Thai Journal of Mathematics
Volume 14 (2016) Number 1 : 13–19
http://thaijmath.in.cmu.ac.th
ISSN 1686-0209

On Some Classes of Four Dimensional Regular Matrices

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Abstract : In 1981, Rath and Tripathy presented some classes of regular matrices such that every bounded sequence is limitable by some member of each class of ordinary conservative matrices using ordinary sequences. The goal of this paper is to present multidimensional analogues of their results.

Keywords : Double sequence; regular matrix; matrix class.
2010 Mathematics Subject Classification : 40B05; 40C05.

1 Introduction

By Ω, we denote the set of all real or complex valued double sequences, i.e.,

$$\Omega = \{x = (x_{mn}) : x_{mn} \in \mathbb{C} \text{ for all } m, n \in \mathbb{N}\},$$

which is a vector space with co-ordinatewise addition and scalar multiplication of double sequences, where $\mathbb{N}$ and $\mathbb{C}$ denote the set of positive integers and the complex field, respectively. Any vector subspace of Ω is called as a double sequence space. The space $\mathcal{M}_u$ of all bounded double sequences is defined by

$$\mathcal{M}_u = \left\{ x = (x_{mn}) \in \Omega : \|x\|_\infty = \sup_{m,n\in\mathbb{N}} |x_{mn}| < \infty \right\}$$

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which is a Banach space with the norm $\|\cdot\|_\infty$. Consider the sequence $x = (x_{mn}) \in \Omega$. If for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ and $\ell \in \mathbb{C}$ such that

$$|x_{mn} - \ell| < \varepsilon$$

for all $m, n > n_0$ then we call that the double sequence x is convergent in the Pringsheim's sense to the limit ℓ and write $\lim_{m,n} x_{mn} = \ell$. By $\mathcal{C}_p$, we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space $\mathcal{C}_p$ but not in the space $\mathcal{M}_u$ (for example, $x = (x_{kl})$ defined by $x_{kl} = l$ if $k = 1$, otherwise $x_{kl} = 0$). So, we may mention the space $\mathcal{C}_{bp}$ of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. By $\mathcal{C}_{bp0}$, we denote the space of the double sequences which are both convergent to zero in the Pringsheim's sense and bounded. Type of convergence of double sequence is more than one, so we denote convergence by υ-convergence for $\upsilon \in \{p, bp\}$.

Let X be the space of double sequences, converging with respect to some linear convergence rule $\upsilon-\lim : X \to \mathbb{C}$. The sum of a double series $\sum_{i,j} x_{ij}$ with respect to this rule is defined by $\upsilon-\sum_{ij} x_{ij} = \upsilon-\lim_{m,n} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}$. Let X, Y be two spaces of double sequences, converging with respect to the linear convergence rules $\upsilon_1 - \lim$ and $\upsilon_2 - \lim$, respectively, and $A = (a_{mnkl})$ also be a four dimensional matrix of real or complex numbers. Define the set

$$X_A^{(\upsilon_2)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \upsilon_2 - \sum_{k,l} a_{mnkl} x_{kl} \text{ exists and } Ax \in X \right\}. \quad (1.1)$$

$\beta(\upsilon)-$dual $X^{\beta(\upsilon)}$ with respect to the $\upsilon-$convergence for $\upsilon \in \{p, bp\}$ of a double sequence space X is defined by

$$X^{\beta(\upsilon)} = \left\{ (a_{ij}) \in \Omega : \upsilon - \sum_{i,j} a_{ij} x_{ij} \text{ exists for all } (x_{ij}) \in X \right\}$$

Then, we say, with the notation of (1.1), that A maps the space X into the space Y if $Y \subset X_A^{(\upsilon_2)}$ and denote the set of all four-dimensional matrices, mapping the space X into the space Y, by $(X : Y)$. It is trivial that for any matrix $A \in (X : Y)$, $(a_{mnkl})_{k,l\in\mathbb{N}}$ is in the $\beta(\upsilon_2)-$dual $X^{\beta(\upsilon_2)}$ of the space X for all $m, n \in \mathbb{N}$. An infinite matrix A is said to be $\mathcal{C}_\upsilon$-conservative if $\mathcal{C}_\upsilon \subset (\mathcal{C}_\upsilon)_A$.

For more details on double sequences and 4-dimensional matrices, we refer to [2, 8, 9, 10, 11, 12].

A matrix A is said to be *RH*-regular if it maps every bounded convergent sequence into a convergent sequence with the same limit.

Lemma 1.1 ([2, 8])**.** *The necessary and sufficient conditions for A to be RH-*

regular are

$$\lim_{m,n} a_{mnkl} = 0, \text{ for each } k, l \in \mathbb{N},$$

$$\lim_{m,n} \sum_{k,l}^{\infty,\infty} a_{mnkl} = 1,$$

$$\lim_{m,n} \sum_{k}^{\infty} |a_{mnkl}| = 0 \text{ for each } l \in \mathbb{N},$$

$$\lim_{m,n} \sum_{l}^{\infty} |a_{mnkl}| = 0 \text{ for each } k \in \mathbb{N},$$

$$\sum_{k,l}^{\infty,\infty} |a_{mnkl}| \text{ is convergence}$$

and

$$\text{there exist positive numbers } B \text{ and } C \text{ such that } \sum_{k,l>C} |a_{mnkl}| < B.$$

By using the definitions of limit inferior, limit superior, Patterson [5] introduced the core of a double sequence with the notion of the regularity of four dimensional matrices and in [6] proved an invariant core theorem. Mursaleen [3] and Mursaleen and Edely [4] defined the almost strong regularity of matrices for double sequences and apply these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the M-core of x. Çakan and Altay [1] investigated statistical core for double sequences and studied an inequality related to the statistical and P-core of bounded double sequences.

In 1981, Rath and Tripathy [7] presented some classes of regular matrices such that every bounded sequence is limitable by some member of each class of ordinary conservative matrices using ordinary sequences. The goal of this paper is to present multidimensional analogues of their results.

2 Main Results

For each $\alpha \in I$ such that $I = [0,1]$ we denote by $Q_2(\alpha)$ the class of RH-regular matrices $A = (a_{mnkl})$ with real entry such that $a_{mnkl} \geq 0$ for all $m, n, k, l \in \mathbb{N}$ and in addition, for each $m, n \in \mathbb{N}$, there are $k, l \in \mathbb{N}$ such that $a_{mnkl} = \alpha$ and also for each $k, l \in \mathbb{N}$, there are $m, n \in \mathbb{N}$ such that $a_{mnkl} = \alpha$.

Theorem 2.1. *Let $x = (x_{kl})$ any bounded double sequence. For every $\alpha \in [0, \frac{1}{2}]$ and $\lambda \in I$, there is $A \in Q_2(\alpha)$ limiting $x = (x_{kl})$ to the value $\lambda a + (1-\lambda)b$, where $L(x) = \limsup_{kl} x_{kl}$, $l(x) = \liminf_{kl} x_{kl}$, $a = \alpha L(x) + (1-\alpha)l(x)$ and $b = \alpha l(x) + (1-\alpha)L(x)$.*

Proof. We have

$$\lambda a + (1-\lambda)b = cl(x) + (1-c)L(x),$$

where $c = \lambda + \alpha - 2\lambda\alpha \in I$. A null double sequence ε_{kl} can be found that for every $k, l \in \mathbb{N}$,

$$l(x) - \varepsilon_{kl} \leq x_{kl} \leq L(x) + \varepsilon_{kl}$$

and so we write

$$x_{kl} = \mu_{kl}(l(x) - \varepsilon_{kl}) + \lambda_{kl}(L(x) + \varepsilon_{kl}),$$

where $\mu_{kl} \geq 0, \lambda_{kl} \geq 0$ and $\mu_{kl} + \lambda_{kl} = 1$ for each k, l. Writing $\alpha_{kl} = c - \alpha\mu_{kl}$ and $\beta_{kl} = 1 - c - \alpha\lambda_{kl}$, we see that

$$0 \leq \alpha_{kl} \leq 1 - \alpha \text{ and } \beta_{kl} = 1 - \alpha - \alpha_{kl} \geq 0.$$

Let (p_k, q_l) and (r_k, s_l) be increasing double sequences of integers such that $(p_k, q_l) \neq (k, l)$, $(p_k, q_l) \neq (r_k, s_l)$ and $(r_k, s_l) \neq (k, l)$ for each $k, l \in \mathbb{N}$ and

$$\lim_{k,l} x_{p_k,q_l} = l(x), \quad \lim_{k,l} x_{r_k,s_l} = L(x).$$

Define $A \in Q_2(\alpha)$ by

$$a_{klmn} = \begin{cases} \alpha & , \quad (m,n) = (k,l) \\ \alpha_{kl} & , \quad (m,n) = (p_k, q_l) \\ \beta_{kl} & , \quad (m,n) = (r_k, s_l) \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \sum_{m,n=1,1}^{\infty,\infty} a_{klmn} x_{mn} &= \alpha\mu_{kl}(l(x) - \varepsilon_{kl}) + \alpha\lambda_{kl}(L(x) + \varepsilon_{kl}) + \alpha_{kl} l(x) \\ &\quad + \beta_{kl} L(x) + \alpha_{kl}(x_{p_k,q_l} - l(x)) + \beta_{kl}(x_{r_k,s_l} - L(x)) \\ &= cl(x) + (1-c)L(x) - \varepsilon_{kl}\alpha\mu_{kl} + \varepsilon_{kl}\alpha\lambda_{kl} \\ &\quad + \alpha_{kl}(x_{p_k,q_l} - l(x)) + \beta_{kl}(x_{r_k,s_l} - L(x). \end{aligned}$$

We take limit for $k, l \to \infty$, then we obtain

$$Ax \to cl(x) + (1-c)L(x)$$

which is desired. □

Theorem 2.2. *If $\alpha \in I$ and $x = (x_{kl}) \in \mathcal{M}_u$ which is limitable by a matrix $A \in Q_2(\alpha)$, then $a \leq \lim_{m,n} A_{mn}x \leq b$; where a, b are defined as Theorem 2.1 in above.*

Proof. If $\alpha = 0$, $a = l(x)$ and $b = L(x)$ the conclusion follows from [5, Theorem 3.2]. Suppose that $\alpha > 0$. Let (p_k, q_l) and (r_k, s_l) be increasing sequences of integers such that $\lim_{k,l} x_{p_k q_l} = l(x)$, $\lim_{k,l} x_{r_k s_l} = L(x)$.

For each k, l we can find integers $(t_k, z_l), (d_k, h_l)$ such that $a_{t_k z_l p_k q_l} = \alpha$ and $a_{d_k h_l r_k s_l} = \alpha$. By the RH-regularity of A it follows that

$$(t_k, z_l) \to \infty, \quad (d_k, h_l) \to \infty, \qquad \sum_{(m,n)\neq(p_k,q_l)} a_{t_k z_l mn} \to 1 - \alpha$$

as $k, l \longrightarrow \infty$. For any $\varepsilon > 0$, positive integers (m_1, n_1) and (k_1, l_1) can be found that for $(m, n) > (m_1, n_1)$ and $(k, l) > (k_1, l_1)$, $x_{kl} < L(x) + \varepsilon$,

$$\sum_{(m,n)\neq(t_k,z_l)} a_{t_k z_l mn} < 1 - \alpha + \varepsilon.$$

Then for $(k, l) > (k_1, l_1)$,

$$\begin{aligned}\sum_{m,n=1,1}^{\infty,\infty} a_{t_k z_l mn} x_{mn} &= \alpha x_{p_k,q_l} + \sum_{\substack{(m,n)\neq(p_k,q_l)\\ m\leq m_1 \cup n\leq n_1}} a_{t_k z_l mn} x_{mn} + \sum_{\substack{(m,n)\neq(p_k,q_l)\\ (m,n)>(m_1,n_1)}} a_{t_k z_l mn} x_{mn} \\ &< \alpha x_{p_k,q_l} + \sum_{\substack{(m,n)\neq(p_k,q_l)\\ m\leq m_1 \cup n\leq n_1}} a_{t_k z_l mn} x_{mn} \\ &\quad + \sum_{\substack{(m,n)\neq(p_k,q_l)\\ (m,n)>(m_1,n_1)}} a_{t_k z_l mn} x_{mn} + [L(x) + \varepsilon](1 - \alpha + \varepsilon).\end{aligned}$$

If we take limit $k, l \longrightarrow \infty$ then we obtain

$$\lim_{k,l} (Ax)_{kl} \leq \alpha l(x) + (1 - \alpha) L(x).$$

Proceeding with $(r_k, s_l), (d_k, h_l)$ in place of $(p_k, q_l), (t_k, z_l)$, we can similarly show that

$$\lim_{k,l} (Ax)_{kl} \geq \alpha L(x) + (1 - \alpha) l(x).$$

□

Theorem 2.3. *If $\alpha > \frac{1}{2}$ and $A \in Q_2(\alpha)$, then A cannot limit any bounded divergent double sequence.*

Proof. Suppose that A limits bounded real double sequence $x = (x_{kl})$. Let $\liminf_{k,l} x_{kl}$ and $\limsup_{k,l} x_{kl}$ be equal to l and L, respectively. In view of Theorem 2.2, we have

$$\alpha L + (1 - \alpha) l \leq \alpha l + (1 - \alpha) L$$

that is,

$$(2\alpha - 1)L \leq (2\alpha - 1)l$$

and the conclusion immediately follows. □

Acknowledgement : The authors express their gratitude to the referee for making some useful comments on the first draft of the manuscript which improved the presentation of the paper.

References

[1] C. Çakan, B. Altay, Statistically boundedness and statistical core of double sequences, J. Math. Anal. Appl. 317 (2006) 690–697.

[2] H.J. Hamilton, Transformations of multiple of sequences, Duke Math. J. 2 (1936) 29–60.

[3] Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, J. Math. Anal. Appl. 293 (2) (2004) 523–531.

[4] Mursaleen and O. H. H. Edely, Almost convergence and a core theorem for double sequences, J. Math. Anal. Appl. 293 (2) (2004) 532–540.

[5] R.F. Patterson, Double sequence core theorems, Int. J. Math. & Math. Sci. 22 (4) (1999) 785–793.

[6] R. F. Patterson, Invariant core theorems for double sequences, Southeast Asian Bull. Math. 24 (3) (2000) 421–429.

[7] D. Rath, B.K. Tripathy, On some classes of regular matrices, Duke Math. J. 84 (1981) 530–533.

[8] G. M. Robison, Divergent double sequences and series, Trans. Amer. Math. Soc. 28 (1) (1926) 50–73.

[9] M. Zeltser, Matrix transformations of double sequences, Acta Comment. Univ. Tartu. Math. 4 (2000) 39–51.

[10] M. Zeltser, Investigation of double sequence spaces by soft and hard analytical methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, 2001.

[11] M. Zeltser, On conservative and coercive SM-methods, Proc. Est. Acad. Sci. Phys. Math. 50 (2) (2001) 76–85.

[12] M. Zeltser, On conservative matrix methods for double sequence spaces, Acta Math. Hung. 95 (3) (2002) 225–242.

(Received 17 July 2012)
(Accepted 23 January 2014)