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Feyzi Basar and Yurdal Sever

Abstract

The spaces BS, BS(t), CS_p , CS_{bp} , CS_r and BV of double sequences have recently been studied by Altay and Başar [J. Math. Anal. Appl. 309(1)(2005), 70–90]. In this work, following Altay and Başar [1], we introduce the Banach space L_q of double sequences corresponding to the well-known space ℓ_q of single sequences and examine some properties of the space L_q . Furthermore, we determine the $\beta(v)$ -dual of the space and establish that the α - and γ -duals of the space L_q coincide with the $\beta(v)$ -dual; where $1 \leq q < \infty$ and $v \in \{p, bp, r\}$.

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ABSTRACT. The spaces \mathcal{BS} , $\mathcal{BS}(t)$, \mathcal{CS}_p , \mathcal{CS}_{bp} , \mathcal{CS}_r and \mathcal{BV} of double sequences have recently been studied by Altay and Başar [J. Math. Anal. Appl. **309**(1)(2005), 70–90]. In this work, following Altay and Başar [1], we introduce the Banach space \mathcal{L}_q of double sequences corresponding to the well-known space ℓ_q of single sequences and examine some properties of the space \mathcal{L}_q . Furthermore, we determine the $\beta(v)$ -dual of the space and establish that the α - and γ -duals of the space \mathcal{L}_q coincide with the $\beta(v)$ -dual; where $1 \leq q < \infty$ and $v \in \{p, bp, r\}$.

1. INTRODUCTION

By w and Ω , we denote the set of all real valued single and double sequences which are the vector spaces with coordinatewise addition and scalar multiplication. Any vector subspaces of w and Ω are called as the *single* and *double sequence spaces*, respectively. The space \mathcal{M}_u of all bounded double sequences is defined by

$$\mathcal{M}_u := \left\{ x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty \right\},$$

which is a Banach space with the norm $\|\cdot\|_\infty$; where \mathbb{N} denotes the set of all positive integers. Consider a sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $l \in \mathbb{R}$ such that

$$|x_{mn} - l| < \varepsilon$$

for all $m, n > n_0$ then we call that the double sequence x is *convergent* in the *Pringsheim's sense* to the limit l and write $p\text{-}\lim x_{mn} = l$; where \mathbb{R} denotes the real field. By \mathcal{C}_p we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space \mathcal{C}_p but not in the space \mathcal{M}_u . So, we can consider the space \mathcal{C}_{bp} of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A sequence in the space \mathcal{C}_p is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and the set of all such sequences denoted by \mathcal{C}_r . Also by \mathcal{C}_{bp0} and \mathcal{C}_{r0} , we denote the spaces of all double sequences converging to 0 contained

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in the sequence spaces \mathcal{C}_{bp} and \mathcal{C}_r , respectively. Móricz [7] proved that \mathcal{C}_{bp} , \mathcal{C}_{bp0} , \mathcal{C}_r and \mathcal{C}_{r0} are Banach spaces with the norm $\|\cdot\|_\infty$.

Let us consider the isomorphism T which plays an essential role for the present study, defined by Zeltser [11, p. 36] as

$$(1.1) \quad \begin{aligned} T : \Omega &\longrightarrow w \\ x &\longmapsto z = (z_i) := (x_{\psi^{-1}(i)}), \end{aligned}$$

where $\psi : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is a bijection defined by

$$\begin{aligned} \psi[(1, 1)] &= 1, \\ \psi[(1, 2)] &= 2, \quad \psi[(2, 2)] = 3, \quad \psi[(2, 1)] = 4, \\ &\vdots \\ \psi[(1, n)] &= (n-1)^2 + 1, \quad \psi[(2, n)] = (n-1)^2 + 2, \quad \dots, \\ \psi[(n, n)] &= (n-1)^2 + n, \quad \psi[(n, n-1)] = n^2 - n + 2, \quad \dots, \quad \psi[(n, 1)] = n^2, \\ &\vdots \end{aligned}$$

Let us consider a double sequence $x = (x_{mn})$ and define the sequence $s = (s_{mn})$ via x by

$$(1.2) \quad s_{mn} := \sum_{i,j=1}^{m,n} x_{ij} ; (m, n \in \mathbb{N}),$$

which will be used throughout. For the sake of brevity, here and in what follows, we abbreviate the summations $\sum_{i=1}^\infty \sum_{j=1}^\infty$, $\sum_{i=1}^m \sum_{j=1}^n$ and $\sum_{i=1}^n \sum_{j=1}^m$ by $\sum_{i,j}$, $\sum_{i,j=1}^{m,n}$ and $\sum_{i,j=1}^n$, respectively. Then the pair (x, s) and the sequence $s = (s_{mn})$ are called as a *double series* and the *sequence of partial sums* of the double series, respectively. Let λ be the space of double sequences, converging with respect to some linear convergence rule $v - \lim : \lambda \rightarrow \mathbb{R}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $v - \sum_{i,j} x_{ij} := v - \lim s_{mn}$.

Let us define the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in \Omega : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in \Omega : p - \lim_{m,n \rightarrow \infty} |x_{mn} - l|^{t_{mn}} = 0 \text{ for some } l \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in \Omega : p - \lim_{m,n \rightarrow \infty} |x_{mn}|^{t_{mn}} = 0 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in \Omega : \sum_{m,n} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \quad \text{and} \quad \mathcal{C}_{0bp}(t) := \mathcal{C}_{0p}(t) \cap \mathcal{M}_u(t);$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$. In the case $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t)$, $\mathcal{C}_p(t)$, $\mathcal{C}_{0p}(t)$, $\mathcal{L}_u(t)$, $\mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets \mathcal{M}_u , \mathcal{C}_p , \mathcal{C}_{0p} , \mathcal{L}_u , \mathcal{C}_{bp} and \mathcal{C}_{0bp} , respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Çolak [4, 5] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t)$, $\mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the α -, β -, γ -duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [11] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [8] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [9] and Mursaleen and Edely [10] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M -core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the M -core of x . More recently, Altay and Başar [1] have defined the spaces \mathcal{BS} , $\mathcal{BS}(t)$, \mathcal{CS}_p , \mathcal{CS}_{bp} , \mathcal{CS}_r and \mathcal{BV} of double sequences consisting of all double series whose sequence of partial sums are in the spaces \mathcal{M}_u , $\mathcal{M}_u(t)$, \mathcal{C}_p , \mathcal{C}_{bp} , \mathcal{C}_r and \mathcal{L}_u , respectively, and also examined some properties of those sequence spaces and determined the α -duals of the spaces \mathcal{BS} , \mathcal{BV} , \mathcal{CS}_{bp} and the $\beta(\vartheta)$ -duals of the spaces \mathcal{CS}_{bp} and \mathcal{CS}_r of double series.

In the present paper, we introduce the space \mathcal{L}_q

$$\mathcal{L}_q := \left\{ (x_{ij}) \in \Omega : \sum_{i,j} |x_{ij}|^q < \infty \right\}, \quad (1 \leq q < \infty)$$

of double sequences corresponding to the space ℓ_q of single sequences and examine some properties of the space.

2. THE DOUBLE SEQUENCE SPACE \mathcal{L}_q

In this section, we give the theorem which states that \mathcal{L}_q is a sequence space and is a Banach space with the norm $\|\cdot\|_q$, firstly. Subsequent to giving some inclusion relations concerning the space \mathcal{L}_q , we establish that the α - and γ -duals of a space of double sequences are identical whenever it is solid, and \mathcal{L}_q is solid if $q > 1$ and determine the $\beta(v)$ -dual of the space \mathcal{L}_q for $v \in \{p, bp, r\}$ which coincides with the α - and γ -duals of the space \mathcal{L}_q .

Theorem 2.1. *The set \mathcal{L}_q becomes a linear space with the coordinatewise addition and scalar multiplication and \mathcal{L}_q is a Banach space with the norm*

$$(2.1) \quad \|x\|_q = \left(\sum_{i,j} |x_{ij}|^q \right)^{1/q},$$

where $1 \leq q < \infty$.

Proof. The proof of the first part of the theorem is a routine verification and so we omit the detail.

Furthermore, the statement "a sequence space ν is a Banach space with the norm $\|\cdot\|_\nu$ if and only if the sequence space $T^{-1}(\nu) = \lambda$ is a Banach space with the norm $\|\cdot\|_\lambda$ " holds by Boos [3, Corollary 6.3.41]. Therefore, the restriction of the transformation defined by (1.1) to the space \mathcal{L}_q which is norm preserving isomorphism yields the fact that $\mathcal{L}_q = T^{-1}(\ell_q)$ is also a Banach space with the norm $\|\cdot\|_q$ defined by (2.1) because of ℓ_q is a Banach space.

This step concludes the proof. □

Theorem 2.2. *Let $1 \leq q < s < \infty$. Then, the inclusions $\mathcal{L}_q \subset \mathcal{L}_s \subset \mathcal{C}_{r0} \subset \mathcal{M}_u$ hold.*

Proof. Let us take any $x = (x_{ij}) \in \mathcal{L}_q$. Then, $\sum_{\max\{i,j\} > n_0} |x_{ij}|^q < \varepsilon < 1$ for sufficiently large $n_0 \in \mathbb{N}$. Since $q < s$, it is obvious that $|x_{ij}|^q \geq |x_{ij}|^s$ for all $i, j \in \mathbb{N}$ such that $\max\{i, j\} > n_0$. Thus,

$$\begin{aligned} \sum_{i,j} |x_{ij}|^s &= \sum_{i,j=1}^{n_0} |x_{ij}|^s + \sum_{\max\{i,j\} > n_0} |x_{ij}|^s \\ &\leq A + \sum_{\max\{i,j\} > n_0} |x_{ij}|^q \\ &\leq A + \varepsilon, \end{aligned}$$

which leads us to the fact that $x \in \mathcal{L}_s$, where $A = \sum_{i,j=1}^{n_0} |x_{ij}|^s$. Hence, $\mathcal{L}_q \subset \mathcal{L}_s$.

Besides one can easily deduce, by means of the suitable restrictions of the isomorphism T defined by (1.1) and taking into account the fact that the space \mathcal{C}_{r0} consists of all sequences $x = (x_{mn})$ such that $\lim_{\max\{m,n\} \rightarrow \infty} x_{mn} = 0$, from the known inclusions $\ell_s \subset c_0 \subset \ell_\infty$ for $1 \leq s < \infty$ that

$$T^{-1}(\ell_s) = \mathcal{L}_s \subset \mathcal{C}_{r0} = T^{-1}(c_0) \subset T^{-1}(\ell_\infty) = \mathcal{M}_u.$$

This step completes the proof. □

The α -dual λ^α , $\beta(v)$ -dual $\lambda^{\beta(v)}$ with respect to the v -convergence for $v \in \{p, bp, r\}$ and the γ -dual λ^γ of a double sequence space λ are respectively defined by

$$\lambda^\alpha := \left\{ (a_{ij}) \in \Omega : \sum_{i,j} |a_{ij}x_{ij}| < \infty \text{ for all } (x_{ij}) \in \lambda \right\},$$

$$\lambda^{\beta(v)} := \left\{ (a_{ij}) \in \Omega : v - \sum_{i,j} a_{ij}x_{ij} \text{ exists for all } (x_{ij}) \in \lambda \right\}$$

and

$$\lambda^\gamma := \left\{ (a_{ij}) \in \Omega : \sup_{k,l \in \mathbb{N}} \left| \sum_{i,j=1}^{k,l} a_{ij}x_{ij} \right| < \infty \text{ for all } (x_{ij}) \in \lambda \right\}.$$

It is easy to see for any two spaces λ, μ of double sequences that $\mu^\alpha \subset \lambda^\alpha$ whenever $\lambda \subset \mu$ and $\lambda^\alpha \subset \lambda^\gamma$. Additionally, it is known that the inclusion $\lambda^\alpha \subset \lambda^{\beta(v)}$ holds while the inclusion $\lambda^{\beta(v)} \subset \lambda^\gamma$ does not hold, since the v -convergence of the sequence of partial sums of a double series does not imply its boundedness.

The space λ of double sequences is said to be *solid* if and only if

$$\tilde{\lambda} = \{(u_{kl}) \in \Omega : \exists (x_{kl}) \in \lambda \text{ such that } |u_{kl}| \leq |x_{kl}| \text{ for all } k, l \in \mathbb{N}\} \subset \lambda.$$

The space λ of double sequences is also said to be *monotone* if and only if $m_0\lambda \subset \lambda$, where m_0 is the span of the set of all sequences of zeros and ones and $m_0\lambda = \{ax = (a_{ij}x_{ij}) : a \in m_0, x \in \lambda\}$. If λ is monotone, then $\lambda^\alpha = \lambda^{\beta(v)}$ (cf. Zeltser [11, p. 36]) and λ is monotone whenever λ is solid.

Prior to giving the theorem which asserts that the α - and γ -duals of a solid space of double sequences are identical, we quote two lemmas which are needed in proving the theorem.

Lemma 2.3. [6, Theorem 2, p. 279] *A positive term double series converges to its l.u.b. (that is the l.u.b. of its partial sums) if it is bounded above. Otherwise it diverges to $+\infty$.*

Lemma 2.4. [2, p. 382] *A double series is absolutely convergent if and only if the set*

$$\left\{ \sum_{i,j=1}^{m,n} |x_{ij}| : m, n \in \mathbb{N} \right\}$$

is a bounded set of real numbers.

Now, we may give the theorem

Theorem 2.5. *If a given double sequence space λ is solid, then the equality $\lambda^\alpha = \lambda^\gamma$ holds.*

Proof. To prove the theorem, it is enough to show that the inclusion $\lambda^\gamma \subset \lambda^\alpha$ holds. Suppose that the sequence space λ is solid and take any $y = (y_{kl}) \in \lambda^\gamma$. Then,

$$\sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=1}^{m,n} x_{kl} y_{kl} \right| < \infty$$

for any $x = (x_{kl}) \in \lambda$. Now, define the sequence $z = (z_{kl})$ via the sequence $x = (x_{kl}) \in \lambda$ by $z_{kl} := x_{kl} \operatorname{sgn}(x_{kl} y_{kl})$ for all $k, l \in \mathbb{N}$. Then, $z = (z_{kl}) \in \lambda$ since λ is solid and $|z_{kl}| \leq |x_{kl}|$ for all $k, l \in \mathbb{N}$. Therefore,

$$\begin{aligned} \sup_{m,n \in \mathbb{N}} \sum_{k,l=1}^{m,n} |x_{kl} y_{kl}| &= \sup_{m,n \in \mathbb{N}} \sum_{k,l=1}^{m,n} x_{kl} y_{kl} \operatorname{sgn}(x_{kl} y_{kl}) \\ &= \sup_{m,n \in \mathbb{N}} \left| \sum_{k,l=1}^{m,n} y_{kl} z_{kl} \right| < \infty. \end{aligned}$$

This shows that the positive term double series $\sum_{k,l} |x_{kl} y_{kl}|$ is bounded which is convergent by Lemma 2.3. Therefore, one can see by Lemma 2.4 that $(x_{kl} y_{kl})_{k,l \in \mathbb{N}} \in \mathcal{L}_1$. Since $x \in \lambda$ is arbitrary, y must be in λ^α , i.e., the inclusion $\lambda^\gamma \subset \lambda^\alpha$ holds.

This step terminates the proof. □

As an easy consequence of Theorem 2.5, we have

Corollary 2.6. *If λ is solid then $\lambda^\alpha = \lambda^{\beta(v)} = \lambda^\gamma$.*

One can easily observe that the double sequence space \mathcal{L}_q is solid, if $q > 1$. This yields to us that the double sequence space \mathcal{L}_q is monotone which implies the fact that the α - and the $\beta(v)$ -duals of the space \mathcal{L}_q are identical.

Now, we may give the theorem on the $\beta(v)$ -dual of the space \mathcal{L}_q .

Theorem 2.7. *The $\beta(v)$ -dual of the space \mathcal{L}_q is the space $\mathcal{L}_{q'}$, where $q > 1$ and $q^{-1} + q'^{-1} = 1$.*

Proof. Let $q > 1$ and $q^{-1} + q'^{-1} = 1$. Let us take any $x \in \mathcal{L}_{q'}$ and $y \in \mathcal{L}_q$. Consider the inequalities

$$|x_{mn} y_{mn}| \leq \frac{|x_{mn}|^{q'}}{q'} + \frac{|y_{mn}|^q}{q} \leq |x_{mn}|^{q'} + |y_{mn}|^q$$

satisfied for all $m, n \in \mathbb{N}$. Therefore, we derive that

$$\sum_{m,n} |x_{mn}y_{mn}| \leq \sum_{m,n} |x_{mn}|^{q'} + \sum_{m,n} |y_{mn}|^q < \infty,$$

which leads us to the fact that $x \in \mathcal{L}_q^\alpha$, i.e., the inclusions

$$(2.2) \quad \mathcal{L}_{q'} \subset \mathcal{L}_q^\alpha \subset \mathcal{L}_q^{\beta(v)}$$

hold.

Conversely, take any $y = (y_{mn}) \in \mathcal{L}_q^{\beta(v)}$. For establishing the inclusion $\mathcal{L}_q^{\beta(v)} \subset \mathcal{L}_{q'}$, we use the analogous idea employing by Boos [3, p. 344, Theorem 7.1.11.c] for single sequences. Let us consider the linear functional f_n and the double sequence $y^{[n]}$ defined by

$$f_n : \quad \mathcal{L}_q \quad \longrightarrow \quad \mathbb{R}$$

$$x = (x_{kl}) \quad \longmapsto \quad f_n(x) := \sum_{k,l=1}^n x_{kl}y_{kl}$$

and

$$y^{[n]} = \begin{bmatrix} y_{11} & y_{12} & y_{13} & \cdots & y_{1n} & 0 & \cdots \\ y_{21} & y_{22} & y_{23} & \cdots & y_{2n} & 0 & \cdots \\ y_{31} & y_{32} & y_{33} & \cdots & y_{3n} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots \\ y_{n1} & y_{n2} & y_{n3} & \cdots & y_{nn} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{bmatrix}$$

for every $n \in \mathbb{N}$. Then, since $y^{[n]} \in \mathcal{L}_{q'}$, we obtain by Hölder's inequality that

$$|f_n(x)| \leq \sum_{k,l=1}^n |x_{kl}y_{kl}| = \sum_{k,l} |x_{kl}y_{kl}^{[n]}| \leq \|x\|_q \left\| y^{[n]} \right\|_{q'}$$

for each $x = (x_{kl}) \in \mathcal{L}_q$ which yields the continuity of the linear functionals f_n . Therefore, we have

$$(2.3) \quad \|f_n\| \leq \left\| y^{[n]} \right\|_{q'} \quad \text{for each } n \in \mathbb{N}.$$

Let us consider the sequence $x^{(n)} = \{x_{kl}^{(n)}\}_{k,l \in \mathbb{N}}$ to prove the reverse inequality, defined by

$$x_{kl}^{(n)} := \begin{cases} \frac{|y_{kl}|^{q'}}{y_{kl}} & , \quad (\text{if } y_{kl} \neq 0, \text{ and } k, l \leq n), \\ 0 & , \quad (\text{otherwise}). \end{cases}$$

Then, it is clear that $x^{(n)} \in \mathcal{L}_q$ and one can see that

$$\|x^{(n)}\|_q = \left(\sum_{k,l=1}^n |y_{kl}|^{(q'-1)q} \right)^{1/q} = \left(\sum_{k,l=1}^n |y_{kl}|^{q'} \right)^{1/q} = \left(\|y^{[n]}\|_{q'} \right)^{q'/q}.$$

This leads us to the consequence for all $n \in \mathbb{N}$ that

$$\frac{|f_n(x^{(n)})|}{\|x^{(n)}\|_q} = \frac{\sum_{k,l=1}^n |y_{kl}|^{q'}}{\|x^{(n)}\|_q} = \|y^{[n]}\|_{q'}.$$

Hence,

$$(2.4) \quad \|y^{[n]}\|_{q'} \leq \|f_n\| \text{ for all } n \in \mathbb{N}.$$

Therefore, we have by (2.3) and (2.4) that

$$\|f_n\| = \|y^{[n]}\|_{q'} \text{ for all } n \in \mathbb{N}.$$

By applying the Banach-Steinhaus Theorem, one can observe by our hypothesis that the sequence (f_n) of linear functionals converges pointwise. Since $(\mathcal{L}_q, \|\cdot\|_q)$ and $(\mathbb{C}, |\cdot|)$ are the Banach spaces, the linear functional defined by

$$f_y : \begin{array}{ll} \mathcal{L}_q & \longrightarrow \mathbb{R} \\ x = (x_{kl}) & \longmapsto f_y(x) := \lim_{n \rightarrow \infty} f_n(x) = \sum_{k,l} x_{kl} y_{kl} \end{array}$$

is continuous, and

$$\|f_y\| \leq \sup_{n \in \mathbb{N}} \|f_n\| = \sup_{n \in \mathbb{N}} \|y^{[n]}\|_{q'} < \infty$$

holds. Thus, we have $y \in \mathcal{L}_{q'}$, because of

$$\|f_y\| \leq \sup_{n \in \mathbb{N}} \|y^{[n]}\|_{q'} = \sup_{n \in \mathbb{N}} \left(\sum_{k,l=1}^n |y_{kl}|^{q'} \right)^{1/q'} = \left(\sum_{k,l} |y_{kl}|^{q'} \right)^{1/q'} < \infty.$$

That is to say that the inclusion

$$(2.5) \quad \mathcal{L}_q^{\beta(v)} \subset \mathcal{L}_{q'}$$

holds.

By combining the inclusions (2.2) and (2.5), the desired result immediately follows.

This completes the proof. □

As a direct consequence of Theorem 2.7, we have

Corollary 2.8. *The α -, $\beta(v)$ - and γ -duals of the space \mathcal{L}_q are the space $\mathcal{L}_{q'}$, where $q > 1$ and $q^{-1} + q'^{-1} = 1$.*

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