathcal{C}_{\rm e} := \big\{ x = (x_{kl}) \in \Omega \mid \exists a \in \mathbb{C},\ \forall \varepsilon > 0\ \exists l_0 \in \mathbb{N},\ \forall l \geqq l_0,$$

$$\exists k_l \in \mathbb{N} \ni \forall k \geqq k_l \Rightarrow |x_{kl} - a| < \varepsilon \big\}$$

$$= \Big\{ x = (x_{kl}) \in \Omega \mid \exists a \in \mathbb{C} :\ \lim_l \overline{\lim_k} |x_{kl} - a| = 0 \Big\}.$$

The subspace

$$\mathcal{C}_{\rm be} = \big\{ x \in \mathcal{C}_{\rm e} \mid \forall l \in \mathbb{N} :\ (x_{kl})_k \in l_\infty \big\}$$

of $\mathcal{C}_{\rm e}$, where l_∞ is the space of all bounded sequences.

Definition 1.1 [16]. A real double sequence $x = (x_{kl})$ is said to be e-bounded if $\overline{\lim}_l \overline{\lim}_k |x_{kl}| < \infty$. That is, a real double sequence $x = (x_{kl})$ is said to be e-bounded if there exists $M > 0$ such that

$$\exists l_0 \in \mathbb{N},\ \forall l \geqq l_0,\ \exists k_l \in \mathbb{N} \ni \forall k \geqq k_l \Rightarrow |x_{kl}| < M.$$

Patterson [11] gave the definition of subsequence, the Pringsheim limit inferior and limit superior of double sequences.

Definition 1.2 [11]. Let $x = (x_{kl})$ be a double sequence of real numbers and for each n, let $\alpha_n = \sup_n \{x_{kl} : k, l \geqq n\}$. The Pringsheim limit superior of x is defined as follows:

(i) if $\alpha_n = +\infty$ for each n, then $\text{P-}\limsup x := +\infty$;
(ii) if $\alpha_n < +\infty$ for some n, then $\text{P-}\limsup x := \inf_n\{\alpha_n\}$.

Similarly, let $\beta_n = \inf_n\{x_{kl} : k, l \geqq n\}$. Then the Pringsheim limit inferior of $x = (x_{kl})$ is defined as follows:

(i) if $\beta_n = -\infty$ for each n, then $\text{P-}\liminf x := -\infty$;
(ii) if $\beta_n > -\infty$ for some n, then $\text{P-}\liminf x := \sup_n\{\beta_n\}$.

Definition 1.3. A number α is called an e-limit point of the double sequence $x = (x_{kl})$ provided that there exists a subsequence $y = (y_{kl})$ of $x = (x_{kl})$ that has e-limit α: $\text{e-}\lim_{kl} y_{kl} = \alpha$.

Example 1.4. The following is an example of $x = (x_{kl})$ which is e-convergent; however, x is not P-convergent. Define

$$x_{kl} := \begin{cases} k, & k = l, \\ 1, & k < l, \\ 0, & k > l. \end{cases}$$

Then, it is easy to see that $\text{e-}\lim_{kl} x_{kl} = 0$, whereas $\text{P-}\lim_{kl} x_{kl}$ does not exist.

Let λ be the space of double sequences, converging with respect to some linear convergence rule $\upsilon\text{-}\lim : \lambda \to \mathbb{C}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $\upsilon\text{-}\sum_{ij} x_{ij} = \upsilon\text{-}\lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$. Let λ, μ be two spaces of double sequences, converging with respect to the linear convergence rules $\upsilon_1\text{-}\lim$ and $\upsilon_2\text{-}\lim$, respectively, and let $A = (a_{mnkl})$ also be a four dimensional matrix of complex numbers. Define the set

(1.1)
$$\lambda_A^{(\upsilon_2)} = \left\{ (x_{kl}) \in \Omega : \; Ax = \left(\upsilon_2 - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and } Ax \in \lambda \right\}.$$

Then, we say, with the notation of (1.1), that A maps the space λ into the space μ if $\mu \subset \lambda_A^{(\upsilon_2)}$ and denote the set of all four dimensional matrices, mapping the space λ into the space μ, by $(\lambda : \mu)$. It is trivial that for any matrix $A \in (\lambda : \mu)$, $(a_{mnkl})_{k,l \in \mathbb{N}}$ is in the $\beta(\upsilon_2)$-dual $\lambda^{\beta(\upsilon_2)}$ of the space λ for all $m, n \in \mathbb{N}$. An infinite matrix A is said to be $\mathcal{C}_\upsilon$-conservative if $\mathcal{C}_\upsilon \subset (\mathcal{C}_\upsilon)_A$. For more details on double sequences, 3-dimensional and 4-dimensional matrices, we refer to [6,13,15–18].

We refer the reader to [16] for the basic terminology. Denote by w the vector space of all number sequences

$$\varphi := \{x \in w : \exists\, k_0 \in \mathbb{N} \;\forall\, k > k_0 : x_k = 0\}.$$

We write e^{kl} $(k,l \in \mathbb{N})$ for the double sequence with

$$e^{kl}_{ij} := \begin{cases} 1, & \text{if } (k,l)=(i,j), \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$e = \sum_{k,l} e^{kl}, \quad e^l = \sum_k e^{kl} \ (l \in \mathbb{N}) \quad \text{and} \quad e_k = \sum_l e^{kl} \ (k \in \mathbb{N})$$

and $\Phi = \operatorname{span}\{e^{kl} : k,l \in \mathbb{N}\}$, that is, $\Phi := \{x \in \Omega : \exists\, k_0 \in \mathbb{N} : k \geqq k_0$ or $l \geqq k_0 \Rightarrow x_{kl} = 0\}$.

Theorem 1.5 [16, p. 106]. *A 3-dimensional matrix $B = (b_{mnk})$ maps w into C_{e} if and only if the following conditions hold*:

(i) $b^{(m,n)} := (b_{mnk})_k \in \varphi$ *for every* $m; n \in \mathbb{N}$,

(ii) *for every $k \in \mathbb{N}$, the limit $b_k := \text{e-}\lim_{m,n} b_{mnk}$ exist,*

(iii) *there exists $N \in \mathbb{N}$ such that*

$$\forall\, n \geqq N\ \exists\, K(n) \in \mathbb{N} :\ k > K(n) \Rightarrow b_{mnk} = 0 \quad (m \in \mathbb{N}),$$

(iv) *there exist $N, K \in \mathbb{N}$ such that* $\lim_m b_{mnk} = 0$ $k \geqq K$, $n \geqq N$.

Under these circumstances, $b := (b_k) \in \varphi$ *and* $\text{e-}\lim_{m,n} [Bz]_{mn} = \Sigma_k b_k z_k$ $(z \in w)$.

Theorem 1.6 [16, p. 110]. (a) *A 4-dimensional matrix $A = (a_{mnkl})$ is $\mathcal{C}_{\mathrm{e}}$-conservative if and only if the following conditions hold*:

(i) $a^{(m,n)} = (a_{mnkl})_{kl} \in \Phi$ *for every* $m, n \in \mathbb{N}$,

(ii) *for every $l_0 \in \mathbb{N}$, the matrix $(a_{mnkl})_{m,n,k}$ maps w into $\mathcal{C}_{\mathrm{e}}$,*

(iii) *the limit* $v := \text{e-}\lim_{m,n} \sum_{kl} a_{mnkl}$ *exists,*

(iv) *there exists $n_0 \in \mathbb{N}$ such that*

$$\forall\, n \geqq n_0\ \exists\, L(n) \in \mathbb{N} :\ a^{(m,n)} = \sum_{l=1}^{L(n)} a^{(m,n)} e^l \quad (m \in \mathbb{N}),$$

(v) *there exists $L, N \in \mathbb{N}$ such that* $l \geqq L$, $n \geqq N \Rightarrow \lim_m a_{mnkl} = 0$ $(k \in \mathbb{N})$,

(vi) *there exists $N' \in \mathbb{N}$ and m_n such that*

$$\sup_{\substack{n \geqq N' \\ m \geqq m_n}} \sum_{kl} |a_{mnkl}| < \infty.$$

Under these circumstances, $a = (a_{kl}) = (\text{e-}\lim_{m,n} a_{mnkl}) \in \Phi$, *and*

$$\text{e-}\lim_{m,n} [Ax]_{mn} = \sum_{kl} a_{kl}x_{kl} + \Big(v - \sum_{kl} a_{kl}\Big)\text{e-}\lim_{m,n} x_{mn} \quad (x \in \mathcal{C}_{\mathrm{e}}).$$

(b) $A = (a_{mnkl})$ *is* $\mathcal{C}_{\mathrm{e}}$*-regular if and only if the conditions*; (i)–(vi) *hold with* $a_{kl} = 0$ $(k, l \in \mathbb{N})$ *and* $v = 1$.

By using the definitions of Pringsheim limit inferior, limit superior and the Pringsheim core of a double sequence with the notion of the regularity of four dimensional matrices, Patterson [11] gave some results on core of double sequences. Mursaleen [9], Mursaleen and Edely [10] defined the almost strong regularity of matrices for double sequences, applied these matrices to establish a core theorem, introduced the M-core for double sequences, and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{kl})$ into one whose core is a subset of the M-core of x. Recently, Çakan and Altay [4] investigated statistical core for double sequences and studied an inequality related to the statistical and P-cores of bounded double sequences. Gökhan, Çolak and Mursaleen [5] generalized the Pringsheim core for bounded double sequences and gave some core theorems via matrix classes. Çakan, Altay and Mursaleen [3] introduced σ-convergence of a double sequence and defined the σ-core for double sequences and determined a class of four-dimensional matrices such that P-core$(Ax) \subset \sigma$-core(x) for all $x \in \mathcal{M}_u$. Kumar [7] defined $\mathcal{I}$-limit inferior, $\mathcal{I}$-limit superior and $\mathcal{I}$-core for real double sequences.

In this paper we introduce the concepts of e-limit superior and inferior for real double sequences and prove some fundamental properties of e-limit superior and inferior. In addition to these results we define e-core for double sequences. Also, we show that if A is a nonnegative $\mathcal{C}_{\mathrm{e}}$-regular matrix then the e-core of Ax is contained in the e-core of x, provided that Ax exists.

2. Main result

Definition 2.1. Let $x = (x_{kl})$ be a double sequence of real numbers. e-limit superior of $x = (x_{kl})$ is defined by

$$\text{e-}\limsup x := \begin{cases} \inf B_x, & B_x \neq \emptyset, \\ \infty, & \text{otherwise.} \end{cases}$$

and e-limit inferior of $x = (x_{kl})$ is defined by

$$\text{e-}\liminf x := \begin{cases} \sup A_x, & A_x \neq \emptyset, \\ -\infty, & \text{otherwise,} \end{cases}$$

where

$$A_x := \left\{ a \in \mathbb{R} : \exists\, l_0 \in \mathbb{N},\ \forall\, l \geqq l_0,\ \exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l : x_{kl} > a \right\}$$

and

$$B_x := \left\{ b \in \mathbb{R} : \exists\, l_0 \in \mathbb{N},\ \forall\, l \geqq l_0,\ \exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l : x_{kl} < b \right\}.$$

Clearly, if a real double sequence $x = (x_{kl})$ is e-bounded, then $A_x \neq \emptyset$ and $B_x \neq \emptyset$. Therefore e- $\liminf x$ and e- $\limsup x$ are both finite numbers.

THEOREM 2.2. *Let $x = (x_{kl})$ be a double sequence of real numbers. If $u =$ e-$\limsup x$ is finite, then for every $\varepsilon > 0$ $\exists\, l_0 \in \mathbb{N}$, $\forall\, l \geqq l_0$, $\exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \Rightarrow x_{kl} < u + \varepsilon$.*

PROOF. Let e-$\limsup x = u$. Then $u = \inf B_x$. By the definition of infimum, given $\varepsilon > 0$, there exists $u_\varepsilon \in B_x$ such that $u_\varepsilon \leqq u + \varepsilon$. Since $u_\varepsilon \in B_x$ and taking into consideration the definition of the set B_x, $\exists\, l_0 \in \mathbb{N}$, $\forall\, l \geqq l_0$, $\exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l$ we get $x_{kl} < u_\varepsilon$. Therefore, for every $\varepsilon > 0$ $\exists\, l_0 \in \mathbb{N}$, $\forall\, l \geqq l_0$, $\exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l$ we obtain that $x_{kl} < u + \varepsilon$. $\square$

The proof of the following theorem is the same as above and so we omit it.

THEOREM 2.3. *Let $x = (x_{kl})$ be a double sequence of real numbers. If e-$\liminf x = v$ is finite, then given $\varepsilon > 0$, $\exists\, l_0 \in \mathbb{N}$, $\forall\, l \geqq l_0$, $\exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \Rightarrow x_{kl} > v - \varepsilon$.*

The proof of the following lemma is the same as the proof for convergence in Pringsheim sense and so we omit it.

LEMMA 2.4. *For any real-valued double sequence x,* e-$\limsup(-x) = -($e-$\liminf x)$

THEOREM 2.5. *For any real-valued double sequence x,* e-$\liminf x \leqq$ e-$\limsup x$.

PROOF. If e-$\limsup x = -\infty$, then we have $B_x = \mathbb{R}$ and $A_x = \emptyset$. This implies that e-$\liminf x = -\infty$. If e-$\limsup x = \infty$, then we have nothing to prove. Assume that e-$\limsup x$ is finite. Let $a \in A_x$ and $b \in B_x$. Thus, we can find x_{kl} such that $a < x_{kl} < b$. That is, any member of B_x is greater than all members of A_x. This completes the proof. $\square$

THEOREM 2.6. *For any real-valued double sequence x,*

$$\text{e-}\limsup x = \text{e-}\liminf x = \ell \quad \textit{if and only if} \quad \text{e-}\lim x = \ell.$$

Proof. Let $\text{e-}\lim x = \ell$. Then for any $\varepsilon > 0$, $\exists\, l_0 \in \mathbb{N}$, $\forall\, l \geqq l_0$, $\exists\, k_l \in \mathbb{N}$ $\ni \forall\, k \geqq k_l$

$$\ell - \varepsilon < x_{kl} < \ell + \varepsilon,$$

which implies that $\ell + \varepsilon \in B_x$ and $\ell - \varepsilon \in A_x$. Thus we obtain

$$\ell - \varepsilon \leqq \text{e-}\liminf x = \sup A_x \leqq \text{e-}\limsup x = \inf B_x \leqq \ell + \varepsilon. \tag{2.1}$$

Since ε is arbitrary, $\text{e-}\limsup x = \text{e-}\liminf x = \ell$ holds.

On the other hand, let $\text{e-}\limsup x = \text{e-}\liminf x = \ell$. So, for any $\varepsilon > 0$ $\exists\, l_1 \in \mathbb{N}$, $\forall\, l \geqq l_1$, $\exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \Rightarrow x_{kl} < \ell + \varepsilon$ and $\exists\, l_2 \in \mathbb{N}$, $\forall\, l \geqq l_2$, $\exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \Rightarrow x_{kl} > \ell - \varepsilon$. Let $\ell_0 = \max\{\ell_1, \ell_2\}$. Then $\forall\, l \geqq l_0$, $\exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l$ we get $\ell - \varepsilon < x_{kl} < \ell + \varepsilon$, that is, $|x_{kl} - \ell| < \varepsilon$. This means that $\text{e-}\lim x = \ell$. □

Theorem 2.7. *If $x = (x_{kl})$ and $y = (y_{kl})$ are two* e*-bounded real double sequences, then we have*:

(i) $\text{e-}\limsup(x + y) \leqq \text{e-}\limsup x + \text{e-}\limsup y$,

(ii) $\text{e-}\liminf(x + y) \geqq \text{e-}\liminf x + \text{e-}\liminf y$.

Proof. (i) Since $x = (x_{kl})$ and $y = (y_{kl})$ are e-bounded real double sequences, $\text{e-}\limsup x$ and $\text{e-}\limsup y$ are both finite. Suppose that $\text{e-}\limsup x = \alpha$, $\text{e-}\limsup y = \beta$ and

$$B_{(x+y)} := \big\{ b \in \mathbb{R} : \exists\, l_0 \in \mathbb{N},\ \forall\, l \geqq l_0,\ \exists\, k_l \in \mathbb{N}\ \ni \forall\, k \geqq k_l \Rightarrow x_{kl} + y_{kl} < b \big\}.$$

For given $\varepsilon > 0$,

$$\exists\, l_1 \in \mathbb{N},\ \forall\, l \geqq l_1,\ \exists\, k_l \in \mathbb{N}\ \ni \forall\, k \geqq k_l \Rightarrow x_{kl} < \alpha + \varepsilon/2$$

and

$$\exists\, l_2 \in \mathbb{N},\ \forall\, l \geqq l_2,\ \exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \Rightarrow y_{kl} < \beta + \varepsilon/2.$$

Let $l_0 = \max\{l_1, l_2\}$. Then

$$\forall\, l \geqq l_0,\ \exists\, k_l \in \mathbb{N}\ \ni \forall\, k \geqq k_l \Rightarrow x_{kl} + y_{kl} < \alpha + \beta + \varepsilon.$$

Therefore we get $\alpha + \beta + \varepsilon \in B_{(x+y)}$. So,

$$\text{e-}\limsup(x + y) = \inf B_{(x+y)} \leqq \alpha + \beta + \varepsilon.$$

Since ε is arbitrary, we obtain

$$\text{e-}\limsup(x + y) \leqq \text{e-}\limsup x + \text{e-}\limsup y.$$

(ii) It can be proved by the same way as above. □

THEOREM 2.8. $\text{P-}\liminf x \leqq \text{e-}\liminf x \leqq \text{e-}\limsup x \leqq \text{P-}\limsup x$.

PROOF. Let $\text{P-}\limsup x = \alpha$. Since $\alpha = \inf_n \sup_{k,l \geqq n} x_{kl}$, given $\varepsilon > 0$ there exists n_ε such that

$$\sup_{k,l \geqq n_\varepsilon} x_{kl} < \alpha + \varepsilon.$$

Hence for all $k, l \geqq n_\varepsilon$ we get $x_{kl} < \alpha + \varepsilon$. Therefore $l \geqq l_0 = n_\varepsilon$, $\exists\, k_l = n_\varepsilon \in \mathbb{N} \ni \forall\, k \geqq k_l \Rightarrow x_{kl} < \alpha + \varepsilon$. This means that $\alpha + \varepsilon \in B_x$. So

$$\text{e-}\limsup x = \inf B_x \leqq \alpha + \varepsilon.$$

Hence ε is arbitrary and we obtain $\text{e-}\limsup x \leqq \alpha$. Similarly, it can be shown that $\text{P-}\liminf x \leqq \text{e-}\liminf x$. □

EXAMPLE 2.9. The following is an example of a sequence $x = (x_{kl})$ which has finite e-lim sup and e-lim inf; however, P-lim sup and P-lim inf are not finite. Define

$$x_{kl} := \begin{cases} k, & k = l, \\ -k, & k = l + 1, \\ 1, & k < l + 1 \text{ and } k + l \text{ is even}, \\ -1, & k < l + 1 \text{ and } k + l \text{ is odd}, \\ 0, & k > l. \end{cases}$$

Then, it is easy to see that $A_x = (-\infty, -1)$ and $B_x = (1, +\infty)$. So, $\text{e-}\limsup_{kl} x_{kl} = 1$ and $\text{e-}\liminf_{kl} x_{kl} = -1$ but $\text{P-}\limsup_{kl} x_{kl} = +\infty$ and $\text{P-}\liminf_{kl} x_{kl} = -\infty$.

In analogy to the P-core [11], statistical core [4] and $\mathcal{I}$-core [7] we define the e-core of double sequences as follows.

DEFINITION 2.10. For any e-bounded real double sequence $x = (x_{kl})$, the e-core of x is defined as the closed interval $[\text{e-}\liminf x, \text{e-}\limsup x]$. In case x is not e-bounded, e-core of the sequence x is defined by either $(-\infty, \text{e-}\limsup x]$, $[\text{e-}\liminf x, \infty)$ or $(-\infty, \infty)$. e-core(x) will denote e-core of the sequence $x = (x_{kl})$.

From Theorem 2.8, it is clear that e-core$(x) \subset P$-core(x), for any real double sequence x.

THEOREM 2.11. *Let* $x = (x_{kl}), y = (y_{kl})$ *be* e-*bounded double sequences. If* $\text{e-}\lim_{kl} |x_{kl} - y_{kl}| = 0$, *then* e-core$(x) =$ e-core(y).

Proof. Suppose that e-$\limsup x = \alpha$, e-$\limsup y = \beta$ and e-$\lim_{kl} |x_{kl} - y_{kl}| = 0$. Then, for each $\varepsilon > 0$

$$\exists\, l_1 \in \mathbb{N},\ \forall\, l \geqq l_1,\ \exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \ \Rightarrow\ y_{kl} - \varepsilon/2 < x_{kl} < y_{kl} + \varepsilon/2,$$

$$\exists\, l_2 \in \mathbb{N},\ \forall\, l \geqq l_2,\ \exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \ \Rightarrow\ y_{kl} < \beta + \varepsilon/2$$

and

$$\exists\, l_3 \in \mathbb{N},\ \forall\, l \geqq l_3,\ \exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \ \Rightarrow\ x_{kl} < \alpha + \varepsilon/2.$$

Let $l_0 = \max\{l_1, l_2, l_3\}$. Then

$$\forall\, l \geqq l_0,\ \exists\, k_l \in \mathbb{N}\ \ni \forall\, k \geqq k_l \ \Rightarrow\ x_{kl} < \beta + \varepsilon$$

and

$$\forall\, l \geqq l_0,\ \exists\, k_l \in \mathbb{N}\ \ni \forall\, k \geqq k_l \ \Rightarrow\ y_{kl} < \alpha + \varepsilon.$$

Therefore we get $\beta + \varepsilon \in B_x$ and $\alpha + \varepsilon \in B_y$. So, $\alpha = \inf B_x \leqq \beta + \varepsilon$ and $\beta = \inf B_y \leqq \alpha + \varepsilon$. Since ε is arbitrary, we obtain $\alpha \leqq \beta$ and $\beta \leqq \alpha$. This means that $\alpha = \beta$. Similarly, it can be shown that e-$\liminf x =$ e-$\liminf y$. Therefore, e-core$(x) =$ e-core(y). $\square$

Theorem 2.12. *Let A be a 4-dimensional $\mathcal{C}_{\mathrm{e}}$-regular matrix with positive real entries. Then,*

$$\text{e-}\limsup Ax \leqq \text{e-}\limsup x \tag{2.2}$$

for all real-valued bounded double sequences $x = (x_{kl})$.

Proof. Let $x = (x_{kl})$ be a double bounded sequence and let A be a $\mathcal{C}_{\mathrm{e}}$-regular summability matrix. We need to show that e-$\limsup(Ax) \leqq$ e-$\limsup x$. Suppose that e-$\limsup x = \ell$. So, for any $\varepsilon > 0$ $\exists\, P_1 \in \mathbb{N}$, $\forall\, l \geqq P_1$, $\exists\, k_l \in \mathbb{N} \ni \forall\, k \geqq k_l \ \Rightarrow\ x_{kl} < \ell + \varepsilon$.

$$\sum_{k,l=1,1}^{\infty,\infty} a_{mnkl} x_{kl} = \sum_{l=P_1}^{\infty} \sum_{k=k_l}^{\infty} a_{mnkl} x_{kl}$$

$$+ \sum_{l=P_1}^{\infty} \sum_{k=1}^{k=k_l-1} a_{mnkl} x_{kl} + \sum_{l=1}^{l=P_1-1} \sum_{k=1}^{\infty} a_{mnkl} x_{kl}$$

$$\leqq (\ell + \varepsilon) \sum_{l=P_1}^{\infty} \sum_{k=k_l}^{\infty} a_{mnkl} + \|x\| \sum_{l=P_1}^{\infty} \sum_{k=1}^{k=k_l-1} a_{mnkl} + \|x\| \sum_{l=1}^{l=P_1-1} \sum_{k=1}^{\infty} a_{mnkl}$$

Taking into account the condition of e-regularity and taking $\text{e-}\limsup$ of both side, we get

$$\text{e-}\limsup Ax \leqq l + \varepsilon.$$

Since ε is arbitrary, we have (2.2). □

From Theorem 2.12 and Lemma 2.4 we obtain the following result.

COROLLARY 2.13. *Let A be a 4-dimensional $\mathcal{C}_{\mathrm{e}}$-regular matrix with positive real entries. Then,*

$$\text{e-core}(Ax) \subset \text{e-core}(x)$$

for all real-valued bounded double sequences $x = (x_{kl})$.

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